Math 242 Exam #2 Key (Fall 2012)

1 For $p_0 = (1,1,0)$, $q_0 = (-2,8,4)$, $r_0 = (1,2,3)$, we have $\overrightarrow{p_0q_0} = \langle -3,7,4\rangle$ and $\overrightarrow{p_0r_0} = \langle 0,1,3\rangle$. Now,

$$\begin{aligned}\mathbf{n} = \overrightarrow{p_0q_0} \times \overrightarrow{p_0r_0} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 7 & 4 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 7 & 4 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 4 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 7 \\ 0 & 1 \end{vmatrix}\mathbf{k} \\ &= 17\mathbf{i} + 9\mathbf{j} - 3\mathbf{k} = \langle 17, 9, -3\rangle,\end{aligned}$$

so if $p = (x,y,z)$, then the equation of the plane is given by

$$\mathbf{n} \cdot \overrightarrow{p_0p} = \langle 17,9,-3\rangle \cdot \langle x-1, y-1, z\rangle = 0,$$

or $17x + 9y - 3z = 26$.

2 We have planes $P : x + 2y - 3z = 1$ and $Q : x + y + z = 2$. Now, the intersection of P and the plane $z = 0$ is the set of points on the line $\ell_0 : x + 2y = 1$, and the intersection of Q and $z = 0$ is the line $\ell_0' : x + y = 2$. So the point that is an element of $\ell_0 \cap \ell_0'$ must be a point in $P \cap Q$. We find this point by finding the solution to the system

$$\begin{cases} x + 2y = 1 \\ x + y = 2 \end{cases}$$

which is $(3,-1)$. Thus $(3,-1,0) \in P \cap Q$ (since we're on the plane $z = 0$).

Next, the intersection of P and the plane $z = 1$ is the line $\ell_1 : x + 2y = 4$, and the intersection of Q and $z = 1$ is the line $\ell_1' : x + y = 1$. Again, a point in $\ell_1 \cap \ell_1'$ is a point in $P \cap Q$. The system

$$\begin{cases} x + 2y = 4 \\ x + y = 1 \end{cases}$$

has solution $(-2,3)$, and thus $(-2,3,1) \in P \cap Q$ (recall we're now on the plane where z is 1).

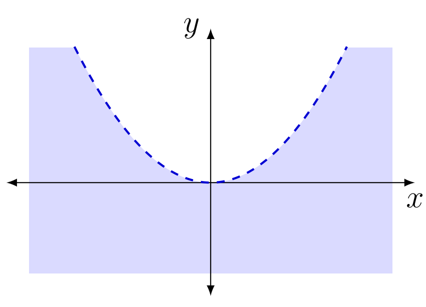
So the line of intersection for P and Q contains points $r_0(3,-1,0)$ and $r_1(-2,3,1)$. Let $\mathbf{v} = \overrightarrow{r_0r_1} = \langle -5,4,1\rangle$. An equation for the line is thus

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 3,-1,0\rangle + t\langle -5,4,1\rangle.$$

3 The function h is a composition of a polynomial function and the natural logarithm function, and so it is continuous on its domain. We have

$$\text{Dom}(h) = \{(x,y) : x^2 - 3y > 0\} = \left\{(x,y) : y < \tfrac{1}{3}x^2\right\},$$

which is the shaded region in $\mathbb{R}^2$ illustrated below.



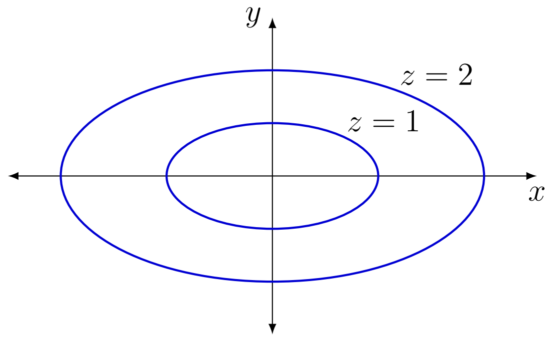
4 The level curve $z = 1$ has equation $1 = \sqrt{x^2 + 4y^2}$, which implies

$$x^2 + \frac{y^2}{1/4} = 1,$$

an ellipse. The level curve $z = 2$ has equation $2 = \sqrt{x^2 + 4y^2}$, which implies

$$\frac{x^2}{4} + y^2 = 1,$$

also an ellipse. Graph is below.



