Math 242 Exam #3 Key (Fall 2013)

1a We have

$$f_x(x,y) = (y^2 + xy + 1)e^{xy} \quad \text{and} \quad f_y(x,y) = (x^2 + xy + 1)e^{xy}$$

Using

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

with $(x_0, y_0) = (2, 0)$, we get

$$z = f_x(2,0)(x-2) + f_y(2,0)(y-0) + f(2,0) = (x-2) + 5y + 2,$$

which simplifies to $x + 5y - z = 0$.

1b The tangent plane serves as a linearization L of the function f in a neighborhood of $(2,0)$, so that $z = f(x,y) \approx L(x,y)$ for (x,y) near $(2,0)$. From (1a) we have $z = x + 5y$, so that

$$L(x,y) = x + 5y,$$

and hence $z = f(1.95, 0.05) \approx L(1.95, 0.05) = 1.95 + 5(0.05) = 2.2$.

2 S is given by $F(x,y,z) = 0$, where

$$F(x,y,z) = x^2 + y^2 - z^2 - 2x + 2y + 3.$$

So $F_x(x,y,z) = 2x - 2$, $F_y(x,y,z) = 2y + 2$, and $F_z(x,y,z) = -2z$. A tangent plane to S at $(a,b,c) \in S$ is given by

$$\nabla F \cdot \langle x-a, y-b, z-c \rangle = 0 \quad \Rightarrow \quad \langle 2a-2, 2b+2, -2c \rangle \cdot \langle x-a, y-b, z-c \rangle = 0,$$

which becomes

$$(a-1)x + (b+1)y - cz = a(a-1) + b(b+1) - c^2.$$

A horizontal plane is a plane with equation $z = k$, where k is some constant. Thus we need $a = 1$ and $b = -1$. Then

$$a^2 + b^2 - c^2 - 2a + 2b + 3 = 0 \quad \Rightarrow \quad c^2 = 1 \quad \Rightarrow \quad c = \pm 1.$$

Therefore the two points on S where the tangent plane is horizontal are $(1,-1,1)$ and $(1,-1,-1)$.

3 First we gather our partial derivatives:

$$\begin{aligned} f_x(x,y) &= -3x^2 - 6x \\ f_y(x,y) &= -3y^2 + 6y \\ f_{xx}(x,y) &= -6x - 6 \\ f_{yy}(x,y) &= -6y + 6 \\ f_{xy}(x,y) &= 0 \end{aligned}$$

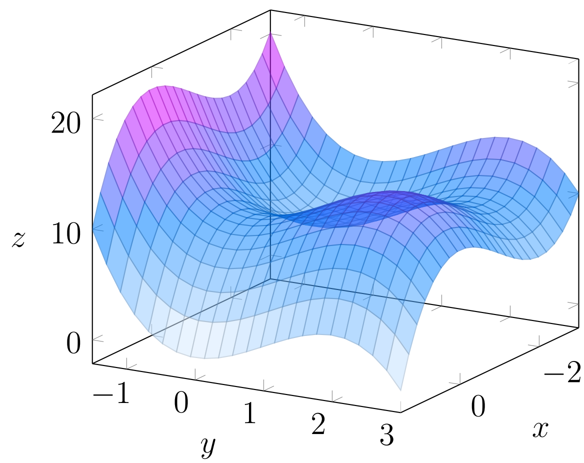
At no point does either f_x or f_y fail to exist, so we search for any point (x,y) for which $f_x(x,y) = f_y(x,y) = 0$. This yields the system

$$\begin{cases} 3x^2 + 6x = 0 \\ 3y^2 - 6y = 0 \end{cases}$$

The system has solutions $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$. We construct a table:

(x,y)	f_{xx}	f_{yy}	f_{xy}	Φ	Conclusion
$(0,0)$	-6	6	0	-36	Saddle Point
$(0,2)$	-6	-6	0	36	Local Maximum
$(-2,0)$	6	6	0	36	Local Minimum
$(-2,2)$	6	-6	0	-36	Saddle Point

Below is a graph of a part of the surface containing the points of interest.



