Math 242 Exam #5 Key (Fall 2013)

1 We have $\mathbf{F} = \langle f, g\rangle$ with $f(x,y) = 2xy$ and $g(x,y) = x^2 - y^2$, and since R is connected and simply connected, and ∂R is simple, closed and piecewise-smooth, by Green's Theorem

$$\oint_{\partial R} \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA = \iint_R (2x - 2x)\,dA = 0.$$

2 We have $\mathbf{F} = \langle f, g\rangle$ with $f(x,y) = 0$ and $g(x,y) = xy$, and since R (the region enclosed by the triangle C) is connected and simply connected, and ∂R (the triangle C) is simple, closed and piecewise-smooth, by the flux form of Green's Theorem

$$\int_C f\,dy - g\,dx = \iint_R (f_x + g_y)\,dA = \iint_R x\,dA = \int_0^2 \int_0^{-2x+4} x\,dydx = \int_0^2 [xy]_0^{-2x+4}\,dx$$
$$= \int_0^2 (-2x^2 + 4x)\,dx = \left[-\frac{2}{3}x^3 + 2x^2\right]_0^2 = \frac{8}{3}.$$

3 We have

$$(\nabla\cdot\mathbf{F})(x,y,z) = D_x(-2y) + D_y(3x) + D_z(z) = 0 + 0 + 1 = 1.$$

4 We have

$$(\operatorname{curl}\mathbf{F})(x,y,z) = (\nabla\times\mathbf{F})(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ x^2 - y^2 & xy & z \end{vmatrix}$$
$$= \begin{vmatrix} D_y & D_z \\ xy & z \end{vmatrix}\mathbf{i} - \begin{vmatrix} D_x & D_z \\ x^2 - y^2 & z \end{vmatrix}\mathbf{j} + \begin{vmatrix} D_x & D_y \\ x^2 - y^2 & xy \end{vmatrix}\mathbf{k}$$
$$= [D_y(z) - D_z(xy)]\mathbf{i} - \left[D_x(z) - D_z(x^2 - y^2)\right]\mathbf{j} + \left[D_x(xy) - D_y(x^2 - y^2)\right]\mathbf{k}$$
$$= 0\mathbf{i} - 0\mathbf{j} + [y - (-2y)]\mathbf{k} = \langle 0, 0, 3y\rangle.$$

5a A parameterization:

$$\mathbf{r}(u,v) = \langle 4\cos u, 4\sin u, v\rangle,\ (u,v) \in [0,\pi]\times[0,7].$$

5b Let Σ denote the surface, and $R = [0,\pi]\times[0,7]$. Surface area:

$$\mathcal{A}(\Sigma) = \iint_\Sigma dS = \iint_R \|(\mathbf{r}_u \times \mathbf{r}_v)(u,v)\|\,dA$$
$$= \iint_R \|\langle -4\sin u, 4\cos u, 0\rangle \times \langle 0,0,1\rangle\|\,dA$$
$$= \iint_R \|\langle 4\cos u, 4\sin u, 0\rangle\|\,dA = \int_0^\pi \int_0^7 4\,dvdu = 28\pi.$$

6 Here C is the circle in the xy-plane centered at the origin with radius $2\sqrt{3}$. As in homework, we give C the positive (i.e. counterclockwise) orientation, and so a parameterization for C is

$$\mathbf{r}(t) = 2\sqrt{3}\langle \cos t, \sin t, 0\rangle, \quad t \in [0, 2\pi].$$

A convenient choice for a surface Σ that has C as its boundary would be the planar region enclosed by C, which is the circular disk in the xy-plane with radius $2\sqrt{3}$. A parameterization for Σ is

$$\boldsymbol{\rho}(u, v) = \langle v\cos u, v\sin u, 0\rangle, \quad (u, v) \in R = [0, 2\pi] \times \left[0, 2\sqrt{3}\,\right].$$

Clearly Σ is orientable (i.e. it has two identifiable "sides"). We have

$$(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 0 \end{vmatrix} = \langle 0, 0, -v\rangle$$

for any $(u, v) \in \operatorname{Int}(R) = (0, 2\pi) \times (0, 2\sqrt{3}\,)$, and so

$$\hat{\mathbf{n}}(u, v) = \frac{\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v}{\|\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v\|}(u, v) = \frac{\langle 0, 0, -v\rangle}{v} = \langle 0, 0, -1\rangle\,.$$

Thus $\hat{\mathbf{n}} : \operatorname{Int}(R) \to \mathbb{R}^3$ is continuous on $\operatorname{Int}(R)$, and it has continuous extension to R by setting $\hat{\mathbf{n}}(u, v) = -\mathbf{k}$ for all $(u, v) \in \partial R = C$. Indeed, since $\hat{\mathbf{n}}$ is a constant function equal to $-\mathbf{k}$ on all R, we simply define $\hat{\mathbf{n}} = -\mathbf{k}$.

It must be determined which orientation $\mathbf{n}$ of Σ, $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$, is consistent with the orientation of C. That is, which orientation $\mathbf{n}$ of Σ is such that $(\mathbf{T} \times \mathbf{N})(t) = \mathbf{n}$ for all $t \in [0, 2\pi]$, where $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and principal unit normal vectors for C, respectively. We calculate

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \langle -\sin t, \cos t, 0\rangle$$

and

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle -\cos t, -\sin t, 0\rangle.$$

Now,

$$(\mathbf{T} \times \mathbf{N})(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle 0, 0, 1\rangle\,,$$

which we see agrees with $-\hat{\mathbf{n}}(u, v)$ for all $(u, v) \in R$, and so we give Σ the orientation $-\hat{\mathbf{n}}$.

Next, we have

$$(\nabla \times \mathbf{F})(x, y, z) = \langle 1, -1, -2\rangle$$

Finally by Stokes' Theorem, substituting $-\hat{\mathbf{n}}$ for $\mathbf{n}$, we obtain

$$\begin{aligned}
\oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} &= -\iint_{\Sigma} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \\
&= -\iint_{R} (\nabla \times \mathbf{F})(\boldsymbol{\rho}(u, v)) \cdot \hat{\mathbf{n}}(u, v)\|(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)(u, v)\|\, dA \\
&= -\iint_{R} (\nabla \times \mathbf{F})(\boldsymbol{\rho}(u, v)) \cdot (\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)(u, v)\, dA \\
&= -\iint_{R} \langle 1, -1, -2\rangle \cdot \langle 0, 0, -v\rangle\, dA
\end{aligned}$$

$$= -\int_0^{2\sqrt{3}} \int_0^{2\pi} 2v \, du dv = -24\pi.$$

7 Let $D = \overline{B}_1(\mathbf{0})$, which is the solid ball with radius 1 and center at the origin. Clearly D is connected and simply connected. The boundary of D is

$$\partial D = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\},$$

which is the sphere with radius 1 and center at the origin. Thus ∂D is a smooth, closed, and orientable surface. The field $\mathbf{F}$ has scalar components

$$f(x, y, z) = 2z - y, \quad g(x, y, z) = x, \quad h(x, y, z) = -2x,$$

which have continuous first partials on $\mathbb{R}^3$. Finally, to find the "outward" flux of $\mathbf{F}$ across the sphere means to give the sphere the positive orientation $\mathbf{n} = \hat{\mathbf{n}}$. With all of the hypotheses of the Divergence Theorem being satisfied, we calculate

$$(\nabla \cdot \mathbf{F})(x, y, z) = D_x(2z - y) + D_y(x) + D_z(-2x) = 0,$$

and finally

$$\oint\!\!\!\oint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV = \iiint_D (0) dV = 0.$$

That is, the net outward flux is zero.