Math 242 Exam #1 Key (Fall 2013)

1a Two such vectors are

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 6, -4\rangle}{\sqrt{52}} = \frac{1}{\sqrt{13}}\langle 3, -2\rangle \quad \text{and} \quad -\hat{\mathbf{v}} = \frac{1}{\sqrt{13}}\langle -3, 2\rangle$$

1b We need a such that $\|\mathbf{w}\| = 1$. Now,

$$1 = \|\mathbf{w}\| = \sqrt{a^2 + a^2/9} = \sqrt{10a^2/9} = \frac{\sqrt{10}}{3}|a| \;\Rightarrow\; |a| = \frac{3}{\sqrt{10}} \;\Rightarrow\; a = \pm\frac{3}{\sqrt{10}}.$$

2 We have the force vectors

$$\begin{aligned}
\mathbf{F}_1 &= 400\langle\cos(-30°), \sin(-30°)\rangle = \big\langle 200\sqrt{3}, -200\big\rangle, \\
\mathbf{F}_2 &= 280\langle\cos(45°), \sin(45°)\rangle = \big\langle 280/\sqrt{2}, 280/\sqrt{2}\,\big\rangle, \\
\mathbf{F}_3 &= 350\langle\cos(135°), \sin(135°)\rangle = \big\langle -350/\sqrt{2}, 350/\sqrt{2}\,\big\rangle.
\end{aligned}$$

Adding these forces gives

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \approx \langle 296.9, 245.5\rangle,$$

and so $\|\mathbf{F}\| = 385.0$ N, directed at an angle of

$$\theta \approx \tan^{-1}\left(\frac{245.5}{296.9}\right) \approx 39.6°.$$

3 The crosswind is $\mathbf{w} = \langle 0, 4, 0\rangle$, the updraft is $\mathbf{u} = \langle 0, 0, 10\rangle$, and thus the terminal velocity is

$$\mathbf{v} = \langle 0, 0, -60\rangle + \mathbf{u} + \mathbf{w} = \langle 0, 4, -50\rangle.$$

The probe's speed is thus

$$\|\mathbf{v}\| = \sqrt{50^2 + 4^2} = 2\sqrt{629} \approx 50.2 \text{ m/s},$$

directed at an angle of

$$\theta = \tan^{-1}\left(\frac{4}{50}\right) \approx 4.6°$$

north of vertical (i.e. 85.4° with respect to the ground, drifting north).

4 Complete the square for each variable:

$$(x^2 - 6x) + (y^2 + 6y) + (z^2 - 8z) = 2 \;\Rightarrow\; (x^2 - 6x + 9) + (y^2 + 6y + 9) + (z^2 - 8z + 16) = 2 + 9 + 9 + 16,$$

and so

$$(x-3)^2 + (y+3)^2 + (z-4)^2 = 36.$$

The solution set is a sphere with center at $(3, -3, 4)$ and radius 6.

5a $\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 5^2} = \sqrt{30}$ and $\|\mathbf{v}\| = \sqrt{(-1)^2 + 4^2 + (-3)^2} = \sqrt{26}$.

5b Since $\mathbf{u}\cdot\mathbf{v} = (2)(-1)+(-1)(4)+(5)(-3) = -21$ and $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = (\sqrt{26}\,)^2 = 26$, we have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = -\frac{21}{26}\langle -1, 4, -3\rangle = \left\langle \frac{21}{26}, -\frac{42}{13}, \frac{63}{26}\right\rangle.$$

5c We have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-21}{\sqrt{30}\sqrt{26}} = \frac{-21}{2\sqrt{195}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{-21}{2\sqrt{195}}\right) \approx 138.8^\circ.$$

6 Let $o = (0,0,0)$, $p = (2,4,8)$, and $q = (1,4,10)$. We have

$$\overrightarrow{op}\times\overrightarrow{oq} = \langle 2,4,8\rangle \times \langle 1,4,10\rangle = \left\langle \begin{vmatrix} 4 & 8 \\ 4 & 10 \end{vmatrix}, -\begin{vmatrix} 2 & 8 \\ 1 & 10 \end{vmatrix}, \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} \right\rangle = \langle 8, -12, 4\rangle,$$

and so the area of P is

$$\mathcal{A}(P) = \|\overrightarrow{op}\times\overrightarrow{oq}\| = \|\langle 8,-12,4\rangle\| = \sqrt{224} = 4\sqrt{14}.$$

