Math 242 Exam #1 Key (Fall 2014)

1 We need a such that $\|\mathbf{w}\| = 1$. Now,

$$1 = \|\mathbf{w}\| = \sqrt{a^2 + a^2/16} = \sqrt{17a^2/16} = \frac{\sqrt{17}}{4}|a| \quad \Rightarrow \quad |a| = \frac{4}{\sqrt{17}} \quad \Rightarrow \quad a = \pm\frac{4}{\sqrt{17}}.$$

2 We have the force vectors

$$\begin{aligned}
\mathbf{F}_1 &= 400\langle\cos(-30°), \sin(-30°)\rangle = \langle 200\sqrt{3}, -200\rangle, \\
\mathbf{F}_2 &= 280\langle\cos(45°), \sin(45°)\rangle = \langle 280/\sqrt{2}, 280/\sqrt{2}\,\rangle, \\
\mathbf{F}_3 &= 350\langle\cos(135°), \sin(135°)\rangle = \langle -350/\sqrt{2}, 350/\sqrt{2}\,\rangle.
\end{aligned}$$

Adding these forces gives

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \approx \langle 296.9, 245.5\rangle,$$

and so $\|\mathbf{F}\| = 385.0$ N, directed at an angle of

$$\theta \approx \tan^{-1}\left(\frac{245.5}{296.9}\right) \approx 39.6°.$$

3 The crosswind is $\mathbf{w} = \langle 0, 4, 0\rangle$, the updraft is $\mathbf{u} = \langle 0, 0, 10\rangle$, and thus the terminal velocity is

$$\mathbf{v} = \langle 0, 0, -60\rangle + \mathbf{u} + \mathbf{w} = \langle 0, 4, -50\rangle.$$

The probe's speed is thus

$$\|\mathbf{v}\| = \sqrt{50^2 + 4^2} = 2\sqrt{629} \approx 50.2 \text{ m/s},$$

directed at an angle of

$$\theta = \tan^{-1}(4/50) \approx 4.6°$$

north of vertical (i.e. 85.4° with respect to the ground, drifting north).

4 Complete the square for each variable:

$$(x^2-8x)+(y^2+14y)+(z^2-18z) \geq 65 \quad \Rightarrow \quad (x^2-8x+16)+(y^2+14y+49)+(z^2-18z+81) \geq 211,$$

and so

$$(x-4)^2 + (y+7)^2 + (z-9)^2 \geq 211.$$

The solution set is the region strictly outside the sphere with center $(4, -7, 9)$ and radius $\sqrt{211}$.

5a $\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 8^2} = \sqrt{69}$ and $\|\mathbf{v}\| = \sqrt{(-2)^2 + 4^2 + (-3)^2} = \sqrt{29}$.

5b Since $\mathbf{u}\cdot\mathbf{v} = (2)(-2) + (-1)(4) + (8)(-3) = -32$ and $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = (\sqrt{29}\,)^2 = 29$, we have

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = -\frac{32}{29}\langle -2, 4, -3\rangle = \left\langle \frac{64}{29}, -\frac{128}{29}, \frac{96}{29}\right\rangle.$$

5c We have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-32}{\sqrt{69}\sqrt{29}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{-32}{\sqrt{2001}}\right) \approx 135.7^\circ.$$

6 Let $\mathbf{u} = \langle x, y, z\rangle$. Then

$$\frac{\langle x,y,z\rangle\cdot\langle 0,2,0\rangle}{\langle 0,2,0\rangle\cdot\langle 0,2,0\rangle}\langle 0,2,0\rangle = \frac{\langle 1,2,4\rangle\cdot\langle 0,2,0\rangle}{\langle 0,2,0\rangle\cdot\langle 0,2,0\rangle}\langle 0,2,0\rangle,$$

which gives

$$\frac{2y}{4}\langle 0,2,0\rangle = \frac{4}{4}\langle 0,2,0\rangle,$$

and finally $\langle 0,y,0\rangle = \langle 0,2,0\rangle$. So $y = 2$, and we conclude that $\mathbf{u} = \langle x, 2, z\rangle$ for $x, z \in \mathbb{R}$. As position vectors this gives the position of all points on the plane $y = 2$.

