Math 242 Exam #5 Key (Fall 2015)

1 Since

$$\partial_y\big(ye^x + \sin y\big) = e^x + \cos y \quad \text{and} \quad \partial_x\big(e^x + x\cos y\big) = e^x + \cos y$$

are equal, $\mathbf{F}$ is conservative. There exists a function $\varphi(x,y)$ such that $\varphi_x(x,y) = ye^x + \sin y$ and $\varphi_y(x,y) = e^x + x\cos y$. Now,

$$\varphi_x(x,y) = ye^x + \sin y \quad \Rightarrow \quad \varphi(x,y) = \int \varphi_x(x,y)\,dx = ye^x + x\sin y + c(y),$$

where $c(y)$ denotes an arbitrary (differentiable) function of y. Thus

$$\varphi_y(x,y) = e^x + x\cos y \quad \Rightarrow \quad e^x + x\cos y + c'(y) = e^x + x\cos y \quad \Rightarrow \quad c'(y) = 0 \quad \Rightarrow \quad c(y) = k,$$

where k is an arbitrary constant. We can set $k = 0$ to obtain

$$\varphi(x,y) = ye^x + x\sin y$$

as a potential function for $\mathbf{F}$.

2 The vector field $\mathbf{F}(x,y,z) = \nabla(xyz)$ is conservative, with potential function $\varphi(x,y,z) = xyz$. The Fundamental Theorem of Line Integrals gives

$$\int_C \nabla(xyz)\cdot d\mathbf{r} = \varphi(\mathbf{r}(\pi)) - \varphi(\mathbf{r}(0)) = \varphi(-1,0,1) - \varphi(1,0,0) = 0 - 0 = 0.$$

3 Get R be the region enclosed by the square C (which includes the points on C itself). Let I be the given line integral. By Green's Theorem,

$$\begin{aligned} I &= \iint_R \big[\partial_x(x^3 + xy) - \partial_y(2y^2 - 2x^2y)\big]\,dA = \iint_R (5x^2 - 3y)\,dA \\ &= \int_{-1}^{1}\int_{-1}^{1}(5x^2 - 3y)\,dx\,dy = \int_{-1}^{1}\left(\frac{10}{3} - 6y\right)dy = \frac{20}{3}. \end{aligned}$$

4a Setting $\mathbf{F} = \langle f, g, h\rangle$, we have

$$(\operatorname{div}\mathbf{F})(x,y,z) = (\nabla\cdot\mathbf{F})(x,y,z) = \partial_x f(x,y,z) + \partial_y g(x,y,z) + \partial_z h(x,y,z) = 2y + 12xz^2.$$

4b Again setting $\mathbf{F} = \langle f, g, h\rangle$,

$$\begin{aligned} (\operatorname{curl}\mathbf{F})(x,y,z) = (\nabla\times\mathbf{F})(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} \\ &= \begin{vmatrix} \partial_y & \partial_z \\ g & h \end{vmatrix}\mathbf{i} - \begin{vmatrix} \partial_x & \partial_z \\ f & h \end{vmatrix}\mathbf{j} + \begin{vmatrix} \partial_x & \partial_y \\ f & g \end{vmatrix}\mathbf{k} \\ &= \big(\partial_y h - \partial_z g\big)\mathbf{i} - \big(\partial_x h - \partial_z f\big)\mathbf{j} + \big(\partial_x g - \partial_y f\big)\mathbf{k} \\ &= \big\langle h_y - g_z, f_z - h_x, g_x - f_y\big\rangle = \langle 0 - 0, 4z^3 - 4z^3, 2x - 2x\rangle \end{aligned}$$

$$= \langle 0,0,0\rangle = \mathbf{0}.$$

5a For each $z \in [0,8]$ we have $x^2+y^2 = z/2$, a circle of radius $\sqrt{z/2}$ with center on the z-axis. Such a circle we may parametrize by

$$\left\langle \sqrt{\frac{z}{2}}\cos t, \sqrt{\frac{z}{2}}\sin t \right\rangle, \quad t \in [0,2\pi].$$

Let $z = v$ and $t = u$. Then

$$\mathbf{r}(u,v) = \left\langle \sqrt{\frac{v}{2}}\cos u, \sqrt{\frac{v}{2}}\sin u, v \right\rangle, \quad (u,v) \in [0,2\pi]\times[0,8]$$

is a parametrization of the surface Σ.

