Math 242 Exam #4 Key (Fall 2013)

1 We have

$$\mathcal{V} = \int_{-2}^{2}\int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}}\int_{x-3}^{3-x} dzdydx = \int_{-2}^{2}\int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}}(6-2x)\,dydx$$

$$= \int_{-2}^{2} 2(6-2x)\sqrt{1+x^2/4}\,dx = 2\int_{-2}^{2}(3-x)\sqrt{4-x^2}\,dx.$$

Trigonometric substitution: let $x = 2\sin\theta$, so

$$\mathcal{V} = 2\int_{-\pi/2}^{\pi/2}(3-2\sin\theta)\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = 24\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta - 16\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta$$

$$= 12\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 12\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 12\pi.$$

2 Letting I be the integral, we have

$$I = \int_1^6\int_0^{4-2y/3}\left[\frac{x}{y}\right]_0^{12-2y-3z} dzdy = \int_1^6\int_0^{4-2y/3}\frac{12-2y-3z}{y}\,dzdy$$

$$= \int_1^6\left[\frac{12z}{y} - 2z - \frac{3z^2}{2y}\right]_0^{4-2y/3} dy = \int_1^6\left(\frac{24}{y} + \frac{2y}{3} - 8\right)dy = 24\ln 6 - \frac{85}{3}$$

3 The region D is shown at left in the figure below. It will be convenient to work in cylindrical coordinates, where $x = r\cos\theta$ and $y = r\sin\theta$ so that the equation of the paraboloid becomes

$$z = x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2,$$

and the equation of the plane remains $z = 36$.

The intersection of the surfaces $z = 36$ and $z = x^2 + y^2$ is the set of points

$$\{(x, y, 25) : x^2 + y^2 = 36\},$$

which is a curve that projects onto the xy-plane as a circle of radius 6 centered at the origin. Thus, the projection of D onto the xy-plane is a region R that is a closed disc with radius 6 centered at the origin, shown at right in the figure below.

Now, a point in R may have a θ-coordinate value ranging anywhere from $\theta = 0$ to $\theta = 2\pi$; that is, if $(r, \theta) \in R$, then $0 \le \theta \le 2\pi$.

If we fix $\theta \in [0, 2\pi]$, then a point $(r, \theta) \in R$ must lie on the line segment joining $o = (0, 0)$ and $a = (6, \theta)$, shown at right in the figure below. That is, given $\theta \in [0, 2\pi]$, a point $(r, \theta) \in R$ can have r-coordinate value ranging anywhere from $r = 0$ to $r = 6$, which is to say $0 \le r \le 6$.

Finally, fixing $\theta \in [0, 2\pi]$ and $r \in [0, 6]$, we consider the limits on z in order for (r, θ, z) to be a point that lies in D. We find that generally z must be such that (r, θ, z) is above the paraboloid $z = r^2$ and below the plane $z = 36$, which is to say $r^2 \le z \le 36$.

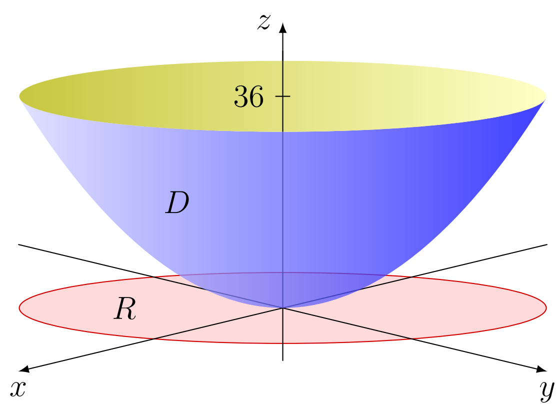
Thus we find that the region $E \subseteq \mathbb{R}^3_{r\theta z}$ for which $T_{\text{cyl}}(E) = D$ is

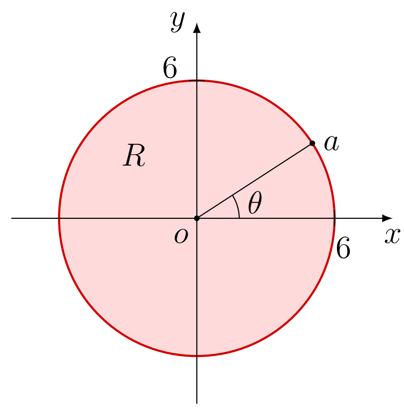
$$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 6,\ r^2 \le z \le 36\},$$

and so

$$\begin{aligned}\mathcal{V}(D) &= \iiint_D dV = \iiint_E r\,dV = \int_0^{2\pi}\int_0^6\int_{r^2}^{36} r\,dz dr d\theta \\ &= \int_0^{2\pi}\int_0^6 (36r - r^3)\,dr d\theta = \int_0^{2\pi}\left[18r^2 - \frac{1}{4}r^4\right]_0^6 d\theta \\ &= \int_0^{2\pi} 324\,d\theta = 324\cdot 2\pi = 648\pi\end{aligned}$$

is the volume of the region D.





