Math 242 Exam #3 Key (Fall 2015)

1 The surface S is given by $F(x,y,z)=0$ for $F(x,y,z)=\tan^{-1}(xy)-z$. We have

$$\nabla F(x,y,z)=\left\langle \frac{y}{1+x^2y^2}, \frac{x}{1+x^2y^2}, -1\right\rangle,$$

and the equation of the tangent plane at $(1,1,\pi/4)$ is given by

$$0=\nabla F(1,1,\pi/4)\cdot\langle x-1,y-1,z-\pi/4\rangle=\left\langle \tfrac{1}{2},\tfrac{1}{2},-1\right\rangle\cdot\left\langle x-1,y-1,z-\tfrac{\pi}{4}\right\rangle,$$

or $2x+2y-4z=4-\pi$.

2 First we gather our partial derivatives:

$$f_x(x,y)=3x^2+6y, \qquad f_y(x,y)=-3y^2+6x,$$
$$f_{xx}(x,y)=6x, \qquad f_{yy}(x,y)=-6y,$$
$$f_{xy}(x,y)=6.$$

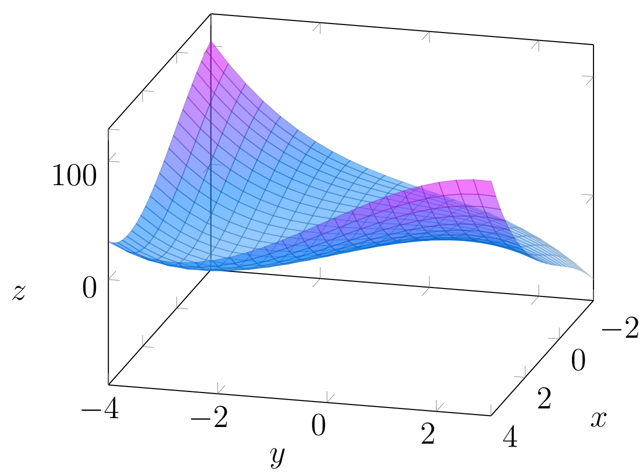
At no point does f_x or f_y fail to exist, so we search for any (x,y) for which $f_x(x,y)=f_y(x,y)=0$. This yields the system

$$\begin{cases} 3x^2+6y=0 \\ -3y^2+6x=0 \end{cases}$$

The solutions are $(0,0)$ and $(2,-2)$, which are the critical points. We construct a table:

(x,y)	f_{xx}	f_{yy}	f_{xy}	Φ	Conclusion
$(0,0)$	0	0	6	-36	Saddle Point
$(2,-2)$	12	12	6	108	Local Minimum

Below is a graph of a part of the surface containing the points of interest.



3 We have $f_x(x,y)=4x-4$, $f_y(x,y)=6y$, $f_{xx}(x,y)=4$, $f_{yy}(x,y)=6$, $f_{xy}(x,y)=0$, and $\Phi(x,y)=24$. Now, $f_x(x,y)=f_y(x,y)=0$ is only satisfied if $(x,y)=(1,0)$, and so there is a local minimum for f at $(1,0)$, with $f(1,0)=0$.

Now we look at the boundary of R, denoted by ∂R, which is a circle of radius 1 centered at $(1,0)$, and so $(x,y) \in \partial R$ implies that $(x-1)^2 + y^2 = 1$ with $x \in [0,2]$. In particular we have $y^2 = 1-(x-1)^2$ for $(x,y) \in \partial R$, and so

$$f(x,y) = 2x^2 - 4x + 3y^2 + 2 = 2x^2 - 4x + 3[1-(x-1)^2] + 2 = 2 + 2x - x^2$$

for all $(x,y) \in \partial R$. Let $g(x) = 2 + 2x - x^2$ for $x \in [0,2]$. To find the extrema of f on ∂R, we find the extrema of g on $[0,2]$. We have $g'(x) = 2 - 2x$, so $g'(x) = 0$ if and only if $x = \frac{1}{2}$. We evaluate: $g(0) = 2$, $g(2) = 2$, and $g(\frac{1}{2}) = 2\frac{3}{4}$. By the Closed Interval Method the global maximum of $g : [0,2] \to \mathbb{R}$ is $g(\frac{1}{2}) = 2\frac{3}{4}$, and the global minimum is $g(0) = g(2) = 2$. Recalling that $y^2 = 1-(x-1)^2$ for $(x,y) \in \partial R$, $x = \frac{1}{2}$ implies $y = \pm\frac{\sqrt{3}}{2}$, so

