Math 242 Exam #5 Key (Fall 2012)

1 We have $\mathbf{F} = \langle f, g\rangle$ with $f(x,y) = 2xy$ and $g(x,y) = x^2 - y^2$, and since R is connected and simply connected, and ∂R is simple, closed and piecewise-smooth, by Green's Theorem

$$\oint_{\partial R} \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA = \iint_R (2x - 2x)\,dA = 0.$$

2 We have $\mathbf{F} = \langle f, g\rangle$ with $f(x,y) = 0$ and $g(x,y) = xy$, and since R is connected and simply connected, and ∂R is simple, closed and piecewise-smooth, by Green's Theorem

$$\oint_{\partial R} \mathbf{F}\cdot \mathbf{n} = \iint_R (f_x + g_y)\,dA = \iint_R x\,dA = \int_0^2\int_0^{-2x+4} x\,dydx = \int_0^2 [xy]_0^{-2x+4}\,dx$$
$$= \int_0^2 (-2x^2 + 4x)\,dx = \left[-\frac{2}{3}x^3 + 2x^2\right]_0^2 = \frac{8}{3}.$$

3 We have

$$\begin{aligned}(\operatorname{div}\mathbf{F})(x,y,z) = (\nabla\cdot\mathbf{F})(x,y,z) &= \langle D_x, D_y, D_z\rangle \cdot \langle yz\sin x, xz\cos y, xy\cos z\rangle \\ &= D_x(yz\sin x) + D_y(xz\cos y) + D_z(xy\cos z) \\ &= yz\cos x - xz\sin y - xy\sin z\end{aligned}$$

4 Totally trivial:

$$(\operatorname{curl}\mathbf{F})(x,y,z) = (\nabla\times\mathbf{F})(x,y,z) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ D_x & D_y & D_z \\ 0 & z^2-y^2 & yz\end{vmatrix}$$
$$= \begin{vmatrix} D_y & D_z \\ z^2-y^2 & yz\end{vmatrix}\mathbf{i} - \begin{vmatrix} D_x & D_z \\ 0 & yz\end{vmatrix}\mathbf{j} + \begin{vmatrix} D_x & D_y \\ 0 & z^2-y^2\end{vmatrix}\mathbf{k}$$
$$= \left[D_y(yz) - D_z(z^2-y^2)\right]\mathbf{i} - \left[D_x(yz) - D_z(0)\right]\mathbf{j} + \left[D_x(z^2-y^2) - D_y(0)\right]\mathbf{k}$$
$$= -z\mathbf{i} = \langle -z, 0, 0\rangle.$$

5a A parameterization:

$$\mathbf{r}(u,v) = \langle 4\cos u, 4\sin u, v\rangle,\ (u,v)\in[0,\pi]\times[0,7].$$

5b Let Σ denote the surface, and $R = [0,\pi]\times[0,7]$. Surface area:

$$\begin{aligned}\mathcal{A}(\Sigma) = \iint_\Sigma dS &= \iint_R \|(\mathbf{r}_u\times\mathbf{r}_v)(u,v)\|\,dA \\ &= \iint_R \|\langle -4\sin u, 4\cos u, 0\rangle \times \langle 0,0,1\rangle\|\,dA \\ &= \iint_R \|\langle 4\cos u, 4\sin u, 0\rangle\|\,dA = \int_0^\pi\int_0^7 4\,dvdu = 28\pi.\end{aligned}$$

6a Since $z^2 = x^2 + y^2$ for $z \geq 0$, we have

$$z = \sqrt{x^2 + y^2}.$$

Thus if we let $x = u$ and $y = v$, we arrive at the parameterization

$$\mathbf{r}(u, v) = \langle u, v, \sqrt{u^2 + v^2} \rangle, \quad (u, v) \in R,$$

where $R = \{(u, v) : 0 \leq u^2 + v^2 \leq 1\}$ since $0 \leq z \leq 1$ implies that $0 \leq \sqrt{u^2 + v^2} \leq 1$.

6b First we need to find an orientation $\mathbf{n} : R \to \mathbb{R}^3$ for Σ such that, for each $(u, v) \in R$, the unit vector $\mathbf{n}(u, v)$ has a positive z-component. From

$$\mathbf{r}_u(u, v) = \left\langle 1, 0, \frac{u}{\sqrt{u^2 + v^2}} \right\rangle \quad \text{and} \quad \mathbf{r}_v(u, v) = \left\langle 0, 1, \frac{v}{\sqrt{u^2 + v^2}} \right\rangle$$

we have

$$(\mathbf{r}_u \times \mathbf{r}_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2 + v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2 + v^2}} \end{vmatrix} = \left\langle -\frac{u}{\sqrt{u^2 + v^2}}, -\frac{v}{\sqrt{u^2 + v^2}}, 1 \right\rangle$$