4 Integrate with respect to y first:

$$\iint_R x^5 e^{x^3 y} dA = \int_0^{\ln 2}\int_0^1 x^5 e^{x^3 y} dy dx = \int_0^{\ln 2} x^2(e^{x^3}-1)dx$$
$$= \int_0^{\ln 2} x^2 e^{x^3} dx - \int_0^{\ln 2} x^2 dx = \frac{e^{\ln^3 2}-1}{3} - \frac{\ln^3 2}{3}$$
$$= \frac{e^{\ln^3 2} - 1 - \ln^3 2}{3}.$$

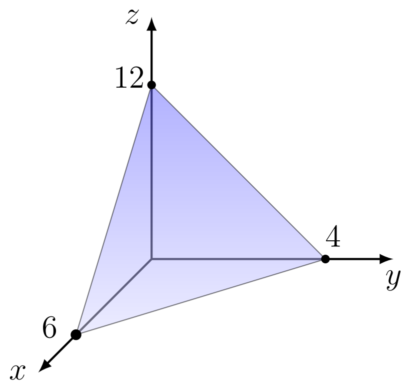
5 The region $D \subseteq \mathbb{R}^3$ is a tetrahedron in the first octant as shown in the stereoscopic figure below, with region $R \subseteq \mathbb{R}^2$ being the bottom side of D in the xy-plane. We have

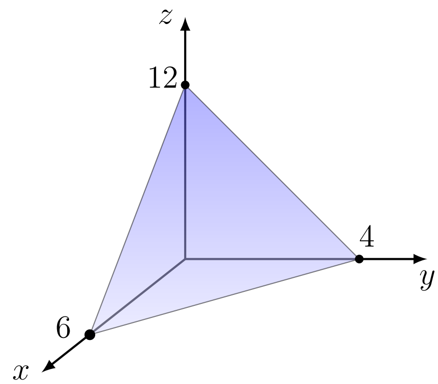
$$R = \left\{(x,y) : 0 \le x \le 6 \text{ and } 0 \le y \le -\tfrac{2}{3}x + 4\right\}.$$

At any point $(x,y) \in R$ we find that the height of D is $h(x,y) = 12 - 2x - 3y$, and so the volume of D is

$$\mathcal{V}(D) = \iint_R h = \int_0^6 \int_0^{-\frac{2}{3}x+4} (12-2x-3y) dy dx$$
$$= \int_0^6 \left[12y - 2xy - \frac{3}{2}y^2\right]_0^{-\frac{2}{3}x+4} dx = \int_0^6 \left(\frac{2}{3}x^2 - 8x + 24\right) dx$$

$$= \left[\frac{2}{9}x^3 - 4x^2 + 24x\right]_0^6 = 48.$$





6 The area of the enclosed region R is

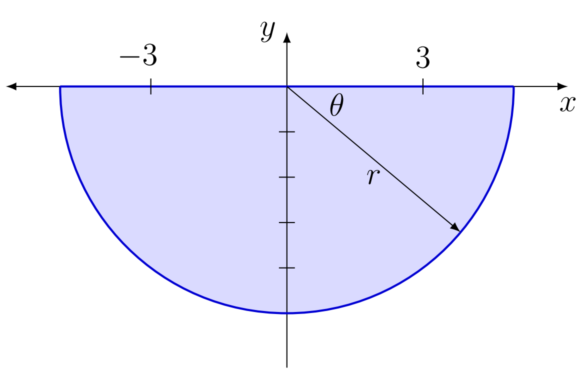
$$\mathcal{A}(R) = \iint_R dA = \int_{-1}^{2}\int_{x^2}^{x+2} dydx = \int_{-1}^{2}(x + 2 - x^2)dx = \frac{9}{2}$$

7 The sketch of R in the xy-plane is below. The region

$$S = \{(r, \theta) : 0 \le r \le 5 \text{ and } \pi \le \theta \le 2\pi\}$$

in the $r\theta$-plane is such that $T_{\text{pol}}(S) = R$, and therefore

$$\begin{aligned}\iint_R 2xy\, dA &= \iint_S 2(r\cos\theta)(r\sin\theta)r\, dA = \int_\pi^{2\pi}\int_0^5 2(r\cos\theta)(r\sin\theta)r\, drd\theta \\ &= \int_\pi^{2\pi}\int_0^5 2r^3\cos\theta\sin\theta\, drd\theta = \int_\pi^{2\pi}\cos\theta\sin\theta\left[\frac{1}{2}r^4\right]_0^5 d\theta \\ &= \frac{625}{2}\int_\pi^{2\pi}\cos\theta\sin\theta\, d\theta = \frac{625}{4}\int_\pi^{2\pi}\sin(2\theta)\, d\theta = 0.\end{aligned}$$



8 The volume of the enclosed region D is

$$\begin{aligned}\mathcal{V}(D) &= \iint_R h = \int_0^{2\pi}\int_0^{25}\left(25 - \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}\right) r\,dr d\theta \\ &= \int_0^{2\pi}\int_0^{25}(25 - r)r\,dr d\theta = \int_0^{2\pi}\left[\frac{25}{2}r^2 - \frac{1}{3}r^3\right]_0^{25} d\theta \\ &= \int_0^{2\pi}\frac{15,625}{3}d\theta = \frac{15,625}{3}\pi.\end{aligned}$$