Math 242 Exam #2 Key (Fall 2013)

1 Let P be the plane $2x+y-z=1$, and let Q be the plane parallel to P and passing through the point $(0,2,-2)$. Writing the equation for P as

$$2x+(y-1)-z=0$$

and comparing to $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, we see that the normal vector for P is $\mathbf{n}_P=\langle a,b,c\rangle=\langle 2,1,-1\rangle$. Now, if $\mathbf{n}_Q$ is the normal vector for Q, then $Q \parallel P$ implies that we must have $\mathbf{n}_Q \parallel \mathbf{n}_P$. Thus we may choose $\mathbf{n}_Q=\mathbf{n}_P$. So Q has normal vector $\langle 2,1,-1\rangle$ and point $(0,2,-2)$, and therefore an equation for Q is

$$2(x-0)+(y-2)-(z+2)=0,$$

which simplifies to become $2x+y-z=4$. Tah-dah!

2 The domain is

$$\text{Dom}(f)=\{(x,y,z)\in\mathbb{R}^3 : y\neq z\},$$

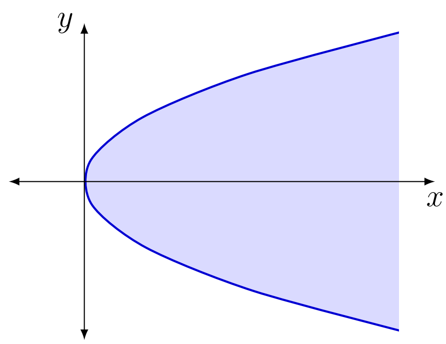
which consists of all of $\mathbb{R}^3$ except for points on the plane $y=z$. As for the range, the only value the fraction $3/(y-z)$ cannot attain is 0, and so

$$\text{Ran}(f)=(-\infty,0)\cup(0,\infty).$$

3 The function h is a composition of a polynomial function and the square root function (which is a radical function), and so it is continuous on its domain. We have

$$\text{Dom}(h)=\{(x,y) : x-y^2\geq 0\}=\left\{(x,y) : x\geq y^2\right\},$$

which is the shaded region in $\mathbb{R}^2$ illustrated below.



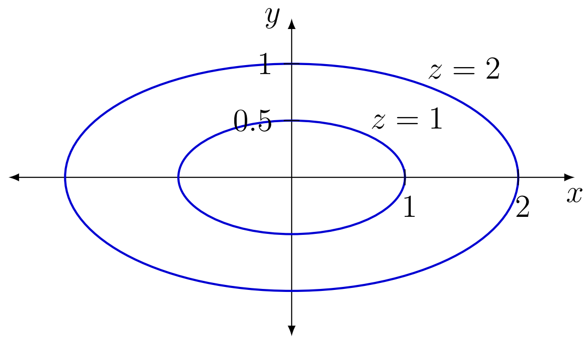
4 The level curve $z=1$ has equation $1=\sqrt{x^2+4y^2}$, which implies

$$x^2+\frac{y^2}{1/4}=1,$$

an ellipse. The level curve $z=2$ has equation $2=\sqrt{x^2+4y^2}$, which implies

$$\frac{x^2}{4}+y^2=1,$$

also an ellipse. Graph is below.



5 We have

$$\lim_{(x,y)\to(-1,1)} \frac{x^2 - y^2}{x^2 - xy - 2y^2} = \lim_{(x,y)\to(-1,1)} \frac{(x-y)(x+y)}{(x-2y)(x+y)} = \lim_{(x,y)\to(-1,1)} \frac{x-y}{x-2y} = \frac{-1-1}{-1-2(1)} = \frac{2}{3}.$$

6 Along the path $y = x$ the limit becomes

$$\lim_{x\to 0} \frac{|x-x|}{|x+x|} = \lim_{x\to 0} \frac{0}{2|x|} = \lim_{x\to 0}(0) = 0.$$

Along the path $y = 2x$ the limit becomes

$$\lim_{x\to 0} \frac{|x-2x|}{|x+2x|} = \lim_{x\to 0} \frac{|-x|}{|3x|} = \lim_{x\to 0} \frac{|x|}{3|x|} = \lim_{x\to 0} \frac{1}{3} = \frac{1}{3}.$$

So, we converge on two different values along two different paths that approach the origin, and therefore the limit does not exist by the Two-Path Test.

