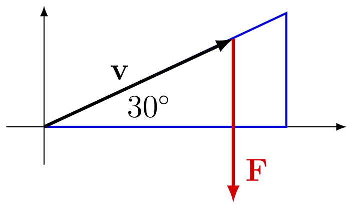
Math 242 Exam #1 Key (Fall 2016)

1 There exist precisely two vectors that fit the bill: $7\mathbf{u} = \langle 14, -35\rangle$ and $-7\mathbf{u} = \langle -14, 35\rangle$.

2 The magnitude of the force of gravity is given by mass times acceleration, or in this case 980 N, and so the force vector is $\mathbf{F} = \langle 0, -980\rangle$. The picture is thus:



Here $\mathbf{v} = \langle\sqrt{3}, 1\rangle$ is a vector parallel to the surface of the ramp. The component of $\mathbf{F}$ parallel to the surface of the ramp, $\mathbf{F}_{\parallel}$, is thus parallel to $\mathbf{v}$. We have

$$\mathbf{F}_{\parallel} = \operatorname{proj}_{\mathbf{v}} \mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = -245\left\langle\sqrt{3}, 1\right\rangle.$$

The component of $\mathbf{F}$ perpendicular to the ramp, $\mathbf{F}_{\perp}$, must be such that $\mathbf{F}_{\parallel} + \mathbf{F}_{\perp} = \mathbf{F}$, and so

$$\mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{\parallel} = \left\langle 245\sqrt{3}, -735\right\rangle = 245\left\langle\sqrt{3}, -3\right\rangle.$$

3 Let $p = (1,2,3)$, $q = (4,7,1)$, and $r = (x,y,2)$. We need $\overrightarrow{pr}$ to be parallel to $\overrightarrow{pq}$. That is, we need $\overrightarrow{pr} = k(\overrightarrow{pq})$ for some $k \neq 0$. Since $\overrightarrow{pr} = \langle x-1, y-2, -1\rangle$ and $\overrightarrow{pq} = \langle 3, 5, -2\rangle$, it follows that we need

$$\langle x-1, y-2, -1\rangle = \langle 3k, 5k, -2k\rangle.$$

This in particular requires that $-2k = -1$, or $k = 1/2$. Now,

$$x - 1 = 3k \quad\Rightarrow\quad x - 1 = \tfrac{3}{2} \quad\Rightarrow\quad x = \tfrac{5}{2},$$

and

$$y - 2 = 5k \quad\Rightarrow\quad y - 2 = \tfrac{5}{2} \quad\Rightarrow\quad y = \tfrac{9}{2}.$$

4a Each of the points O, P, and Q are a distance of 2 from the others, and so the spheres each have radius. The sphere centered at C is shown to be symmetrically placed atop the lower three spheres, and so C is a distance of 2 from O, P, and Q. Thus C, O, P, and Q form a tetrahedron with edges of length 2, and so C must lie directly over the geometric center T of the triangle $\triangle OPQ$.

Now, symmetry demands that T lie on the x-axis and so have coordinates $(x,0,0)$. Symmetry also demands that $d(T,Q) = d(T,P) = d(T,O)$, which solves to give $x = 2/\sqrt{3}$, and so $T = \left(\frac{2}{\sqrt{3}}, 0, 0\right)$. Since C lies directly above T, we have $C = \left(\frac{2}{\sqrt{3}}, 0, z\right)$ for some $z > 0$. This, together with the fact that $d(C,O) = 2$, solves to give $z = \frac{2\sqrt{6}}{3}$, and therefore $C = \left(\frac{2}{\sqrt{3}}, 0, \frac{2\sqrt{6}}{3}\right)$.

4b $\mathbf{r}_{PQ} = \overrightarrow{PQ} = \langle 0, 2, 0\rangle$ and

$$\mathbf{r}_{PC} = \overrightarrow{PC} = \left\langle \tfrac{2}{\sqrt{3}} - \sqrt{3}, 1, \tfrac{2\sqrt{2}}{3}\right\rangle = \left\langle -\tfrac{1}{\sqrt{3}}, 1, \tfrac{2\sqrt{2}}{3}\right\rangle.$$

5a $\|\mathbf{u}\| = \sqrt{2^2+1^2+9^2} = \sqrt{86}$ and $\|\mathbf{v}\| = \sqrt{2^2+5^2+3^2} = \sqrt{38}$. Now,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{28}{\sqrt{86}\sqrt{38}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{14}{\sqrt{817}}\right) \approx 60.67^\circ.$$

5b We have

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{14}{19}\langle -2,-5,3\rangle.$$

6 The area of the parallelogram is

$$\|\mathbf{u}\times\mathbf{v}\| = \left\| \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{k} \right\| = \sqrt{(-2)^2+5^2+(-3)^2} = \sqrt{38}.$$

7 The direction vector of the line must be parallel to $\mathbf{u}\times\mathbf{v}$, and since

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} 1 & -5 \\ 4 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -5 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}\mathbf{k} = \langle 20,0,4\rangle,$$

a suitable direction vector for the line is $\langle 5,0,1\rangle$. Parametrization:

$$\mathbf{r}(t) = \langle -3,4,2\rangle + t\langle 5,0,1\rangle = \langle 5t-3, 4, t+2\rangle, \quad t\in\mathbb{R}.$$

8 Since $\mathbf{r}'(t) = \langle 3,7,2t\rangle$, the tangent *vector* to the curve at the point corresponding to $t=1$ is $\mathbf{r}'(1) = \langle 3,7,2\rangle$. The point itself is $\mathbf{r}(1) = \langle 2,9,1\rangle$. So the tangent line in question has direction $\langle 3,7,2\rangle$ and point $\langle 2,9,1\rangle$. Parametrization:

$$\mathbf{r}(t) = \langle 2,9,1\rangle + t\langle 3,7,2\rangle = \langle 3t+2, 7t+9, 2t+1\rangle, \quad t\in\mathbb{R}.$$

9 From $\mathbf{r}'(t) = \langle \cos t - \sin t, -\cos t - \sin t\rangle$ we have

$$\|\mathbf{r}'(t)\| = \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2} = \sqrt{2},$$

and so the length of the curve is

$$\int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}.$$

10 From $\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle$ we have

$$\|\mathbf{r}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}.$$

Now,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle$$

implies

$$\mathbf{T}'(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{a^2+b^2}}\langle -a\cos t, -a\sin t, 0\rangle,$$

and then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{a^2\cos^2 t}{a^2+b^2} + \frac{a^2\sin^2 t}{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}.$$

Finally, the curvature is

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{a}{a^2+b^2},$$

and it's seen that $\kappa(t) \to 0$ as $b \to \infty$.