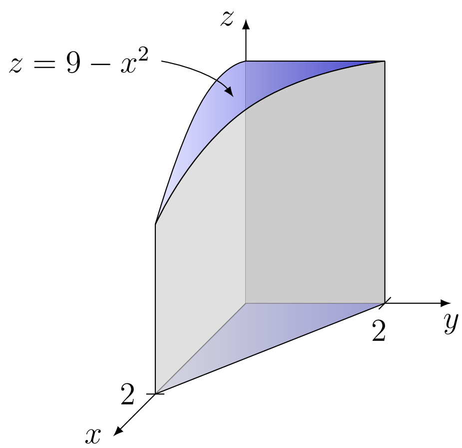
Math 242 Exam #4 Key (Fall 2015)

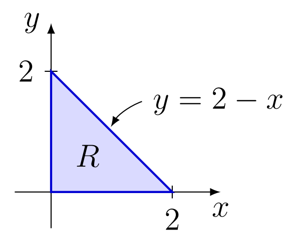
1 For any $(x, y, z) \in D$ we have $0 \leq z \leq 9 - x^2$. We can evaluate $\iiint_D dV$ in the order $dz\,dy\,dx$ (other orders are possible). See the figure below.

To determine the limits of integration for y and x, project D onto the xy-plane to obtain the region R shown at right in the figure. There it can be seen that if $(x, y) \in R$, then $0 \leq y \leq 2 - x$ for $0 \leq x \leq 2$, and so the limits of integration for y will be 0 and $2 - x$, and the limits of integration for x will be 0 and 2. We obtain

$$\begin{aligned} \mathcal{V}(D) &= \iiint_D dV = \int_0^2 \int_0^{2-x} \int_0^{9-x^2} dz\,dy\,dx \\ &= \int_0^2 \int_0^{2-x} (9 - x^2)\,dy dx = \int_0^2 \left[9y - x^2 y\right]_0^{2-x} dx \\ &= \int_0^2 \left[9(2-x) - x^2(2-x)\right] dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{9}{2}x^2 + 18x\right]_0^2 = \frac{50}{3}. \end{aligned}$$

It can be instructive to try determining the volume of D by integrating in the orders $dz\,dx\,dy$ and $dy\,dz\,dx$.

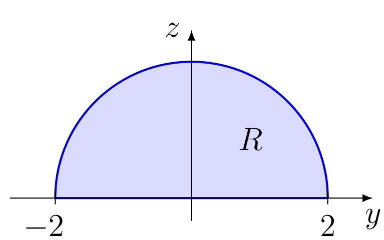




2 On the yz-plane the region of integration is

$$R = \left\{(y, z) : 0 \leq z \leq \sqrt{4 - y^2},\ -2 \leq y \leq 2\right\},$$

the top half of a circular disc of radius 2:



This region is also expressible as

$$R = \left\{(y, z) : -\sqrt{4 - z^2} \leq y \leq \sqrt{4 - z^2},\ 0 \leq z \leq 2\right\},$$

and so the integral becomes

$$\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\,dz\,dx.$$

3 The mass m of the cone is

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^6 \int_0^{6-r} (8-z) r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^6 \left(30r - 2r^2 - \tfrac{1}{2}r^3\right) dr\,d\theta \\ &= \int_0^{2\pi} \left[15r^2 - \tfrac{2}{3}r^3 - \tfrac{1}{8}r^4\right]_0^6 d\theta = \int_0^{2\pi} 234\,d\theta = 468\pi. \end{aligned}$$

4 The intersection of the hyperboloid with the plane $z = 0$ is the curve given by

$$0 = \sqrt{17} - \sqrt{1 + x^2 + y^2} \;\Rightarrow\; 1 + x^2 + y^2 = 17 \;\Rightarrow\; x^2 + y^2 = 16,$$

which is a circle in the xy-plane (i.e. the plane $z = 0$) with radius 4, centered at the origin. The bounded region D in rectangular coordinates thus corresponds to a region

$$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 4,\ 0 \le z \le \sqrt{17} - \sqrt{1 + r^2}\}$$

in cylindrical coordinates, noting that $x^2 + y^2 = r^2$. Volume V is thus

$$\begin{aligned} V &= \iiint_E r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{17}-\sqrt{1+r^2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^4 \left(r\sqrt{17} - r\sqrt{1+r^2}\right) dr\,d\theta = \int_0^{2\pi} \left(\frac{7\sqrt{17}+1}{3}\right) d\theta = \frac{(14\sqrt{17}+2)\pi}{3}. \end{aligned}$$

