Math 242 Exam #4 Key (Fall 2014)

1 We have

$$V = \int_0^2 \int_{2x^2}^8 \int_0^{2-y/4} dz\,dy\,dx = \int_0^2 \int_{2x^2}^8 (2-y/4)\,dy\,dx = \int_0^2 \left[2y - \frac{y^2}{8}\right]_{2x^2}^8 dx$$
$$= \int_0^2 \left(\frac{x^4}{2} - 4x^2 + 8\right) dx = \left[\frac{x^5}{10} - \frac{4x^3}{3} + 8x\right]_0^2 = \frac{128}{15}.$$

2 Let I be the integral. Making the substitution $u = y^2 - z$ along the way, we have

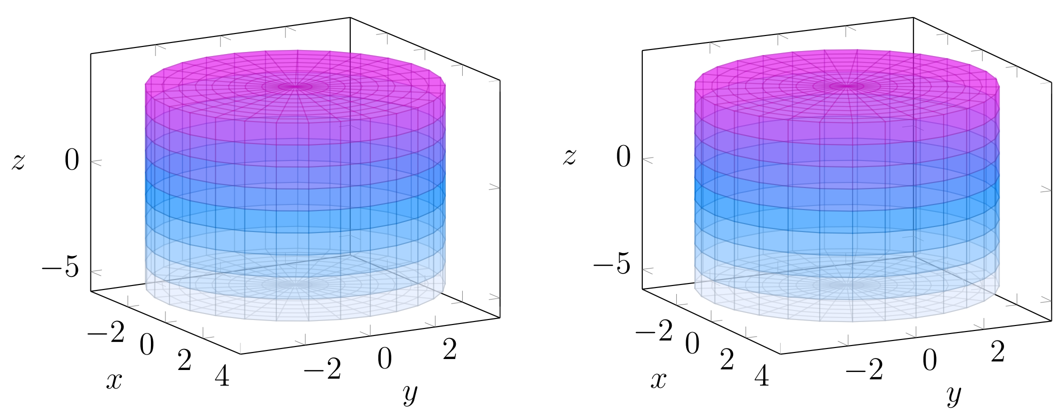
$$I = \int_1^{\ln 8} \int_1^{\sqrt{z}} \left[e^{x+y^2-z}\right]_{\ln y}^{\ln 2y} dy\,dz = \int_1^{\ln 8} \int_1^{\sqrt{z}} y e^{y^2-z} dy\,dz$$
$$= \int_1^{\ln 8} \int_1^{\sqrt{z}} \frac{1}{2} e^u du\,dz = \frac{1}{2}\int_1^{\ln 8} [e^u]_{1-z}^0 dz = \frac{1}{2}\int_1^{\ln 8} (1 - e^{1-z}) dz$$
$$= \frac{1}{2}\left[z + e^{1-z}\right]_1^{\ln 8} = \frac{\ln 8}{2} - \frac{e}{16} - 1.$$

3 The region D is shown in the stereoscopic figure below. In $r\theta z$-space D corresponds to the region

$$E = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 4,\ -5 \le z \le 4\}.$$

We have

$$\iiint_D \sqrt{x^2+y^2}\,dV = \iiint_E r\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}\,dV = \int_0^{2\pi} \int_0^4 \int_{-5}^4 r^2\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^4 9r^2\,dr\,d\theta = \int_0^{2\pi} \left[3r^3\right]_0^4 d\theta = \int_0^{2\pi} 192\,d\theta = 384\pi.$$



4a A fine parameterization would be

$$\mathbf{r}(t) = \langle 0,1,2\rangle(1-t) + \langle -3,7,-1\rangle t = \langle -3t, 1+6t, 2-3t\rangle, \quad t \in [0,1].$$

4b We have $\mathbf{r}'(t) = \langle -3, 6, -3 \rangle$, so that $\|\mathbf{r}'(t)\| = 3\sqrt{6}$. Now,

$$\begin{aligned}\int_C (xz - y^2)\,ds &= 3\sqrt{6}\int_0^1 \left[(-3t)(2-3t) - (1+6t)^2\right] dt \\ &= -3\sqrt{6}\int_0^1 (27t^2 + 18t + 1)\,dt = -57\sqrt{6}.\end{aligned}$$

5 Making the substitution $u = t^2 - 1$ along the way, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 \mathbf{F}(t^2, t^3)\cdot\langle 2t, 3t^2\rangle\,dt \\ &= \int_0^1 \langle e^{t^2-1}, t^5\rangle\cdot\langle 2t, 3t^2\rangle\,dt = \int_0^1 (2te^{t^2-1} + 3t^7)\,dt \\ &= \int_0^1 2te^{t^2-1}\,dt + \int_0^1 3t^7\,dt = \int_{-1}^0 e^u\,du + \frac{3}{8}\left[t^8\right]_0^1 = \frac{11e-8}{8e}.\end{aligned}$$

6 We have $\mathbf{F} = \langle f, g\rangle$ with $f(x,y) = ye^x + \sin y$ and $g(x,y) = e^x + x\cos y$. It is easy to check that

$$f_y(x,y) = e^x + \cos y = g_x(x,y),$$

and so $\mathbf{F}$ is indeed conservative. We now find a function $\varphi(x,y)$ such that $\nabla\varphi = \mathbf{F}$, or $\langle\varphi_x, \varphi_y\rangle = \langle f, g\rangle$. We have

$$\varphi_x(x,y) = f(x,y) \quad\Rightarrow\quad \varphi(x,y) = \int (ye^x + \sin y)\,dx = ye^x + x\sin y + c(y),$$

where $c(y)$ is some arbitrary (differentiable) function of y. But then

$$e^x + x\cos y = g(x,y) = \varphi_y(x,y) = e^x + x\cos y + c'(y),$$

giving $c'(y) = 0$, and hence $c(y) = c$ (i.e. $c(y)$ must be independent of y and hence a constant c). Now $\varphi(x,y) = ye^x + x\sin y + c$ for arbitrary constant c. Letting $c = 0$ for convenience, we obtain

$$\varphi(x,y) = ye^x + x\sin y$$

as a potential function for $\mathbf{F}$.

7 The curve C goes from $\mathbf{a} = \langle 0, 0\rangle$ to $\mathbf{b} = \langle \ln 2, 2\pi\rangle$, and the fact that it's a line segment will be irrelevant. Letting $\varphi(x,y) = e^{-x}\cos y$, the Fundamental Theorem of Line Integrals gives

$$\begin{aligned}\int_C \nabla(e^{-x}\cos y)\cdot d\mathbf{r} &= \int_C \nabla\varphi\cdot d\mathbf{r} = \varphi(\mathbf{b}) - \varphi(\mathbf{a}) = \varphi(\ln 2, 2\pi) - \varphi(0,0) \\ &= e^{-\ln 2}\cos(2\pi) - e^{-0}\cos(0) = \frac{1}{2} - 1 = -\frac{1}{2}.\end{aligned}$$