Math 242 Exam #3 Key (Fall 2012)

1a We have

$$f_x(x,y) = -\frac{2y}{(x-y)^2} \quad \text{and} \quad f_y(x,y) = \frac{2x}{(x-y)^2}$$

Using

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$

with $(x_0,y_0) = (3,2)$, we get

$$z = -4(x-3) + 6(y-2) + 5,$$

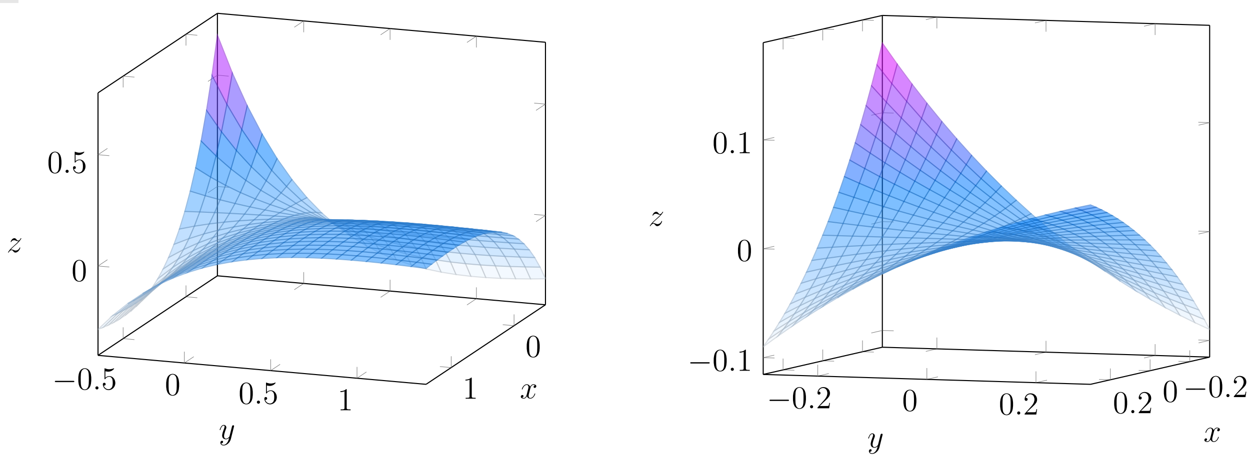
which simplifies to $4x - 6y + z = 5$.

1b The tangent plane serves as a linearization L of the function f in a neighborhood of $(3,2)$, so that $L(x,y) \approx f(x,y)$ for (x,y) near $(3,2)$. From (1a) we have

$$L(x,y) = -4x + 6y + 5,$$

and so $f(2.95, 2.05) \approx L(2.95, 2.05) = 5.5$.

2



First we gather our partial derivatives:

$$\begin{aligned}
f_x(x,y) &= (y - xy)e^{-x-y} \\
f_y(x,y) &= (x - xy)e^{-x-y} \\
f_{xx}(x,y) &= (xy - 2y)e^{-x-y} \\
f_{yy}(x,y) &= (xy - 2x)e^{-x-y} \\
f_{xy}(x,y) &= (1 - x + xy - y)e^{-x-y}
\end{aligned}$$

At no point does either f_x or f_y fail to exist, so we search for any point (x,y) for which $f_x(x,y) = f_y(x,y) = 0$. This yields the system

$$\begin{cases} y - xy = 0 \\ x - xy = 0 \end{cases}$$

We see we must have $x = xy = y$. Putting $x = y$ into the 1st equation yields $x - x^2 = 0$, which has solutions $x = 0, 1$. When $x = 0$ we obtain (from the 1st equation) $y = 0$; and when $x = 1$

we obtain (from the 2nd equation) $y = 1$. Thus we have solutions $(0, 0)$ and $(1, 1)$, which are critical points.

From $f_{xx}(0, 0) = f_{yy}(0, 0) = 0$ and $f_{xy}(0, 0) = 1$ we have $\Phi(0, 0) = -1 < 0$, and therefore f has a saddle point at $(0, 0)$ by the Second Derivative Test.

From $f_{xx}(1, 1) = f_{yy}(1, 1) = -e^{-2}$ and $f_{xy}(1, 1) = 0$ we have $\Phi(0, 0) = e^{-4} > 0$, and therefore f has a local maximum at $(1, 1)$ by the Second Derivative Test.

