Math 242 Exam #2 Key (Fall 2015)

1 We have $z = 4 - x - y$ from the second equation, which when put into the first equation gives

$$1 = x + 2y - z = x + 2y - (4 - x - y) = 2x + 3y - 4 \quad \Rightarrow \quad 2x = 5 - 3y \quad \Rightarrow \quad x = \tfrac{5}{2} - \tfrac{3}{2}y,$$

and thus

$$z = 4 - x - y = 4 - \left(\tfrac{5}{2} - \tfrac{3}{2}y\right) - y = \tfrac{3}{2} + \tfrac{1}{2}y.$$

Letting $y(t) = t$, it follows that $x(t) = \frac{5}{2} - \frac{3}{2}t$ and $z(t) = \frac{3}{2} + \frac{1}{2}t$. A parametrization for the line is thus

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \left\langle \tfrac{5}{2} - \tfrac{3}{2}t, t, \tfrac{3}{2} + \tfrac{1}{2}t \right\rangle, \quad t \in (-\infty, \infty).$$

2 The domain is

$$\mathrm{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : 49 - x^2 - y^2 > 0\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 49\},$$

which consists of all of the points in $\mathbb{R}^2$ in the open disc with radius 7 centered at the origin. As for the range, we have $f(0, 0) = -\frac{12}{7}$, and then the value of $f(x, y)$ decreases without bound as (x, y) gets nearer the boundary of the disc. For instance,

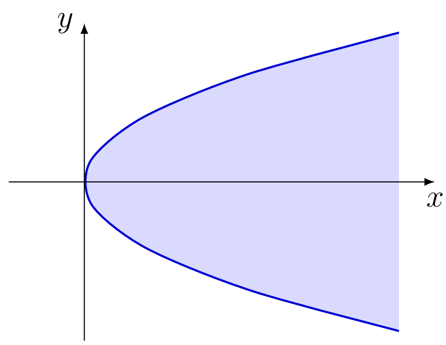
$$\lim_{x \to 7^-} f(x, 0) = \lim_{x \to 7^-} \frac{-12}{\sqrt{49 - x^2}} = -\infty,$$

and therefore $\mathrm{Ran}(f) = \left(-\infty, -\frac{12}{7}\right]$.

3 The function h is a composition of a polynomial function and the square root function (which is a radical function), and so it is continuous on its domain. We have

$$\mathrm{Dom}(h) = \{(x, y) : x - y^2 \geq 0\} = \left\{(x, y) : x \geq y^2\right\},$$

which is the shaded region in $\mathbb{R}^2$ illustrated below.



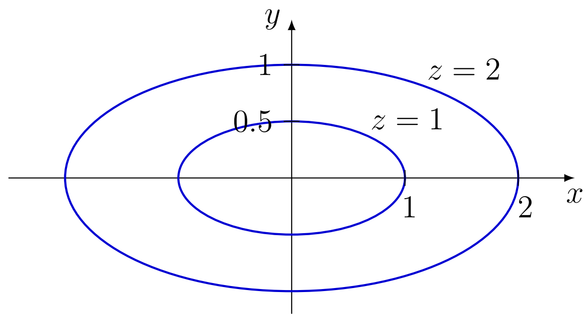
4 The level curve $z = 1$ has equation $1 = \sqrt{x^2 + 4y^2}$, which implies

$$x^2 + \frac{y^2}{1/4} = 1,$$

an ellipse. The level curve $z = 2$ has equation $2 = \sqrt{x^2 + 4y^2}$, which implies

$$\frac{x^2}{4} + y^2 = 1,$$

also an ellipse. Graph is below.



5 We have

$$\lim_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2-xy-2y^2} = \lim_{(x,y)\to(-1,1)} \frac{(x-y)(x+y)}{(x-2y)(x+y)} = \lim_{(x,y)\to(-1,1)} \frac{x-y}{x-2y} = \frac{-1-1}{-1-2(1)} = \frac{2}{3}.$$

6 Along the path $y = x$ the limit becomes

$$\lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{x}{x^2+1} = 0.$$

Along the path $y = \sqrt{x}$ the limit becomes

$$\lim_{x\to 0^+} \frac{x(\sqrt{x})^2}{x^2+(\sqrt{x})^4} = \lim_{x\to 0^+} \frac{x^2}{x^2+x^2} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}.$$

So, we converge on two different values along two different paths that approach the origin, and therefore the limit does not exist by the Two-Path Test.