4 D is given as $\{(\rho,\varphi,\theta) : 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi, 0 \le \rho \le 1\}$ in spherical coordinates, so

$$\iiint_D e^{-(x^2+y^2+z^2)^{3/2}} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{-\rho^3}\rho^2 \sin\varphi\, d\rho d\varphi d\theta.$$

Substitution: let $u = -\rho^3$, so that $\rho^2 d\rho = -\frac{1}{3}du$ and we obtain

$$\begin{aligned}\iiint_D e^{-(x^2+y^2+z^2)^{3/2}} dV &= \int_0^{2\pi}\int_0^{\pi} -\frac{\sin\varphi}{3}\left(\int_0^{-1} e^u\,du\right) d\varphi d\theta = \frac{1-e^{-1}}{3}\int_0^{2\pi}\int_0^{\pi}\sin\varphi\, d\varphi d\theta \\ &= \frac{1-e^{-1}}{3}\int_0^{2\pi} 2\,d\theta = \frac{1-e^{-1}}{3}\cdot 4\pi = \frac{4\pi(e-1)}{3e}\end{aligned}$$

5 The line may be parameterized by $\mathbf{r}(t) = \langle t, 3t\rangle$, $-\infty < t < \infty$, with tangent vector at $\mathbf{r}(t)$ given by $\mathbf{r}'(t) = \langle 1,3\rangle$. We must find a vector field $\mathbf{F}$ such that, for each t, $\mathbf{F}(\mathbf{r}(t))$ is orthogonal (i.e. normal) to $\mathbf{r}'(t)$. That is, we must have

$$\mathbf{F}(t,t)\cdot\langle 1,3\rangle = 0.$$

There are many possibilities: $\mathbf{F}(x,y) = \langle -3,1\rangle$, or $\mathbf{F}(x,y) = \langle -3c, c\rangle$ for any $c \ne 0$, or $\mathbf{F}(x,y) = \langle -3x, x\rangle$. (A trivial solution would be $\mathbf{F}(x,y) = \langle 0,0\rangle$, but we are looking for a nonzero vector field here.)

6a A viable parameterization:

$$\mathbf{r}(t) = \langle t, t^2\rangle, \quad 0 \le t \le 8.$$

6b Making the substitution $u = 1 + 4t^2$, we have

$$\int_C (xy)^{1/3} ds = \int_0^8 (t \cdot t^2)^{1/3} \|\mathbf{r}'(t)\| dt = \int_0^8 t\sqrt{1+4t^2} dt = \int_1^{257} \frac{\sqrt{u}}{8} du = \frac{257^{3/2} - 1}{12}.$$

7 Here $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t\rangle$, so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \mathbf{F}(4\cos t, 4\sin t) \cdot \langle -4\sin t, 4\cos t\rangle \, dt \\ &= \int_0^\pi \langle -4\sin t, 4\cos t\rangle \cdot \langle -4\sin t, 4\cos t\rangle \, dt \\ &= \int_0^\pi 16 \, dt = 16\pi.\end{aligned}$$

8 For brevity let $\mathbf{x} = \langle x, y, z\rangle$. We have $\mathbf{F} = \langle f, g, h\rangle$ with $f(\mathbf{x}) = 2xy^3z^4$, $g(\mathbf{x}) = 3x^2y^2z^4$, and $h(\mathbf{x}) = 4x^2y^3z^3$. Since

$$f_y(\mathbf{x}) = 6xy^2z^4 = g_x(\mathbf{x}), \; f_z(\mathbf{x}) = 8xy^3z^3 = h_x(\mathbf{x}), \; g_z(\mathbf{x}) = 12x^2y^2z^3 = h_y(\mathbf{x}),$$

it follows that $\mathbf{F}$ is conservative.

We now find φ. From $\nabla\varphi = \mathbf{F}$ we have

$$\varphi_x(\mathbf{x}) = 2xy^3z^4, \; \varphi_y(\mathbf{x}) = 3x^2y^2z^4, \; \varphi_z(\mathbf{x}) = 4x^2y^3z^3.$$

Hence

$$\varphi(\mathbf{x}) = \int \varphi_x(\mathbf{x}) \, dx = x^2y^3z^4 + c(y, z).$$

Differentiating this with respect to y gives $\varphi_y(\mathbf{x}) = 3x^2y^2z^4 + c_y(y, z)$, which, when compared to $\varphi_y(\mathbf{x}) = 3x^2y^2z^4$, informs us that $c_y(y, z) = 0$ and therefore $c(y, z) = c(z)$ (that is, the function c must not be a function of y).

At this point we have $\varphi(\mathbf{x}) = x^2y^3z^4 + c(z)$. This implies that $\varphi_z(\mathbf{x}) = 4x^2y^3z^3 + c'(z)$, which, when compared to $\varphi_z(\mathbf{x}) = 4x^2y^3z^3$, informs us that $c'(z) = 0$. So $c(z) = c$, where c is an arbitrary constant. Choosing c be zero, we obtain $\varphi(\mathbf{x}) = x^2y^3z^4$.

9 The curve C goes from $\mathbf{a} = \langle 0, 0\rangle$ to $\mathbf{b} = \langle \ln 2, 2\pi\rangle$, and the fact that it's a line segment will be irrelevant. Letting $\varphi(x, y) = e^{-x}\cos y$, the Fundamental Theorem of Line Integrals gives

$$\begin{aligned}\int_C \nabla(e^{-x}\cos y) \cdot d\mathbf{r} &= \int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(\mathbf{b}) - \varphi(\mathbf{a}) = \varphi(\ln 2, 2\pi) - \varphi(0, 0) \\ &= e^{-\ln 2}\cos(2\pi) - e^{-0}\cos(0) = \frac{1}{2} - 1 = -\frac{1}{2}.\end{aligned}$$