Math 242 Exam #3 Key (Fall 2016)

1 The surface Σ is given by $F(x,y,z)=0$ for $F(x,y,z)=xy\sin z-1$. We have

$$\nabla F(x,y,z)=\langle y\sin z, x\sin z, xy\cos z\rangle,$$

and the equation of the tangent plane at $\left(-2,-1,\frac{5\pi}{6}\right)$ is given by

$$0=\nabla F\left(-2,-1,\tfrac{5\pi}{6}\right)\cdot\left\langle x+2,y+1,z-\tfrac{5\pi}{6}\right\rangle=\left\langle -\tfrac{1}{2},-1,\sqrt{3}\right\rangle\cdot\left\langle x+2,y+1,z-\tfrac{5\pi}{6}\right\rangle,$$

or

$$3x+6y-6z\sqrt{3}=-12-5\pi\sqrt{3}.$$

2 First we gather our partial derivatives:

$$f_x(x,y)=-3x^2+4y, \qquad f_y(x,y)=4x-4y,$$
$$f_{xx}(x,y)=-6x, \qquad f_{yy}(x,y)=-4,$$
$$f_{xy}(x,y)=4.$$

At no point does f_x or f_y fail to exist, so we search for any (x,y) for which $f_x(x,y)=f_y(x,y)=0$. This yields the system

$$\begin{cases}-3x^2+4y=0\\ \quad 4x-4y=0\end{cases}$$

The solutions are $(0,0)$ and $\left(\frac{4}{3},\frac{4}{3}\right)$, which are the critical points. Defining $\Phi=f_{xx}f_{yy}-f_{xy}^2$, we construct a table:

(x,y)	f_{xx}	f_{yy}	f_{xy}	Φ	Conclusion
$(0,0)$	0	0	4	-16	Saddle Point
$\left(\frac{4}{3},\frac{4}{3}\right)$	-8	-4	4	16	Local Maximum

3 This is problem 13.8.59 in the textbook (also this was done in class). The point on the curve is $\left(\frac{1}{2},\frac{1}{4}\right)$, and the point on the line is $\left(\frac{7}{8},-\frac{1}{8}\right)$.

4 Set $g(x,y)=2(x-1)^2+4(y-1)^2-1$, so the constraint is $g(x,y)=0$. Find all $(x,y)\in\mathbb{R}^2$ for which there can be found some $\lambda\in\mathbb{R}$ such that the system

$$\begin{cases}f_x(x,y)=\lambda g_x(x,y)\\ f_y(x,y)=\lambda g_y(x,y)\\ \;g(x,y)=0\end{cases}$$

has a solution. Explicitly the system is

$$\begin{cases}2=4\lambda(x-1)\\ 1=8\lambda(y-1)\\ 1=2(x-1)^2+4(y-1)^2\end{cases}\tag{1}$$

Clearly no solution can have $x = 1$, so the 1st equation can be written as $\lambda = \frac{1}{2(x-1)}$. Substitute this into the 2nd equation to get

$$1 = \frac{8(y-1)}{2(x-1)} \quad \Rightarrow \quad x = 4y - 3.$$

Put this last result into the 3rd equation:

$$2[(4y-3)-1]^2 + 4(y-1)^2 = 1 \quad \Rightarrow \quad 36(y-1)^2 = 1 \quad \Rightarrow \quad |y-1| = \frac{1}{6} \quad \Rightarrow \quad y = 1 \pm \frac{1}{6},$$

so $y = \frac{5}{6}, \frac{7}{6}$. Since $x = 4y - 3$, we obtain two points: $\left(\frac{1}{3}, \frac{5}{6}\right)$ and $\left(\frac{5}{3}, \frac{7}{6}\right)$. We evaluate $f(x, y) = 2x + y + 10$ at these points: $f\left(\frac{1}{3}, \frac{5}{6}\right) = 11.5$ and $f\left(\frac{5}{3}, \frac{7}{6}\right) = 14.5$. Therefore the minimum values of $f(x, y)$ subject to the constraint $g(x, y) = 0$ is 11.5 at $\left(\frac{1}{3}, \frac{5}{6}\right)$, and the maximum value is 14.5 at $\left(\frac{5}{3}, \frac{7}{6}\right)$.

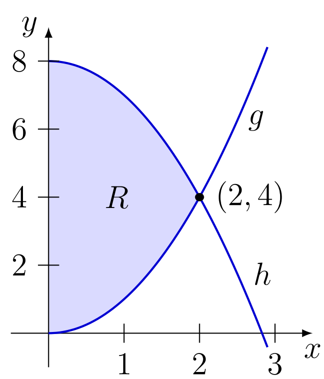
5 By Fubini's Theorem, making the substitution $u = 1 + xy$ in the inner integral,

$$\begin{aligned}\iint_R \frac{x}{(1+xy)^2} dA &= \int_0^4 \int_1^3 \frac{x}{(1+xy)^2}\, dy\, dx = \int_0^4 \int_{x+1}^{3x+1} \frac{1}{u^2}\, du\, dx \\ &= \int_0^4 \left(\frac{1}{x+1} - \frac{1}{3x+1} \right) dx \\ &= \left[\ln(x+1) - \tfrac{1}{3}\ln(3x+1) \right]_0^4 = \ln\left(\tfrac{5}{\sqrt[3]{13}}\right).\end{aligned}$$