$$f\left(\tfrac{1}{2}, \pm\tfrac{\sqrt{3}}{2}\right) = g\left(\tfrac{1}{2}\right) = 2\tfrac{3}{4};$$

and $x = 0, 2$ implies $y = 0$, so

$$f(0,0) = g(0) = 2 \quad \text{and} \quad f(2,0) = g(2) = 2.$$

The global maximum of f on the closed disc R is $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = 2\frac{3}{4}$, and the global minimum of f on R is $f(1,0) = 0$.

4 Letting $g(x,y) = x^6 + y^6 - 1$, the given constraint is expressible as $g(x,y) = 0$. We find every $(x,y) \in \mathbb{R}^2$ for which there can be found some $\lambda \in \mathbb{R}$ that results in the system

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ \;g(x,y) = 0 \end{cases}$$

having a solution. We write the system as

$$\begin{cases} x = 3\lambda x^5 \\ y = 3\lambda y^5 \\ 1 = x^6 + y^6 \end{cases} \tag{1}$$

and note that $x = y = 0$ is not possible.

Suppose $x \neq 0$ and $y \neq 0$. Then the first two equations give $x^{-4} = 3\lambda = y^{-4}$, implying that $x^4 = y^4$, and hence $x^6 = y^6$. Now the 3rd equation in the system becomes $2x^6 = 1$, and we obtain four points:

$$p_1 = \left(2^{-1/6}, 2^{-1/6}\right), \quad p_2 = \left(-2^{-1/6}, -2^{-1/6}\right), \quad p_3 = \left(-2^{-1/6}, 2^{-1/6}\right), \quad p_4 = \left(2^{-1/6}, -2^{-1/6}\right).$$

Next suppose that $x = 0$ and $y \neq 0$. Then the system (1) reduces to $y = 3\lambda y^5$ and $y^6 = 1$, which results in two more points: $p_5 = (0,1)$ and $p_6 = (0,-1)$.

Finally suppose $x \neq 0$ and $y = 0$. The system (1) reduces to $x = 3\lambda x^5$ and $x^6 = 1$, giving another two points: $p_7 = (1,0)$ and $p_8 = (-1,0)$.

We now evaluate f at each of the eight points we have found. We have $f(p_n) = 2/\sqrt[3]{2} \approx 1.587$ for $n = 1,2,3,4$, and $f(p_n) = 1$ for $n = 5,6,7,8$. Therefore the maximum value of f subject to the constraint g is $2/\sqrt[3]{2}$ at the points p_1, p_2, p_3, p_4, and the minimum value of f is 1 at the points p_5, p_6, p_7, p_8.

5 The region R is given by

$$R = \left\{ (x,y) : 0 \le x \le 4 \text{ and } 0 \le y \le \sqrt{x} \right\}.$$

Then, making the substitution $u = 1 + x^2$ along the way, we obtain

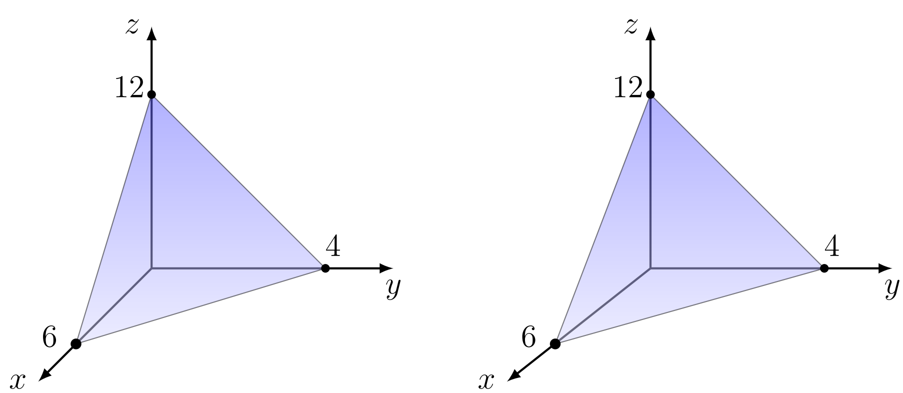
$$\begin{aligned}\iint_R \frac{y}{1+x^2} dA &= \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dy dx = \int_0^4 \frac{1}{1+x^2} \left[\frac{1}{2} y^2 \right]_0^{\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^4 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^{17} \frac{1/2}{u} du = \frac{1}{4} \big[\ln |u| \big]_1^{17} = \frac{\ln 17}{4}.\end{aligned}$$

