Math 242 Exam #2 Key (Fall 2016)

1 A normal vector for plane is $\langle 1,-3,4\rangle \times \langle 4,0,-2\rangle = \langle 6,18,12\rangle$, or alternatively $\mathbf{n} = \langle 1,3,2\rangle$. The plane consists of all $\mathbf{x}$ for which $\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)=0$, where $\mathbf{x}_0 = \langle 1,0,1\rangle$. Thus:

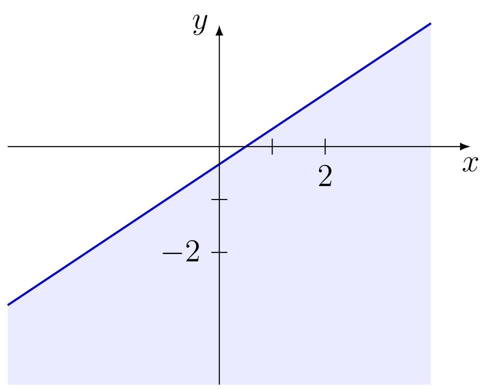
$$\langle 1,3,2\rangle \cdot \langle x-1, y, z-1\rangle = 0,$$

or $x+3y+2z=3$.

2 xy-trace is $x^2+y^2/4=0$ (an ellipse), xz-trace is $z=\pm x$ (two lines), and yz-trace is $z=\pm y/2$ (two lines).

3 The domain is

$$\text{Dom}(f) = \{(x,y)\in\mathbb{R}^2 : 2x-3y\geq 1\} = \left\{(x,y)\in\mathbb{R}^2 : y \leq \frac{2x-1}{3}\right\}.$$



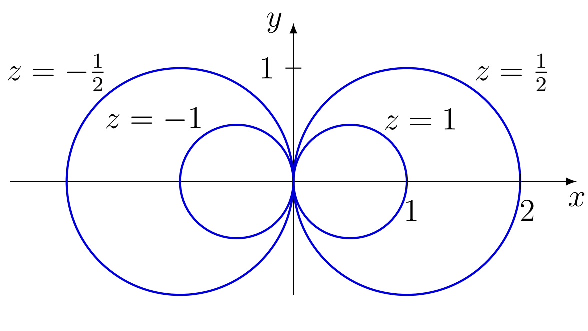
4 Level curve $F(x,y)=\pm 1$ has equation

$$\frac{x}{x^2+y^2} = \pm 1 \quad\Rightarrow\quad x^2+y^2=\pm x \quad\Rightarrow\quad (x\pm\tfrac{1}{2})^2+y^2=\tfrac{1}{4},$$

a circle with center $(\pm\frac{1}{2},0)$ and radius $\frac{1}{2}$. Level curve $F(x,y)=\pm\frac{1}{2}$ has equation

$$\frac{x}{x^2+y^2} = \pm\frac{1}{2} \quad\Rightarrow\quad x^2+y^2=\pm 2x \quad\Rightarrow\quad (x\pm 1)^2+y^2=1,$$

a circle with center $(\pm 1,0)$ and radius 1.



5 We have

$$\lim_{(x,y)\to(1,1)} \frac{x^2+xy-2y^2}{2x^2-xy-y^2} = \lim_{(x,y)\to(1,1)} \frac{(x-y)(x+2y)}{(2x+y)(x-y)} = \lim_{(x,y)\to(1,1)} \frac{x+2y}{2x+y} = \frac{1+2(1)}{2(1)+1} = 1.$$

6 Along the path $y = x$ the limit becomes

$$\lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{x}{x^2+1} = 0.$$

Along the path $y = \sqrt{x}$ the limit becomes

$$\lim_{x\to 0^+} \frac{x(\sqrt{x})^2}{x^2+(\sqrt{x})^4} = \lim_{x\to 0^+} \frac{x^2}{x^2+x^2} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}.$$

The limits on the chosen paths are not equal, therefore the limit does not exist by the Two-Path Test.