In the figure at left above, it is not at all obvious at a glance that there is a local maximum present, but it is there! The figure at right zooms in on $(0, 0, 0)$ to at least make the saddle point clear.

3 We have $f_x(x, y) = -2x$, $f_y(x, y) = -8y$, $f_{xx}(x, y) = -2$, $f_{yy}(x, y) = -8$, $f_{xy}(x, y) = 0$, and thus $\Phi(x, y) = 16$. Setting $f_x(x, y) = f_y(x, y) = 0$ yields the system $-2x = 0$ & $-8y = 0$, which gives $(0, 0)$ as the only critical point, which is a point that lies in R. Since $f_{xx}(0, 0) = -2 < 0$ and $\Phi(0, 0) = 8 > 0$, f has a local maximum at $(0, 0)$.

Along the top side of R we have $y = 1$, which yields the function $f_1(x) = 2 - x^2$ for $x \in [-2, 2]$. Using the Closed Interval Method on f_1 in $[-2, 2]$, the global maximum of f_1 occurs at $x = 0$ (corresponding to point $(0, 1)$ for f), and the global minimum at $x = \pm 2$ (corresponding to points $(\pm 2, 1)$ for f).

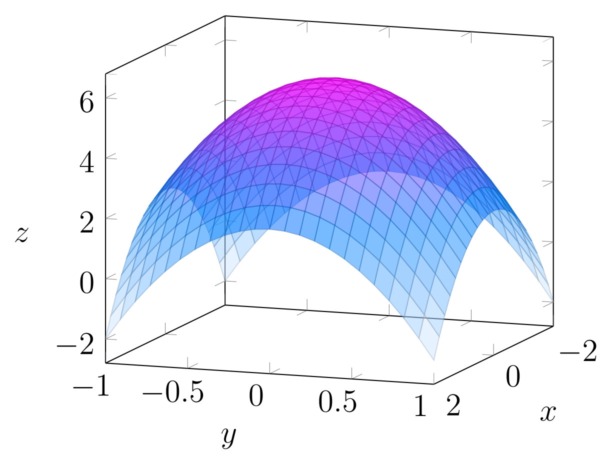
Along the bottom of R we have $y = -1$, which yields the function $f_2(x) = 2 - x^2$ for $x \in [-2, 2]$. The global maximum of f_2 occurs at $x = 0$ (corresponding to point $(0, -1)$ for f), and the global minimum at $x = \pm 2$ (corresponding to points $(\pm 2, -1)$ for f).

Along the left side of R we have $x = -2$, which yields the function $f_3(y) = 2 - 4y^2$ for $y \in [-1, 1]$. Using the Closed Interval Method on f_3 in $[-1, 1]$, the global maximum of f_3 occurs at $y = 0$ (corresponding to point $(-2, 0)$ for f), and the global minimum at $y = \pm 1$ (corresponding to points $(-2, \pm 1)$ for f).

Along the right side of R we have $x = 2$, which yields the function $f_4(y) = 2 - 4y^2$ for $y \in [-1, 1]$. The global maximum of f_4 occurs at $y = 0$ (corresponding to point $(2, 0)$ for f), and the global minimum at $y = \pm 1$ (corresponding to points $(2, \pm 1)$ for f).

Any point in R that corresponds to a point where any of the functions f_i has an extremum is a point where f itself has an extremum. Thus to find the global extrema of f we evaluate f at all these points as well as all critical points. We have: $f(\pm 2, \pm 1) = -2$, $f(0, \pm 1) = 2$, $f(\pm 2, 0) = 2$, and $f(0, 0) = 6$.

Therefore f has a global minimum at the points $(\pm 2, \pm 1)$, and a global maximum at $(0, 0)$.



4 By Fubini's Theorem we have

$$\iint_R e^{x+2y}\,dA = \int_1^{\ln 3}\int_0^{\ln 2} e^{x+2y}\,dxdy = \int_1^{\ln 3} e^{2y}\left(\int_0^{\ln 2} e^x\,dx\right)dy$$
$$= \int_1^{\ln 3} e^{2y}\left[e^x\right]_0^{\ln 2} dy = \int_1^{\ln 3} e^{2y}\,dy = \frac{1}{2}\left[e^{2y}\right]_1^{\ln 3} = \frac{1}{2}(9-e^2) = \frac{9-e^2}{2}.$$

