Math 242 Exam #3 Key (Fall 2014)

1 The surface S is given by $F(x,y,z)=0$ for

$$F(x,y,z) = xy^2 + 3x - z^2 - 4.$$

Now,

$$\nabla F(x,y,z) = \langle y^2+3, 2xy, -2z\rangle,$$

and since the equation of the tangent plane at $(2,1,-2)$ is given by

$$\nabla F(2,1,-2)\cdot\langle x-2, y-1, z+2\rangle = 0,$$

we get

$$\langle 4,4,4\rangle\cdot\langle x-2, y-1, z+2\rangle = 0,$$

and finally $x+y+z=1$.

2 S is given by $F(x,y,z)=0$, where

$$F(x,y,z) = x^2+y^2-z^2-2x+2y+3.$$

So $F_x(x,y,z) = 2x-2$, $F_y(x,y,z) = 2y+2$, and $F_z(x,y,z) = -2z$. A tangent plane to S at $(a,b,c)\in S$ is given by

$$\nabla F\cdot\langle x-a, y-b, z-c\rangle = 0 \quad\Rightarrow\quad \langle 2a-2, 2b+2, -2c\rangle\cdot\langle x-a, y-b, z-c\rangle = 0,$$

which becomes

$$(a-1)x + (b+1)y - cz = a(a-1) + b(b+1) - c^2.$$

A horizontal plane is a plane with equation $z=k$, where k is some constant. Thus we need $a=1$ and $b=-1$. Then

$$a^2+b^2-c^2-2a+2b+3=0 \quad\Rightarrow\quad c^2=1 \quad\Rightarrow\quad c=\pm1.$$

Therefore the two points on S where the tangent plane is horizontal are $(1,-1,1)$ and $(1,-1,-1)$.

3 First we gather our partial derivatives:

$$\begin{aligned} f_x(x,y) &= 2y-2x^3\\ f_y(x,y) &= 2x-2y^3\\ f_{xx}(x,y) &= -6x^2\\ f_{yy}(x,y) &= -6y^2\\ f_{xy}(x,y) &= 2 \end{aligned}$$

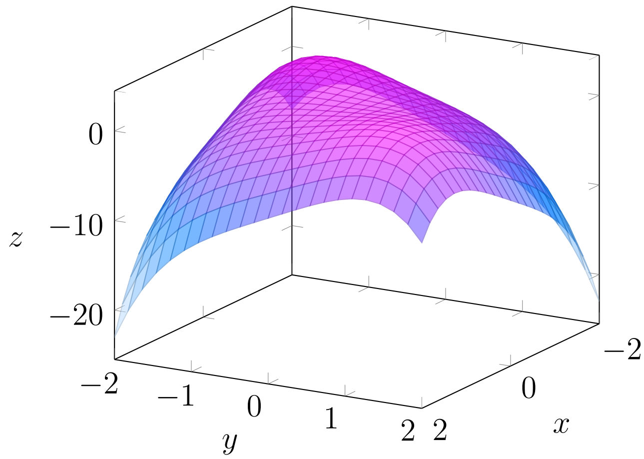
At no point does either f_x or f_y fail to exist, so we search for any point (x,y) for which $f_x(x,y) = f_y(x,y) = 0$. This yields the system

$$\begin{cases} -2x^3+2y=0\\ -2y^3+2x=0 \end{cases}$$

The first equation gives $y=x^3$, which when put into the second equation yields $x^9-x=0$, or $x(x^8-1)=0$. The solutions are $(0,0)$, $(1,1)$, and $(-1,-1)$, which are the critical points. We construct a table:

(x,y)	f_{xx}	f_{yy}	f_{xy}	Φ	Conclusion
$(0,0)$	0	0	2	-4	Saddle Point
$(1,1)$	-6	-6	2	32	Local Maximum
$(-1,-1)$	-6	-6	2	32	Local Maximum

Below is a graph of a part of the surface containing the points of interest.



4 The region R is given by

$$R = \{(x,y) : 0 \leq x \leq 4 \text{ and } 0 \leq j \leq \sqrt{x}\}.$$

Then, making the substitution $u = 1 + x^2$ along the way, we obtain

$$\begin{aligned}\iint_R \frac{y}{1+x^2} dA &= \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dy dx = \int_0^4 \frac{1}{1+x^2} \left[\frac{1}{2} y^2\right]_0^{\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^4 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^{17} \frac{1/2}{u} du = \frac{1}{4} \big[\ln|u|\big]_1^{17} = \frac{\ln 17}{4}.\end{aligned}$$