Alternatively we may replace v with $2v^2$ to obtain the parametrization

$$\mathbf{r}(u,v) = \langle v\cos u, v\sin u, 2v^2\rangle, \quad (u,v) \in [0,2\pi]\times[0,2] \tag{1}$$

(note the correspondingly altered domain).

5b Using the parametrization (1) above, so that

$$\mathbf{r}_u(u,v) = \langle -v\sin u, v\cos u, 0\rangle \quad \text{and} \quad \mathbf{r}_v(u,v) = \langle \cos u, \sin u, 4v\rangle,$$

we find the area of Σ to be

$$\begin{aligned}\mathcal{A} &= \iint_\Sigma dS = \iint_R \|(\mathbf{r}_u\times\mathbf{r}_v)(u,v)\|\,dA = \iint_R v\sqrt{16v^2+1}\,dA \\ &= \int_0^2\int_0^{2\pi} v\sqrt{16v^2+1}\,du\,dv = \int_0^2 2\pi v\sqrt{16v^2+1}\,dv = \frac{\pi}{24}\big(65\sqrt{65}-1\big).\end{aligned}$$

6 A suitable parametrizatin for the plane $z = 4-x-y$ is

$$\mathbf{r}(u,v) = \langle u, v, 4-u-v\rangle, \quad (u,v)\in R,$$

where $R = \{(u,v) : 0 \le v \le 4-u,\ 0\le u\le 4\}$. From

$$\mathbf{r}_u(u,v) = \langle 1,0,-1\rangle \quad \text{and} \quad \mathbf{r}_v(u,v) = \langle 0,1,-1\rangle$$

we obtain

$$(\mathbf{r}_u\times\mathbf{r}_v)(u,v) = \mathbf{r}_u(u,v)\times\mathbf{r}_v(u,v) = \langle 1,1,1\rangle,$$

so

$$\hat{\mathbf{n}}(u,v) = \frac{(\mathbf{r}_u\times\mathbf{r}_v)(u,v)}{\|(\mathbf{r}_u\times\mathbf{r}_v)(u,v)\|} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle. \tag{2}$$

This normal vector to the surface Σ points outward from the origin, and thus is consistent with the assumed counterclockwise orientation of the curve $\partial\Sigma$. Thus we choose the orientation $\mathbf{n}$ of Σ to be $\hat{\mathbf{n}}$ as given by (2). We also have

$$(\nabla\times\mathbf{F})(x,y,z) = \langle 0,-4z,0\rangle,$$

and so finally, by Stokes' Theorem,

$$\begin{aligned}
\oint_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{r} &= \iint_{\Sigma} (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_{R} (\nabla\times\mathbf{F})(\mathbf{r}(u,v))\cdot \mathbf{n}(u,v)\|(\mathbf{r}_u\times\mathbf{r}_v)(u,v)\|\,dA \\
&= \iint_{R} (\nabla\times\mathbf{F})(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)(u,v)\,dA \\
&= \iint_{R} (\nabla\times\mathbf{F})(u,v,4-u-v)\cdot\langle 1,1,1\rangle\,dA \\
&= \iint_{R} \Big\langle 0,-4(4-u-v),0\Big\rangle\cdot\langle 1,1,1\rangle\,dA \\
&= \int_0^4\int_0^{4-u} (4u+4v-16)\,dv\,du \\
&= -2\int_0^4 (u^2-8u+16)\,du = -\frac{128}{3}.
\end{aligned}$$