Our two choices of orientation for Σ are $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$. In order to have positive z-components we choose $\mathbf{n} = \hat{\mathbf{n}}$; that is,

$$\mathbf{n}(u, v) = \hat{\mathbf{n}}(u, v) = \frac{(\mathbf{r}_u \times \mathbf{r}_v)(u, v)}{\|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|} = \left\langle -\frac{u}{\sqrt{2(u^2 + v^2)}}, -\frac{v}{\sqrt{2(u^2 + v^2)}}, \frac{1}{\sqrt{2}} \right\rangle$$

Finally we evaluate the appropriate flux integral, substituting $\hat{\mathbf{n}}$ for $\mathbf{n}$:

$$\begin{aligned} \iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iint_\Sigma \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \hat{\mathbf{n}}(u, v) \|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|\, dA \\ &= \iint_R \mathbf{F}\big(u, v, \sqrt{u^2 + v^2}\big) \cdot (\mathbf{r}_u \times \mathbf{r}_v)(u, v)\, dA \\ &= \iint_R \left\langle u, v, \sqrt{u^2 + v^2} \right\rangle \cdot \left\langle -\frac{u}{\sqrt{u^2 + v^2}}, -\frac{v}{\sqrt{u^2 + v^2}}, 1 \right\rangle dA \\ &= \iint_R (0)\, dA = 0. \end{aligned}$$

The flux of $\mathbf{F}$ across Σ is therefore 0.

7 C is a circle centered at the origin, and so a convenient choice for Σ would be the planar region enclosed by C. A parameterization for Σ is

$$\boldsymbol{\rho}(u, v) = \langle 3v \cos u, 4v \cos u, 5v \sin u \rangle, \quad (u, v) \in [0, 2\pi] \times [0, 1].$$

Note the origin is obtained when $v = 0$, C is obtained when $v = 1$, and concentric circles inside C are obtained for $0 < v < 1$. The parameter u simply stands for t.

Clearly Σ is orientable. Let $R = [0, 2\pi] \times [0, 1]$. Then $\text{Int}(R) = (0, 2\pi) \times (0, 1)$, and for any $(u, v) \in \text{Int}(R)$ we have

$$(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3v \sin u & -4v \sin u & 5v \cos u \\ 3 \cos u & 4 \cos u & 5 \sin u \end{vmatrix} = \langle -20v, 15v, 0 \rangle,$$

and so

$$\hat{\mathbf{n}}(u, v) = \frac{\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v}{\|\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v\|}(u, v) = \frac{\langle -20v, 15v, 0 \rangle}{25v} = \left\langle -\frac{4}{5}, \frac{3}{5}, 0 \right\rangle.$$

Thus $\hat{\mathbf{n}} : \text{Int}(R) \to \mathbb{R}^3$ is continuous on $\text{Int}(R)$, and it has continuous extension to R simply by setting $\hat{\mathbf{n}}(u, v) = \langle -4/5, 3/5, 0 \rangle$ for all $(u, v) \in \partial R$.

It must be determined which orientation, $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$, is consistent with the orientation of C. The unit tangent and principal unit normal vectors for C are

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{5}\langle -3 \sin t, -4 \sin t, 5 \cos t \rangle$$

and

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{5}\langle -3 \cos t, -4 \cos t, -5 \sin t \rangle.$$

Now,

$$(\mathbf{T} \times \mathbf{N})(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{5} \sin t & -\frac{4}{5} \sin t & \cos t \\ -\frac{3}{5} \cos t & -\frac{4}{5} \cos t & -\sin t \end{vmatrix} = \left\langle \frac{4}{5}, -\frac{3}{5}, 0 \right\rangle,$$

which we see agrees with $-\hat{\mathbf{n}}(u, v)$ for all $(u, v) \in R$, and so we give Σ the orientation $-\hat{\mathbf{n}}$.

Next, $\mathbf{F} = \langle f, g, h \rangle$ with $f(x, y, z) = y^2$, $g(x, y, z) = -z^2$, and $h(x, y, z) = x$, and so

$$(\nabla \times \mathbf{F})(x, y, z) = \langle 2z, -1, -2y \rangle.$$

Finally by Stokes' Theorem, substituting $-\hat{\mathbf{n}}$ for $\mathbf{n}$, we obtain

$$\begin{aligned}
\oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} &= -\iint_{\Sigma} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \\
&= -\iint_{R} (\nabla \times \mathbf{F})(\boldsymbol{\rho}(u, v)) \cdot \hat{\mathbf{n}}(u, v) \|(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)(u, v)\|\, dA \\
&= -\iint_{R} \left[(\nabla \times \mathbf{F})(3v \cos u, 4v \cos u, 5v \sin u) \cdot \left\langle \tfrac{4}{5}, -\tfrac{3}{5}, 0 \right\rangle\right](25v)\, dA \\
&= -\iint_{R} 25v \langle 10v \sin u, -1, -8v \cos u \rangle \cdot \left\langle \tfrac{4}{5}, -\tfrac{3}{5}, 0 \right\rangle\, dA \\
&= \int_0^1 \int_0^{2\pi} (15v + 200v^2 \sin u)\, du dv = \int_0^1 30\pi v\, dv = 15\pi.
\end{aligned}$$