

1 Let P be the plane $2x + y - z = 1$, and let Q be the plane parallel to P and passing through the point $(0, 2, -2)$. Writing the equation for P as

$$2x + (y - 1) - z = 0$$

and comparing to $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, we see that the normal vector for P is $\mathbf{n}_P = \langle a, b, c \rangle = \langle 2, 1, -1 \rangle$. Now, if \mathbf{n}_Q is the normal vector for Q , then $Q \parallel P$ implies that we must have $\mathbf{n}_Q \parallel \mathbf{n}_P$. Thus we may choose $\mathbf{n}_Q = \mathbf{n}_P$. So Q has normal vector $\langle 2, 1, -1 \rangle$ and point $(0, 2, -2)$, and therefore an equation for Q is

$$2(x - 0) + (y - 2) - (z + 2) = 0,$$

which simplifies to become $2x + y - z = 4$. Tah-dah!

2 The domain is

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : 4 - x^2 - y^2 > 0\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\},$$

which consists of all of the points in \mathbb{R}^2 in the open disc with radius 2 centered at the origin. As for the range, we have $f(0, 0) = 4$, and then the value of $f(x, y)$ grows without bound as (x, y) gets nearer the boundary of the disc. For instance,

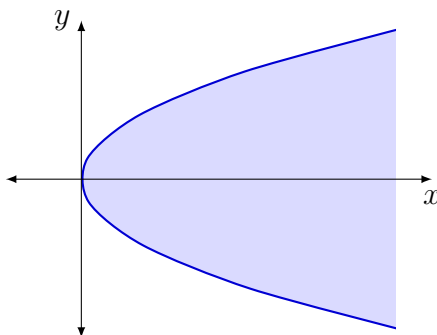
$$\lim_{x \rightarrow 2^-} f(x, 0) = \lim_{x \rightarrow 2^-} \frac{8}{\sqrt{4 - x^2}} = +\infty,$$

and therefore $\text{Ran}(f) = [4, \infty)$.

3 The function h is a composition of a polynomial function and the square root function (which is a radical function), and so it is continuous on its domain. We have

$$\text{Dom}(h) = \{(x, y) : x - y^2 \geq 0\} = \{(x, y) : x \geq y^2\},$$

which is the shaded region in \mathbb{R}^2 illustrated below.



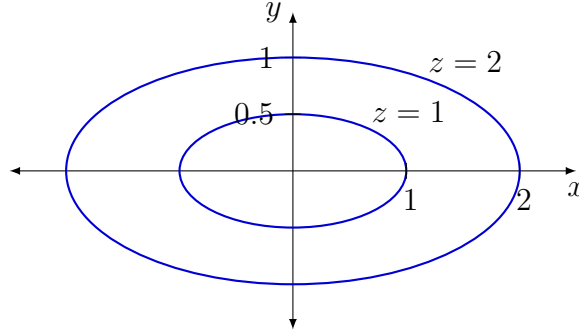
4 The level curve $z = 1$ has equation $1 = \sqrt{x^2 + 4y^2}$, which implies

$$x^2 + \frac{y^2}{1/4} = 1,$$

an ellipse. The level curve $z = 2$ has equation $2 = \sqrt{x^2 + 4y^2}$, which implies

$$\frac{x^2}{4} + y^2 = 1,$$

also an ellipse. Graph is below.



5 We have

$$\lim_{(x,y) \rightarrow (-1,1)} \frac{x^2 - y^2}{x^2 - xy - 2y^2} = \lim_{(x,y) \rightarrow (-1,1)} \frac{(x-y)(x+y)}{(x-2y)(x+y)} = \lim_{(x,y) \rightarrow (-1,1)} \frac{x-y}{x-2y} = \frac{-1-1}{-1-2(1)} = \frac{2}{3}.$$

6 Along the path $y = x$ the limit becomes

$$\lim_{x \rightarrow 0} \frac{-x^3}{x^2 + x^4} = \lim_{x \rightarrow 0} \frac{-x}{x^2 + 1} = 0.$$

Along the path $y = \sqrt{x}$ the limit becomes

$$\lim_{x \rightarrow 0^+} \frac{-x(\sqrt{x})^2}{x^2 + (\sqrt{x})^4} = \lim_{x \rightarrow 0^+} \frac{-x^2}{x^2 + x^2} = - \lim_{x \rightarrow 0^+} \frac{1}{2} = -\frac{1}{2}.$$

So, we converge on two different values along two different paths that approach the origin, and therefore the limit does not exist by the Two-Path Test.