5 By Fubini's Theorem we have

$$\iint_R y^3\sin(xy^2)\,dA = \int_0^{\sqrt{\pi/2}}\int_0^1 y^3\sin(xy^2)\,dxdy = \int_0^{\sqrt{\pi/2}}\left[-\frac{y^3}{y^2}\cos(xy^2)\right]_0^1 dy$$
$$= \int_0^{\sqrt{\pi/2}} -y(\cos y^2 - 1)\,dy = \int_0^{\sqrt{\pi/2}} y\,dy - \int_0^{\sqrt{\pi/2}} y\cos(y^2)\,dy$$
$$= \frac{\pi}{4} - \frac{1}{2}\int_0^{\sqrt{\pi/2}}\left[\sin(y^2)\right]'dy = \frac{\pi}{4} - \frac{1}{2}\left[\sin(y^2)\right]_0^{\sqrt{\pi/2}} = \frac{\pi}{4} - \frac{1}{2}.$$

6 In the first quadrant $y = x^2$ and $y = 8 - x^2$ intersect at $(2, 4)$, which allows us to determine R so that

$$\iint_R (x+y)dA = \int_0^2\int_{x^2}^{8-x^2}(x+y)dydx = \int_0^2\left[xy + \frac{1}{2}y^2\right]_{x^2}^{8-x^2} dx$$
$$= \int_0^2\left[x(8-x^2) + \frac{1}{2}(8-x^2)^2 - x^3 - \frac{1}{2}x^4\right]dx$$
$$= \int_0^2\left(32 + 8x - 8x^2 - 2x^3\right)dx = \frac{152}{3}.$$

7 The order $dydx$ will prove more tractable:

$$\int_0^{1/4}\int_0^{\sqrt{x}} y\cos(16\pi x^2)dydx = \int_0^{1/4}\left[\frac{y^2}{2}\cos(16\pi x^2)\right]_0^{\sqrt{x}} = \int_0^{1/4}\frac{x\cos(16\pi x^2)}{2}\,dx.$$

Now let $u = 16\pi x^2$ to obtain

$$\int_0^{1/4}\frac{x\cos(16\pi x^2)}{2}\,dx = \int_0^{\pi}\frac{\cos u}{x}\cdot\frac{1}{32\pi}\,du = \frac{1}{64\pi}\int_0^{\pi}\cos u\,du = \frac{1}{64\pi}[\sin u]_0^{\pi} = 0.$$

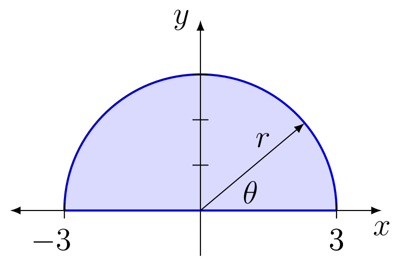
8 The sketch of R in the xy-plane is below. The region

$$S = \{(r,\theta) : 0 \le r \le 3 \text{ and } 0 \le \theta \le \pi\}$$

in the $r\theta$-plane is such that $T_{\text{pol}}(S) = R$, and therefore

$$\iint_R 2xy\,dA = \iint_S 2(r\cos\theta)(r\sin\theta)r\,dA = \int_0^{\pi}\int_0^3 2(r\cos\theta)(r\sin\theta)r\,drd\theta$$

$$= \int_0^\pi \int_0^3 2r^3 \cos\theta \sin\theta \, dr d\theta = \int_0^\pi \cos\theta \sin\theta \left[\frac{1}{2}r^4\right]_0^3 d\theta$$
$$= \frac{81}{2}\int_0^\pi \cos\theta \sin\theta \, d\theta = \frac{81}{4}\int_0^\pi \sin(2\theta)\, d\theta = 0.$$



9 By definition area is given by

$$\mathcal{A} = \int_0^\pi \int_0^{2\cos 3\theta} r \, dr d\theta = \int_0^\pi \left[\frac{1}{2}r^2\right]_0^{2\cos 3\theta} d\theta = 2\int_0^\pi \cos^2 3\theta \, d\theta$$
$$= \int_0^\pi \frac{1+\cos 6\theta}{2} d\theta = \int_0^\pi (1+\cos 6\theta) d\theta = \left[\theta + \frac{\sin 6\theta}{6}\right]_0^\pi = \pi,$$

where along the way we make use of the old trigonometric identity

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}.$$

Note a critical thing: the entire curve is traced out exactly once as θ ranges from 0 to π, so if you integrate with respect to θ from 0 to 2π you will get the area times 2!