7 We have

$$\psi_x(t,x) = 2x\sec(t^3x) + x^2\sec(t^3x)\tan(t^3x)\cdot\frac{\partial}{\partial x}(t^3x) = 2x\sec(t^3x) + t^3x^2\sec(t^3x)\tan(t^3x)$$

and

$$\psi_t(t,x) = x^2\sec(t^3x)\tan(t^3x)\cdot\frac{\partial}{\partial t}(t^3x) = 3t^2x^3\sec(t^3x)\tan(t^3x)$$

8a Both $g_x(0,0)$ and $g_y(0,0)$ do not exist:

$$g_x(0,0) = \lim_{h\to 0} \frac{g(h,0)-g(0,0)}{h} = \lim_{h\to 0} \frac{3-0}{h} = \text{DNE},$$

and

$$g_y(0,0) = \lim_{h\to 0} \frac{g(0,h)-g(0,0)}{h} = \lim_{h\to 0} \frac{-1-0}{h} = \text{DNE}.$$

8b Along the path $y = x$ we have

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x\to 0} \frac{3x^2-x^2}{x^2+x^2} = \lim_{x\to 0}(1) = 1,$$

so right away we see that

$$\lim_{(x,y)\to(0,0)} g(x,y) \neq 0 = g(0,0),$$

and therefore g is not continuous at $(0,0)$.

8c Since g is not continuous at $(0,0)$, it cannot be differentiable at $(0,0)$.

9a From $\nabla f(x,y) = \langle 2x+4y, 4x-2y\rangle$ we get $\nabla f(3,-2) = \langle -2, 16\rangle$.

9b Steepest ascent:

$$\frac{\nabla f(3,-2)}{\|\nabla f(3,-2)\|} = \frac{1}{\sqrt{260}}\langle -2, 16\rangle = \frac{1}{\sqrt{65}}\langle -1, 8\rangle.$$

Steepest descent:

$$-\frac{\nabla f(3,-2)}{\|\nabla f(3,-2)\|} = -\frac{1}{\sqrt{260}}\langle -2, 16\rangle = -\frac{1}{\sqrt{65}}\langle -1, 8\rangle.$$

No change: we need a vector orthogonal to $\langle -1, 8\rangle$, such as $\langle 8, 1\rangle$, then normalize to get

$$\frac{1}{\sqrt{65}}\langle 8, 1\rangle.$$

10 Let C_0 be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$, where $\mathbf{r}(0) = \langle 1, 2\rangle$. Note that

$$f_x(x,y) = y \quad \text{and} \quad f_y(x,y) = x$$

for all $(x,y) \in \mathbb{R}^2$. To remain on the path of steepest descent, for any $t \geq 0$ the tangent vector to C_0 at the point $\mathbf{r}(t)$, which is $\mathbf{r}'(t)$, must be in the direction of

$$-\nabla f(\mathbf{r}(t)) = -\nabla f(x(t), y(t)) = -\big\langle f_x(x(t), y(t)), f_y(x(t), y(t))\big\rangle = \langle -y(t), -x(t)\rangle.$$

Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle -y(t), -x(t)\rangle, \tag{1}$$

from which we obtain the differential equations $x'(t) = -y(t)$ and $y'(t) = -x(t)$. Divide the 2nd equation by the 1st equation to obtain

$$\frac{dy/dt}{dx/dt} = \frac{x}{y} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}.$$

The Separation of Variables technique then yields

$$\int y\,dy = \int x\,dx \quad \Rightarrow \quad \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + c \quad \Rightarrow \quad y^2 = x^2 + c.$$

(Note that c is an arbitrary constant, so $2c$ may be written as c without loss of information.) Since $(1,2)$ is a point on the curve, we have $2^2 = 1^2 + c$, and finally $c = 3$. The equation $y^2 = x^2 + 3$ is therefore an equation whose solution set contains the path of steepest descent. This leads to $y = \pm\sqrt{x^2+3}$, and hence $y = \sqrt{x^2+3}$ since when $x = 1$ we should have $y = 2$.

The question remains: does x increase or decrease from its initial value of 1? Looking at (1), we see that $\mathbf{r}'(0) = \langle -2, -1\rangle$; that is, $x'(0) = -2$, and so x must be decreasing. The path is thus given by $y = \sqrt{x^2+1}$ for $x \le 1$. Letting $x(t) = t$, so that $y(t) = \sqrt{t^2+3}$, we get the vector-valued function

$$\mathbf{r}(t) = \left\langle t, \sqrt{t^2+3}\,\right\rangle, \quad t \le 1.$$

Alternatively we may write

$$\mathbf{r}(t) = \left\langle 1-t, \sqrt{(t-1)^2+3}\,\right\rangle, \quad t \ge 0.$$