7 We have $y(t) = -t+4$, so $y(t) = -2$ implies that $-t+4 = -2$, or $t = 6$. So the line intersects the plane at the point

$$\mathbf{r}(6) = \big\langle 2(6)+1, -6+4, 6-6\big\rangle = \langle 13, -2, 0\rangle.$$

8 Let $\mathbf{v} = \langle 3-1, -3-0, 3-1\rangle = \langle 2,-3,2\rangle$ and $\mathbf{r}_0 = \langle 1,0,1\rangle$. Then an equation (i.e. parameterization) for the line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 1,0,1\rangle + \langle 2t, -3t, 2t\rangle,$$

or

$$\mathbf{r}(t) = \langle 1+2t, -3t, 1+2t\rangle, \quad -\infty < t < \infty.$$

9 From $\mathbf{r}'(t) = \langle 1, 0, -2/t^2\rangle$ we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+4/t^4}}\left\langle 1, 0, -\frac{2}{t^2}\right\rangle.$$

Now we obtain

$$\mathbf{T}(2) = \frac{1}{\sqrt{1+4/2^4}}\left\langle 1, 0, -\frac{2}{2^2}\right\rangle = \left\langle \frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right\rangle$$

10 By definition,

$$\int_0^1 \langle e^{2t}, e^{-t}, t\rangle dt = \left\langle \int_0^1 e^{2t}dt, \int_0^1 e^{-t}dt, \int_0^1 t\,dt \right\rangle = \left\langle \left[\frac{1}{2}e^{2t}\right]_0^1, \left[-e^{-t}\right]_0^1, \left[\frac{1}{2}t^2\right]_0^1 \right\rangle$$
$$= \left\langle \frac{e^2-1}{2},\ 1-\frac{1}{e},\ \frac{1}{2} \right\rangle.$$

11 From $\mathbf{a}(t) = \langle 1, t\rangle$ we integrate to obtain $\mathbf{v}(t) = \langle t + a_1, t^2/2 + a_2\rangle$. Now,

$$\mathbf{v}(0) = \langle a_1, a_2\rangle = \langle 2, -1\rangle,$$

so we have $a_1 = 2$ and $a_2 = -1$, and obtain

$$\mathbf{v}(t) = \langle t+2, t^2/2 - 1\rangle$$

for the velocity function. Integrating this function then yields

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t + b_1, \frac{1}{6}t^3 - t + b_2 \right\rangle.$$

Now,

$$\mathbf{r}(0) = \langle b_1, b_2\rangle = \langle -3, 6\rangle,$$

so $b_1 = -3$ and $b_2 = 6$ and we obtain

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t - 3, \frac{1}{6}t^3 - t + 6 \right\rangle.$$

12 First we obtain

$$\mathbf{r}'(t) = \langle -3\cos^2 t \sin t + 3\sin^2 t \cos t,$$

and then

$$\|\mathbf{r}'(t)\| = \sqrt{9\cos^4 t\cos^2 t + 9\sin^4 t\cos^2 t} = 3\sin t\cos t\sqrt{\cos^2 t + \sin^2 t} = 3\sin t\cos t.$$

Now, letting $u = \sin t$, we have

$$\mathcal{L}(C) = \int_0^{\pi/2} \|\mathbf{r}'(t)\|dt = 3\int_0^{\pi/2} \sin t\cos t\,dt = 3\int_0^1 u\,du = \frac{3}{2}.$$

13 The vector function $\mathbf{r}(t) = \langle t, t^2\rangle$, $t \in (-\infty, \infty)$, yields the same curve. We have

$$\mathbf{r}'(t) = \langle 1, 2t\rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{1+4t^2},$$

so that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t\rangle,$$

which yields

$$\mathbf{T}'(t) = \frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2\rangle,$$

and thus

$$\|\mathbf{T}'(t)\| = \frac{\sqrt{4+16t^2}}{(1+4t^2)^{3/2}} = \frac{2}{1+4t^2}$$

Finally we obtain

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2}{1+4t^2} \cdot \frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}.$$

as the curvature of the curve at the point (t, t^2). The point $(0, 0)$ corresponds to $t = 0$, and so the curvature at $(0, 0)$ is $\kappa(0) = 2$.