6 The region $D \subseteq \mathbb{R}^3$ is a tetrahedron in the first octant as shown in the stereoscopic figure below, with region $R \subseteq \mathbb{R}^2$ being the bottom side of D in the xy-plane. We have

$$R = \left\{ (x,y) : 0 \le x \le 6 \text{ and } 0 \le y \le -\tfrac{2}{3}x + 4 \right\}.$$

At any point $(x,y) \in R$ we find that the height of D is $h(x,y) = 12 - 2x - 3y$, and so the volume of D is

$$\begin{aligned}\mathcal{V}(D) &= \iint_R h = \int_0^6 \int_0^{-\frac{2}{3}x+4} (12 - 2x - 3y) dy dx \\ &= \int_0^6 \left[12y - 2xy - \frac{3}{2} y^2 \right]_0^{-\frac{2}{3}x+4} dx = \int_0^6 \left(\frac{2}{3} x^2 - 8x + 24 \right) dx \\ &= \left[\frac{2}{9} x^3 - 4x^2 + 24x \right]_0^6 = 48.\end{aligned}$$



7 The area of the enclosed region R is

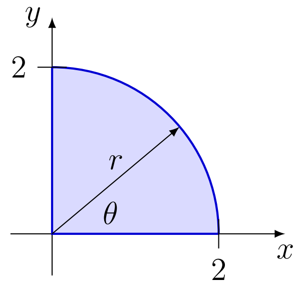
$$\mathcal{A}(R) = \iint_R dA = \int_{-1}^2 \int_{x^2}^{x+2} dy dx = \int_{-1}^2 (x + 2 - x^2) dx = \frac{9}{2}$$

8 The sketch of R in the xy-plane is below. The region

$$S = \{(r, \theta) : 0 \le r \le 2 \text{ and } 0 \le \theta \le \pi/2\}$$

in the $r\theta$-plane is such that $T_{\text{pol}}(S) = R$, and therefore, making the substitution $u = 16 - r^2$, we have

$$\begin{aligned}\iint_R \frac{1}{\sqrt{16-x^2-y^2}}\,dA &= \iint_S \frac{r}{\sqrt{16-r^2}}\,dA = \int_0^{\pi/2}\int_0^2 \frac{r}{\sqrt{16-r^2}}\,dr\,d\theta \\ &= \int_0^{\pi/2}\int_{16}^{12} \frac{-1/2}{\sqrt{u}}\,du\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_{12}^{16} u^{-1/2}\,du\,d\theta \\ &= \frac{\pi}{4}\left[\frac{1}{2}u^{1/2}\right]_{12}^{16} = \frac{\pi}{8}\left(4-2\sqrt{3}\right) = \frac{2-\sqrt{3}}{4}\pi.\end{aligned}$$



9 The graph of $r = 4\cos 3\theta$ in the xy-plane is shown below. The region

$$S = \{(r,\theta) : 0 \le r \le 4\cos 3\theta \text{ and } 0 \le \theta \le \pi\}$$

in the $r\theta$-plane corresponds to the region R enclosed by the curve in the xy-plane. The area of the region R is:

$$\begin{aligned}\mathcal{A}(R) &= \iint_R dA = \int_0^{\pi}\int_0^{4\cos 3\theta} r\,dr\,d\theta = \int_0^{\pi} 8\cos^2 3\theta\,d\theta = \int_0^{\pi}(4+4\cos 6\theta)\,d\theta \\ &= \left[4\theta + \frac{2}{3}\sin 6\theta\right]_0^{\pi} = 4\pi.\end{aligned}$$