5 Let $(x, y) \in C$. A vector parallel to C at the point (x, y) would be $\langle y, -x\rangle$, whereas $\mathbf{F}(x, y) = \langle y, x\rangle$. So $\mathbf{F}(x, y)$ is tangent to C at (x, y) if $\mathbf{F}(x, y) = c\langle y, -x\rangle$ for some $c \neq 0$. That is, $\langle y, x\rangle = c\langle y, -x\rangle$, giving $y = cy$ and $x = -cx$. From $y = cy$ there are two possibilities: $c = 1$ or $y = 0$. If $c = 1$, then $x = -x$ results, and hence $x = 0$. Then, since $x^2 + y^2 = 1$, it follows that $y = \pm 1$, and we obtain two points: $(0, \pm 1)$. If $y = 0$, then $x^2 + y^2 = 1$ implies that $x = \pm 1$, and we obtain another two points: $(\pm 1, 0)$. That is, $\mathbf{F}$ is tangent to C at the four points $(\pm 1, 0)$, $(0, \pm 1)$.

$\mathbf{F}(x, y)$ is normal to C at (x, y) if $\mathbf{F}(x, y) \cdot \langle y, -x\rangle = 0$, which yields $y^2 - x^2 = 0$. Adding this equation to $x^2 + y^2 = 1$ gives $2y^2 = 1$, or $y = \pm 1/\sqrt{2}$. On the other hand $y^2 = x^2$ implies $|x| = |y|$, and so we obtain four points:

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), \quad \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right).$$

Note: this problem can also be resolved by working with a parametrization for C, such as the function $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$.

6a A fine parametrization would be

$$\mathbf{r}(t) = \langle 0, -3, 2\rangle(1 - t) + \langle 1, -7, 4\rangle t = \langle t, -4t - 3, 2t + 2\rangle, \quad t \in [0, 1].$$

6b We have $\mathbf{r}'(t) = \langle 1, -4, 2\rangle$, so that $\|\mathbf{r}'(t)\| = \sqrt{21}$. Now,

$$\begin{aligned}\int_C (xz - y^2)\,ds &= \sqrt{21}\int_0^1 \left[t(2t+2) - (-4t-3)^2\right] dt \\ &= -\sqrt{21}\int_0^1 (14t^2 + 22t + 9)\,dt = -\frac{74\sqrt{21}}{3}.\end{aligned}$$

7 Making the substitution $u = t^2 - 1$ along the way, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_0^1 \mathbf{F}(t^2, t^3)\cdot\langle 2t, 3t^2\rangle\,dt \\ &= \int_0^1 \langle e^{t^2-1}, t^5\rangle\cdot\langle 2t, 3t^2\rangle\,dt = \int_0^1 (2te^{t^2-1} + 3t^7)\,dt \\ &= \int_0^1 2te^{t^2-1}\,dt + \int_0^1 3t^7\,dt = \int_{-1}^0 e^u\,du + \frac{3}{8}\left[t^8\right]_0^1 = \frac{11e-8}{8e}.\end{aligned}$$

8 Here we have $x(t) = 2\cos t$ and $y(t) = 2\sin t$, so $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$, and then

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{n} &= \int_0^{2\pi} \left[f(\mathbf{r}(t))y'(t) - g(\mathbf{r}(t))x'(t)\right]dt \\ &= \int_0^{2\pi} \left[f(2\cos t, 2\sin t)(2\cos t) - g(2\cos t, 2\sin t)(-2\sin t)\right]dt \\ &= \int_0^{2\pi} \left[(2\sin t - 2\cos t)(2\cos t) - (2\cos t)(-2\sin t)\right]dt \\ &= 4\int_0^{2\pi} 2\cos t\sin t\,dt - 4\int_0^{2\pi}\cos^2 t\,dt \\ &= \int_0^{2\pi}\sin(2t)\,dt - 4\int_0^{2\pi}\frac{1+\cos(2t)}{2}\,dt \\ &= 4\left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} - 2\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\ &= 4\cdot 0 - 2\cdot 2\pi = -4\pi.\end{aligned}$$

Thus there is a net flux of 4π *into* the region enclosed by C.