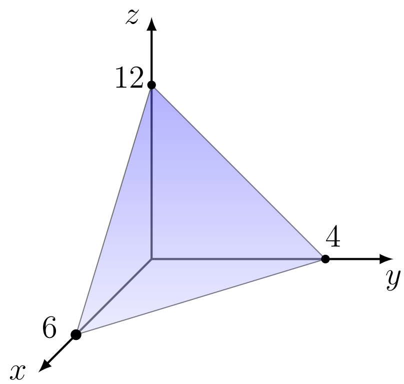
5 The region $D \subseteq \mathbb{R}^3$ is a tetrahedron in the first octant as shown in the stereoscopic figure below, with region $R \subseteq \mathbb{R}^2$ being the bottom side of D in the xy-plane. We have

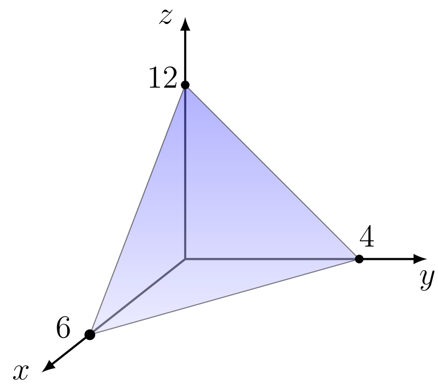
$$R = \left\{(x,y) : 0 \leq x \leq 6 \text{ and } 0 \leq y \leq -\tfrac{2}{3}x + 4\right\}.$$

At any point $(x,y) \in R$ we find that the height of D is $h(x,y) = 12 - 2x - 3y$, and so the volume of D is

$$\begin{aligned}\mathcal{V}(D) &= \iint_R h = \int_0^6 \int_0^{-\frac{2}{3}x+4} (12 - 2x - 3y) dy dx \\ &= \int_0^6 \left[12y - 2xy - \frac{3}{2} y^2\right]_0^{-\frac{2}{3}x+4} dx = \int_0^6 \left(\frac{2}{3}x^2 - 8x + 24\right) dx\end{aligned}$$

$$= \left[\frac{2}{9}x^3 - 4x^2 + 24x\right]_0^6 = 48.$$





6 The area of the enclosed region R is

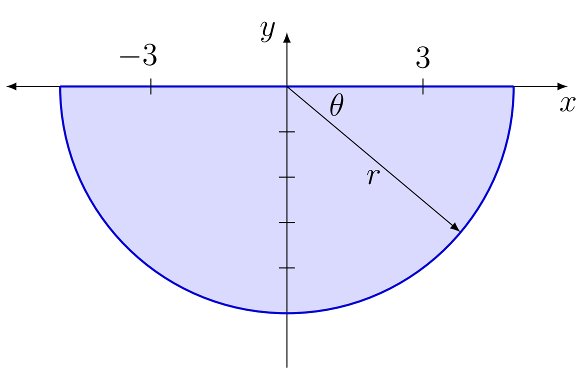
$$\mathcal{A}(R) = \iint_R dA = \int_{-1}^{2}\int_{x^2}^{x+2} dydx = \int_{-1}^{2}(x+2-x^2)dx = \frac{9}{2}$$

7 The sketch of R in the xy-plane is below. The region

$$S = \{(r,\theta) : 0 \le r \le 5 \text{ and } \pi \le \theta \le 2\pi\}$$

in the $r\theta$-plane is such that $T_{\text{pol}}(S) = R$, and therefore

$$\begin{aligned}\iint_R 2xy\,dA &= \iint_S 2(r\cos\theta)(r\sin\theta)r\,dA = \int_\pi^{2\pi}\int_0^5 2(r\cos\theta)(r\sin\theta)r\,drd\theta \\ &= \int_\pi^{2\pi}\int_0^5 2r^3\cos\theta\sin\theta\,drd\theta = \int_\pi^{2\pi}\cos\theta\sin\theta\left[\frac{1}{2}r^4\right]_0^5 d\theta \\ &= \frac{625}{2}\int_\pi^{2\pi}\cos\theta\sin\theta\,d\theta = \frac{625}{4}\int_\pi^{2\pi}\sin(2\theta)\,d\theta = 0.\end{aligned}$$



8 The volume of the enclosed region D is

$$\begin{aligned}\mathcal{V}(D) &= \iint_R h = \int_0^{2\pi} \int_0^{25} \left(25 - \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}\right) r\,dr d\theta \\ &= \int_0^{2\pi} \int_0^{25} (25 - r) r\,dr d\theta = \int_0^{2\pi} \left[\frac{25}{2}r^2 - \frac{1}{3}r^3\right]_0^{25} d\theta \\ &= \int_0^{2\pi} \frac{15,625}{3} d\theta = \frac{15,625}{3}\pi.\end{aligned}$$