7 We have

$$\varphi_t(t, z) = z^2 \sec^2(tz) \cdot (tz)_t = z^2 \sec^2(tz) \cdot z = z^3 \sec^2(tz)$$

and

$$\begin{aligned} \varphi_z(t, z) &= (z^2)_z \tan(tz) + z^2 (\tan(tz))_z = 2z \tan(tz) + z^2 (\sec^2(tz) \cdot t) \\ &= 2z \tan(tz) + tz^2 \sec^2(tz). \end{aligned}$$

8a We have

$$h_x(0, 0) = \lim_{k \rightarrow 0} \frac{h(k, 0) - h(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0,$$

and

$$h_y(0,0) = \lim_{k \rightarrow 0} \frac{h(0,k) - h(0,0)}{k} = \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0.$$

Remark: the original version of the problem only defined $h(0,0) = 0$, so that h was undefined for all points on the line $y = -x$ except the origin. In that case $(0,0)$ is not an interior point of the domain of h , and therefore h could not have partial derivatives at $(0,0)$ according to our definition of a partial derivative. Anyone who noted this would get full credit.

8b Apply the Two-Path Test. Along the path $y = x$ we have:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{5x^2y}{x^3 + y^3} = \lim_{(x,x) \rightarrow (0,0)} \frac{5x^2x}{x^3 + x^3} = \lim_{x \rightarrow 0} \frac{5x^3}{2x^3} = \lim_{x \rightarrow 0} \frac{5}{2} = \frac{5}{2}.$$

From this it is immediate that

$$\lim_{(x,y) \rightarrow (0,0)} h(x,y) \neq 0 = f(0,0),$$

and therefore h is not continuous at $(0,0)$.

8c Since h is not continuous at $(0,0)$, it cannot be differentiable at $(0,0)$.

9 First get the unit vector in the direction of $\langle 1, \sqrt{3} \rangle$:

$$\mathbf{u} = \frac{\langle 1, \sqrt{3} \rangle}{2} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle.$$

Now,

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = \langle e^x \sin y, e^x \cos y \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{e^x \sin y}{2} + \frac{\sqrt{3}e^x \cos y}{2},$$

and so

$$D_{\mathbf{u}}f(0, \pi/4) = \frac{e^0 \sin(\pi/4)}{2} + \frac{\sqrt{3}e^0 \cos(\pi/4)}{2} = \frac{1/\sqrt{2}}{2} + \frac{\sqrt{3} \cdot 1/\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}.$$

10 Let C_0 be given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \geq 0$, where $\mathbf{r}(0) = \langle 2, 0 \rangle$. Note that

$$f_x(x,y) = 3 \quad \text{and} \quad f_y(x,y) = -2$$

for all $(x,y) \in \mathbb{R}^2$. To remain on the path of steepest descent, for any $t \geq 0$ the tangent vector to C_0 at the point $\mathbf{r}(t)$, which is $\mathbf{r}'(t)$, must be in the direction of

$$-\nabla f(\mathbf{r}(t)) = -\nabla f(x(t), y(t)) = -\langle f_x(x(t), y(t)), f_y(x(t), y(t)) \rangle = -\langle 3, -2 \rangle.$$

Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle = \langle -3, 2 \rangle,$$

from which we obtain the differential equations $x'(t) = -3$ and $y'(t) = 2$. Thus $x(t) = -3t + c_1$ and $y(t) = 2t + c_2$ for arbitrary constants c_1, c_2 . So far we have

$$\mathbf{r}(t) = \langle -3t + c_1, 2t + c_2 \rangle.$$

Now,

$$\mathbf{r}(0) = \langle c_1, c_2 \rangle = \langle 2, 0 \rangle$$

shows that $c_1 = 2$ and $c_2 = 0$, and hence

$$\mathbf{r}(t) = \langle -3t + 2, 2t \rangle, \quad t \geq 0$$

is a parameterization for C_0 .