6 First determine where the curves $g(x) = x^2$ and $h(x) = 8 - x^2$ intersect:

$$g(x) = h(x) \quad \Rightarrow \quad x^2 = 8 - x^2 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2.$$

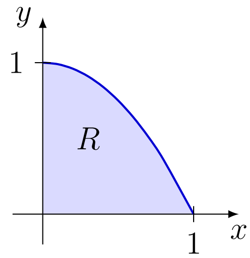
Since R is in Quadrant I, only $x = 2$ is relevant, and this yields the intersection point $(2, 4)$. Thus R is as shown below. In R we have $x^2 \le y \le 8 - x^2$ for each $0 \le x \le 2$, and so



$$\begin{aligned}\iint_R (x+y)\, dA &= \int_0^2 \int_{x^2}^{8-x^2} (x+y)\, dy\, dx = \int_0^2 \left[xy + \frac{1}{2}y^2 \right]_{x^2}^{8-x^2} dx \\ &= \int_0^2 \left[\left(x(8-x^2) + \frac{1}{2}(8-x^2)^2 \right) - \left(x^3 + \frac{1}{2}(x^2)^2 \right) \right] dx\end{aligned}$$

$$= \int_0^2 \left(32 + 8x - 8x^2 - 2x^3\right) dx = \frac{152}{3}.$$

7 On the xy-plane we have $z = 0$, and so the nonplanar boundary of the solid is given on the xy-plane as $1 - y - x^2 = 0$, or $y = 1 - x^2$. Thus the part of the solid lying on the xy-plane is the region R shown below:



At each $(x, y) \in R$ the height of the solid is given by $h(x, y) = 1 - y - x^2$, and so

$$\text{Volume} = \iint_R h = \int_0^1 \int_0^{1-x^2} (1 - y - x^2)\, dy\, dx = \int_0^1 \left(\tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4\right) dx = \frac{1}{15}.$$

8 By definition,

$$\int_0^\infty e^{-x-y} dy = \lim_{t\to\infty} \int_0^t e^{-x-y} dy = \lim_{t\to\infty} e^{-x} \int_0^t e^{-y} dy = \lim_{t\to\infty} e^{-x}\left(1 - e^{-t}\right) = e^{-x},$$

and so

$$\int_0^\infty \int_0^\infty e^{-x-y} dy\, dx = \int_0^\infty e^{-x} dx = \lim_{t\to\infty} \int_0^t e^{-x} dx = \lim_{t\to\infty} \left(1 - e^{-t}\right) = 1.$$

9 Find where the paraboloids $p(x, y) = x^2 + y^2$ and $q(x, y) = 2 - x^2 - y^2$ intersect, which is to say find (x, y) for which $p(x, y) = q(x, y)$:

$$p(x, y) = q(x, y) \quad \Rightarrow \quad x^2 + y^2 = 2 - x^2 - y^2 \quad \Rightarrow \quad x^2 + y^2 = 1.$$

This occurs at $z = 1$, since $z = p(x, y) = x^2 + y^2 = 1$. That is, the paraboloids intersect at the circle in $\mathbb{R}^3$ given by $x^2 + y^2 = 1$ and $z = 1$. Project this circle down to the corresponding circle C in the xy-plane, and let R be the region inside C. For each $(x, y) \in R$ the height of the solid is

$$h(x, y) = \big|p(x, y) - q(x, y)\big| = q(x, y) - p(x, y) = (2 - x^2 - y^2) - (x^2 + y^2) = 2 - 2x^2 - 2y^2.$$

Using polar coordinates, we find that

$$\begin{aligned}\text{Volume} &= \iint_R h(x, y)\, dA = \int_0^{2\pi} \int_0^1 h(r\cos\theta, r\sin\theta) r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (2 - 2r^2) r\, dr\, d\theta = \int_0^{2\pi} \left[r^2 - \tfrac{1}{2}r^4\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi.\end{aligned}$$

10 The region

$$S = \{(r,\theta) : 0 \le r \le 2 + \sin\theta \text{ and } 0 \le \theta \le 2\pi\}$$

in the $r\theta$-plane corresponds to the region R in the xy-plane enclosed by the curve $r = 2 + \sin\theta$ shown below. The area of R is

$$\mathcal{A}(R) = \iint_R dA = \int_0^{2\pi} \int_0^{2+\sin\theta} r\,dr\,d\theta = \int_0^{2\pi} \left(2 + 2\sin\theta + \tfrac{1}{2}\sin^2\theta\right) d\theta.$$

Using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ yields

$$\mathcal{A}(R) = \int_0^{2\pi} \left(\tfrac{9}{4} + 2\sin\theta - \tfrac{1}{4}\cos 2\theta\right) d\theta = \int_0^{2\pi} \frac{9}{4}\,d\theta = \frac{9}{2}\pi.$$