7 We have

$$\varphi_t(t,z) = z^2 \sec^2(tz) \cdot (tz)_t = z^2 \sec^2(tz) \cdot z = z^3 \sec^2(tz)$$

and

$$\begin{aligned} \varphi_z(t,z) &= (z^2)_z \tan(tz) + z^2(\tan(tz))_z = 2z\tan(tz) + z^2(\sec^2(tz)\cdot t) \\ &= 2z\tan(tz) + tz^2\sec^2(tz). \end{aligned}$$

8a The function $(x,y) \mapsto xy$ is continuous on $\mathbb{R}^2$, while the absolute value function $x \mapsto |x|$ is continuous on $\mathbb{R}$. Hence $\varphi(x,y) = |xy|$ is continuous on $\mathbb{R}^2$.

Now, if we define $\psi(x) = \sqrt{x}$, then $g = \psi \circ \varphi$ with domain $\mathbb{R}^2$, and clearly g is continuous on its domain. Hence g is continuous at $(0,0)$.

8b We have

$$g_x(0,0) = \lim_{h\to 0} \frac{g(h,0) - g(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and

$$g_y(0,0) = \lim_{h\to 0} \frac{g(0,h)-g(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

8c By definition g is differentiable at $(0,0)$ if and only if

$$\lim_{(h,k)\to(0,0)} \frac{g(h,k)-g(0,0)-g_x(0,0)h-g_y(0,0)k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} = 0. \qquad (1)$$

Along the path $k=h$ the limit becomes

$$\lim_{(h,k)\to(0,0)} \frac{\sqrt{|h^2|}}{\sqrt{h^2+h^2}} = \lim_{(h,k)\to(0,0)} \frac{|h|}{|h|\sqrt{2}} = \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

This is already enough to show that the limit (1) cannot be 0. Thus g is not differentiable at $(0,0)$.

9 First get the unit vector in the direction of $\langle 1,\sqrt{3}\rangle$:

$$\mathbf{u} = \frac{\langle 1,\sqrt{3}\rangle}{2} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle.$$

Now,

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \mathbf{u} = \langle e^x \sin y, e^x \cos y\rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{e^x \sin y}{2} + \frac{\sqrt{3}e^x \cos y}{2},$$

and so

$$D_{\mathbf{u}}f(0,\pi/4) = \frac{e^0 \sin(\pi/4)}{2} + \frac{\sqrt{3}e^0\cos(\pi/4)}{2} = \frac{1/\sqrt{2}}{2} + \frac{\sqrt{3}\cdot 1/\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

10 Let C_0 be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \ge 0$, where $\mathbf{r}(0) = \langle 2,2\rangle$. Note that

$$f_x(x,y) = f_y(x,y) = 1$$

for all $(x,y)\in\mathbb{R}^2$. To remain on the path of steepest descent, for any $t\ge 0$ the tangent vector to C_0 at the point $\mathbf{r}(t)$, which is $\mathbf{r}'(t)$, must be in the direction of

$$-\nabla f(\mathbf{r}(t)) = -\nabla f(x(t),y(t)) = -\big\langle f_x(x(t),y(t)), f_y(x(t),y(t))\big\rangle = -\langle 1,1\rangle.$$

Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle -1,-1\rangle,$$

from which we obtain the differential equations $x'(t) = -1$ and $y'(t) = -1$. Thus $x(t) = -t + c_1$ and $y(t) = -t + c_2$ for arbitrary constants c_1, c_2. So far we have

$$\mathbf{r}(t) = \langle -t+c_1, -t+c_2\rangle.$$

Now,

$$\mathbf{r}(0) = \langle c_1, c_2\rangle = \langle 2,2\rangle$$

shows that $c_1 = c_2 = 2$, and hence

$$\mathbf{r}(t) = \langle 2 - t, 2 - t \rangle, \quad t \geq 0$$

is a parameterization for C_0.