Math 242 Exam #1 Key (Fall 2015)

1 First, $\overrightarrow{pq} = \langle 7, -6 \rangle$. Now let

$$\mathbf{v} = 10\left(\frac{\overrightarrow{pq}}{\|\overrightarrow{pq}\|}\right) = \frac{10}{\sqrt{7^2+6^2}}\langle 7,-6\rangle = \frac{10}{\sqrt{85}}\langle 7,-6\rangle.$$

Both $\mathbf{v}$ and $-\mathbf{v}$ are parallel to $\overrightarrow{pq}$ with length 10.

2 We have the force vectors

$$\begin{aligned}
\mathbf{F}_1 &= 400\langle\cos(-30^\circ),\sin(-30^\circ)\rangle = 200\big\langle\sqrt{3},-1\big\rangle,\\
\mathbf{F}_2 &= 280\langle\cos(45^\circ),\sin(45^\circ)\rangle = \tfrac{280}{\sqrt{2}}\langle 1,1\rangle = 140\sqrt{2}\,\langle 1,1\rangle,\\
\mathbf{F}_3 &= 350\langle\cos(-135^\circ),\sin(-135^\circ)\rangle = -\tfrac{350}{\sqrt{2}}\big\langle 1,1\big\rangle = -175\sqrt{2}\,\langle 1,1\rangle.
\end{aligned}$$

Adding these forces gives

$$\mathbf{F} = \mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3 = \big\langle 200\sqrt{3}-35\sqrt{2}, -200-35\sqrt{2}\,\big\rangle \approx \langle 296.91, -249.50\rangle,$$

and so $\|\mathbf{F}\| = 387.8$ N, directed at an angle of

$$\theta \approx \tan^{-1}\left(-\frac{249.50}{296.91}\right) \approx -40.0^\circ.$$

3 The crosswind is $\mathbf{w} = \langle 0,6,0\rangle$, the updraft is $\mathbf{u} = \langle 0,0,10\rangle$, and thus the resulting velocity is

$$\mathbf{v} = \langle 0,0,-60\rangle + \mathbf{u} + \mathbf{w} = \langle 0,6,-50\rangle.$$

relative to the ground. The probe's speed is thus

$$\|\mathbf{v}\| = \sqrt{50^2+6^2} = \sqrt{2536} \approx 50.4 \text{ m/s},$$

directed at an angle of

$$\theta = \tan^{-1}(6/50) \approx 6.8^\circ$$

north of vertical (i.e. 83.2° with respect to the ground, drifting north).

4 Center of sphere is at $(\frac{3}{2}, \frac{3}{2}, 7)$, and radius is

$$r = \tfrac{1}{2}d(p,q) = \tfrac{1}{2}\sqrt{(2-1)^2+(3-0)^2+(9-5)^2} = \tfrac{1}{2}\sqrt{26}$$

Note that $r^2 = \frac{13}{2}$. Equation of the sphere is therefore

$$\left(x-\tfrac{3}{2}\right)^2 + \left(y-\tfrac{3}{2}\right)^2 + \left(z-7\right)^2 = \tfrac{13}{2}.$$

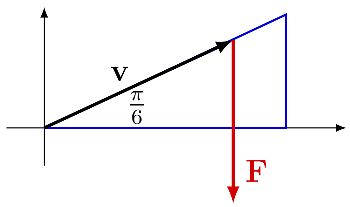
5a $\|\mathbf{u}\| = \sqrt{2^2+1^2+9^2} = \sqrt{86}$ and $\|\mathbf{v}\| = \sqrt{2^2+4^2+3^2} = \sqrt{29}$. Now,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{19}{\sqrt{86}\sqrt{29}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{19}{\sqrt{2494}}\right) \approx 67.64^\circ.$$

5b We have

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{19}{29}\langle -2,4,3\rangle.$$

6 The ramp and force $\mathbf{F} = \langle 0, -12 \rangle$ may be depicted thus:



Here $\mathbf{v} = \langle \sqrt{3}, 1 \rangle$ is a vector parallel to the surface of the ramp. The component of $\mathbf{F}$ parallel to the surface of the ramp, $\mathbf{F}_\parallel$, is thus parallel to $\mathbf{v}$. We have

$$\mathbf{F}_\parallel = \operatorname{proj}_{\mathbf{v}} \mathbf{F} = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = -3 \left\langle \sqrt{3}, 1 \right\rangle = \left\langle -3\sqrt{3}, -3 \right\rangle.$$

The component of $\mathbf{F}$ perpendicular to the ramp, $\mathbf{F}_\perp$, must be such that $\mathbf{F}_\parallel + \mathbf{F}_\perp = \mathbf{F}$, and so

$$\mathbf{F}_\perp = \mathbf{F} - \mathbf{F}_\parallel = \left\langle 3\sqrt{3}, -9 \right\rangle.$$

7 The area of the parallelogram is

$$\|\mathbf{u} \times \mathbf{v}\| = \left\| \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{k} \right\| = \sqrt{(-2)^2 + 5^2 + (-3)^2} = \sqrt{38}.$$

8 The direction vector of the line must be parallel to $\mathbf{u} \times \mathbf{v}$, and since

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} 1 & -5 \\ 4 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -5 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} \mathbf{k} = \langle 20, 0, 4 \rangle,$$

a suitable direction vector for the line is $\langle 5, 0, 1 \rangle$. Parametrization:

$$\mathbf{r}(t) = \langle -3, 4, 2 \rangle + t\langle 5, 0, 1 \rangle = \langle 5t - 3, 4, t + 2 \rangle, \quad t \in \mathbb{R}.$$

9 Since $\mathbf{r}'(t) = \langle 3, 7, 2t \rangle$, the tangent *vector* to the curve at the point corresponding to $t = 1$ is $\mathbf{r}'(1) = \langle 3, 7, 2 \rangle$. The point itself is $\mathbf{r}(1) = \langle 2, 9, 1 \rangle$. So the tangent line in question has direction $\langle 3, 7, 2 \rangle$ and point $\langle 2, 9, 1 \rangle$. Parametrization:

$$\mathbf{r}(t) = \langle 2, 9, 1 \rangle + t\langle 3, 7, 2 \rangle = \langle 3t + 2, 7t + 9, 2t + 1 \rangle, \quad t \in \mathbb{R}.$$

10 From $\mathbf{r}'(t) = \langle \cos t - \sin t, -\cos t - \sin t \rangle$ we have

$$\|\mathbf{r}'(t)\| = \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2} = \sqrt{2},$$

and so the length of the curve is

$$\int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}.$$

11 From $\mathbf{r}'(t) = \langle \sqrt{3} \cos t, \cos t, -2 \sin t \rangle$ we have

$$\|\mathbf{r}'(t)\| = \sqrt{3 \cos^2 t + \cos^2 t + 4 \sin^2 t} = 2.$$

Now,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \frac{\sqrt{3}}{2}\cos t, \frac{1}{2}\cos t, -\sin t \right\rangle$$

implies

$$\mathbf{T}'(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle -\frac{\sqrt{3}}{2}\sin t, -\frac{1}{2}\sin t, -\cos t \right\rangle,$$

and then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{3}{4}\sin^2 t + \frac{1}{4}\sin^2 t + \cos^2 t} = 1.$$

Finally, the curvature is

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{2}.$$