7 We find any $t \in [0, 2\pi]$ for which $2\sin t = 1$, since $2\sin t$ is the y-component of $\mathbf{r}(t)$. Now,

$$2\sin t = 1 \quad\Rightarrow\quad \sin t = \tfrac{1}{2} \quad\Rightarrow\quad t = \tfrac{\pi}{6}, \tfrac{5\pi}{6}.$$

So we obtain two points:

$$\mathbf{r}(\pi/6) = \langle 5\sqrt{3}, 1, 1\rangle \quad\text{and}\quad \mathbf{r}(5\pi/6) = \langle -5\sqrt{3}, 1, 1\rangle.$$

8 Let $\mathbf{v} = \langle -9-(-1), 5-(-8), -3-4\rangle = \langle -8, 13, -7\rangle$ and $\mathbf{r}_0 = \langle -1, -8, 4\rangle$. Then a parameterization for the line segment is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle -1,-8,4\rangle + \langle -8t, 13t, -7t\rangle, \quad 0 \le t \le 1,$$

or

$$\mathbf{r}(t) = \langle -1-8t, -8+13t, 4-7t\rangle, \quad t \in [0,1].$$

9 From $\mathbf{r}'(t) = \langle 1, 0, -2/t^2\rangle$ we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+4/t^4}}\left\langle 1, 0, -\frac{2}{t^2}\right\rangle.$$

Now we obtain

$$\mathbf{T}(2) = \frac{1}{\sqrt{1+4/2^4}}\left\langle 1, 0, -\frac{2}{2^2}\right\rangle = \left\langle \frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right\rangle$$

10 By definition,

$$\int_0^1 \langle e^{2t}, e^{-t}, t\rangle dt = \left\langle \int_0^1 e^{2t}dt, \int_0^1 e^{-t}dt, \int_0^1 t\,dt\right\rangle = \left\langle \left[\frac{1}{2}e^{2t}\right]_0^1, \left[-e^{-t}\right]_0^1, \left[\frac{1}{2}t^2\right]_0^1\right\rangle$$

$$= \left\langle \frac{e^2-1}{2},\ 1-\frac{1}{e},\ \frac{1}{2}\right\rangle.$$

11 From $\mathbf{a}(t) = \langle 1, t\rangle$ we integrate to obtain $\mathbf{v}(t) = \langle t + a_1, t^2/2 + a_2\rangle$. Now,

$$\mathbf{v}(0) = \langle a_1, a_2\rangle = \langle 2, -1\rangle,$$

so we have $a_1 = 2$ and $a_2 = -1$, and obtain

$$\mathbf{v}(t) = \langle t + 2, t^2/2 - 1\rangle$$

for the velocity function. Integrating this function then yields

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t + b_1, \frac{1}{6}t^3 - t + b_2 \right\rangle.$$

Now,

$$\mathbf{r}(0) = \langle b_1, b_2\rangle = \langle -3, 6\rangle,$$

so $b_1 = -3$ and $b_2 = 6$ and we obtain

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t - 3, \frac{1}{6}t^3 - t + 6 \right\rangle.$$

12a C may be parameterized by $\mathbf{r}(t) = \langle t, t^3\rangle$.

12b We have

$$\mathbf{r}'(t) = \langle 1, 3t^2\rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + 9t^4},$$

so that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1 + 9t^4}}\langle 1, 3t^2\rangle,$$

which yields

$$\mathbf{T}'(t) = \frac{6t}{(1 + 9t^4)^{3/2}}\langle -3t^2, 1\rangle,$$

and thus

$$\|\mathbf{T}'(t)\| = \frac{6|t|}{(1 + 9t^4)^{3/2}}\sqrt{1 + 9t^4} = \frac{6|t|}{1 + 9t^4}$$

Finally we obtain

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{6|t|}{1 + 9t^4} \cdot \frac{1}{\sqrt{1 + 9t^4}} = \frac{6|t|}{(1 + 9t^4)^{3/2}}.$$

as the curvature of the curve at the point (t, t^2).

12c Since curvature is never negative, it's easier to find where the square of the curvature is maximized. Let

$$K(t) = \kappa^2(t) = \frac{36t^2}{(1 + 9t^4)^3}$$

and find t for which $K'(t) = 0$. (Note how the absolute value is eliminated.) We have

$$K'(t) = \frac{(1 + 9t^4)^3(72t) - (36t^2)[3(1 + 9t^4)^2 \cdot 36t^3]}{(1 + 9t^4)^6}.$$

Now, $K'(t) = 0$ is only possible when the numerator is zero:

$$K'(t) = 0 \quad \Rightarrow \quad (1+9t^4)^3(72t) - (36t^2)[3(1+9t^4)^2 \cdot 36t^3] = 0 \quad \Rightarrow \quad t - 45t^5 = 0,$$

and so $t(1-45t^4) = 0$. Solutions are $t = 0$ and $t = \pm 1/\sqrt[4]{45}$. Since $\kappa(0) = 0$, we see that $t = 0$ corresponds to a minimum. Curvature is maximum when $t = \pm 45^{-1/4} \approx \pm 0.386$. Thus the graph of f has maximum curvature at the points (x, x^3) for $x = \pm 45^{-1/4}$:

$$\big(45^{-1/4}, 45^{-3/4}\big) \quad \text{and} \quad \big(-45^{-1/4}, -45^{-3/4}\big).$$