7 We have

$$\psi_x(t,x) = 2x\sec(t^3x) + x^2\sec(t^3x)\tan(t^3x)\cdot\frac{\partial}{\partial x}(t^3x) = 2x\sec(t^3x) + t^3x^2\sec(t^3x)\tan(t^3x)$$

and

$$\psi_t(t,x) = x^2\sec(t^3x)\tan(t^3x)\cdot\frac{\partial}{\partial t}(t^3x) = 3t^2x^3\sec(t^3x)\tan(t^3x)$$

8a We have $\psi(h,0) = 0$ and $\psi(0,h) = 0$ for any $h \neq 0$. Now,

$$\psi_x(0,0) = \lim_{h\to 0} \frac{\psi(h,0)-\psi(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and

$$\psi_y(0,0) = \lim_{h\to 0} \frac{\psi(0,h)-\psi(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

8b Along the path $y = x$ we have

$$\lim_{(x,y)\to(0,0)} \psi(x,y) = \lim_{x\to 0} \frac{5x^3}{x^3+x^3} = \frac{5}{2},$$

so right away we see that

$$\lim_{(x,y)\to(0,0)} \psi(x,y) \neq 0 = \psi(0,0),$$

and therefore ψ is not continuous at $(0,0)$.

8c Since ψ is not continuous at $(0,0)$, it cannot be differentiable at $(0,0)$.

9a From $\nabla f(x,y) = \langle 2x+4y, 4x-3y^2\rangle$ we get $\nabla f(3,-1) = \langle 2, 9\rangle$.

9b Steepest ascent and descent:

$$\frac{\nabla f(3,-1)}{\|\nabla f(3,-1)\|} = \frac{1}{\sqrt{85}}\langle 2,9\rangle \quad \text{and} \quad -\frac{\nabla f(3,-1)}{\|\nabla f(3,-1)\|} = -\frac{1}{\sqrt{85}}\langle 2,9\rangle,$$

respectively. No change: we need a vector orthogonal to $\langle 2,9\rangle$, such as $\langle -9,2\rangle$, then normalize to get

$$\frac{1}{\sqrt{85}}\langle -9,2\rangle.$$

10 The path is a curve C given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$, where $\mathbf{r}(0) = \langle 10, 10\rangle$. For C to be the path of steepest ascent (i.e. greatest temperature increase), for any $t \geq 0$ the tangent vector to C at $\mathbf{r}(t)$, which is $\mathbf{r}'(t)$, must be in the direction of

$$\nabla T(\mathbf{r}(t)) = \nabla T(x(t), y(t)) = \big\langle T_x(x(t), y(t)), T_y(x(t), y(t))\big\rangle = \langle -4x(t), -2y(t)\rangle.$$

So we set

$$\mathbf{r}'(t) = \langle -4x(t), -2y(t)\rangle,$$

which gives the differential equations $x'(t) = -4x(t)$ and $y'(t) = -2y(t)$. We treat the first equation using Separation of Variables:

$$\frac{dx}{dt} = -4x \quad \Rightarrow \quad \int \frac{1}{x}\,dx = -\int 4\,dt \quad \Rightarrow \quad \ln|x| = -4t + c,$$

c an arbitrary constant. Since $x > 0$ at the initial point $(10, 10)$, we have $|x| = x$, and thus $\ln x = -4t + c$. Now,

$$\ln x = -4t + c \quad \Rightarrow \quad x(t) = e^{-4t+c} = Ke^{-4t},$$

where $K = e^c$. From the initial condition $x(0) = 10$ we find that $K = 10$, and hence $x(t) = 10e^{-4t}$. A nearly identical routine shows that $y' = -2y$ has solution $y(t) = 10e^{-2t}$. Now we have

$$\mathbf{r}(t) = \langle 10e^{-4t}, 10e^{-2t}\rangle, \quad t \in [0, \infty). \tag{1}$$

This is a good answer.

There's the option of going farther. We see in (1) that $x/10 = e^{-4t}$ and $y/10 = e^{-2t}$, so that $(y/10)^2 = x/10$, and hence $y^2 = 10x$. This eliminates the parameter t, but the whole graph of $y^2 = 10x$ is not exactly C. In fact, x starts at 10, and then decreases in value as t increases since $x'(t) = -40e^{-4t} < 0$. Therefore C is precisely given by.

$$y^2 = 10x, \quad x \in (-\infty, 10].$$