5 We have

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 4y^2}{x - 2y} = \lim_{(x,y)\to(2,1)} \frac{(x - 2y)(x + 2y)}{x - 2y} = \lim_{(x,y)\to(2,1)} (x + 2y) = 2 + 2(1) = 4.$$

6 First approach $(0, 0)$ on the path $(x(t), y(t)) = (t, 0)$ (i.e. the x-axis), so the limit becomes:

$$\lim_{t\to 0} \frac{x(t)y(t) + y^3(t)}{x^2(t) + y^2(t)} = \lim_{t\to 0} \frac{0}{t^2 + 0} = 0.$$

Next, approach $(0, 0)$ on the path $(x(t), y(t)) = (t, t)$ (i.e. the line $y = x$), so the limit becomes:

$$\lim_{t\to 0} \frac{x(t)y(t) + y^3(t)}{x^2(t) + y^2(t)} = \lim_{t\to 0} \frac{t^2 + t^3}{t^2 + t^2} = \lim_{t\to 0} \frac{t^2(1 + t)}{2t^2} = \lim_{t\to 0} \frac{1 + t}{2} = \frac{1}{2}.$$

The limits don't agree, so the original limit cannot exist by the Two-Path Test.

7a We have

$$g_x(x, y) = \ln(x^2 + y^2) + \frac{2x^2}{x^2 + y^2} \quad \text{and} \quad g_y(x, y) = \frac{2xy}{x^2 + y^2}.$$

7b We have

$$h_z(x, y, z) = -3\sin(x + 2y + 3z) \quad \text{and} \quad h_{zy}(x, y, z) = -6\cos(x + 2y + 3z).$$

8a Along the path $y = x$ the limit becomes

$$\lim_{(x,x)\to(0,0)} -\frac{x \cdot x}{x^2 + x^2} = \lim_{(x,x)\to(0,0)} -\frac{1}{2} = -\frac{1}{2},$$

which implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$$

and therefore f is not continuous at $(0,0)$.

8b By an established theorem, since f is not continuous at $(0,0)$ it cannot be differentiable at $(0,0)$.

8c By definition we have

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0, 0+h) - f(0,0)}{h} = \lim_{h\to 0}(0) = 0.$$

Thus, even though f is not differentiable at $(0,0)$, it can have partial derivatives at $(0,0)$.

9 Here $w(t) = f(x,y)$ with $f(x,y) = \cos(2x)\sin(3y)$, $x = x(t) = t/2$ and $y = y(t) = t^4$. By Chain Rule 1 in notes,

$$\begin{aligned} w'(t) = f_x(x,y)x'(t) + f_y(x,y)y'(t) &= -\sin(2x)\sin(3y) + 12t^3\cos(2x)\cos(3y) \\ &= -\sin(t)\sin(3t^4) + 12t^3\cos(t)\cos(3t^4). \end{aligned}$$

10 Here $z(s,t) = f(x,y)$ with $f(x,y) = xy - 2x + 3y$, $x = x(s,t) = \sin s$ and $y = y(s,t) = \tan t$. By Chain Rule 2 in notes,

$$z_s(s,t) = f_x(x,y)x_s(s,t) + f_y(x,y)y_s(s,t) = (y-2)\cos s + (x+3)(0) = (\tan t - 2)\cos s,$$

and

$$z_t(s,t) = f_x(x,y)x_t(s,t) + f_y(x,y)y_t(s,t) = (y-2)(0) + (x+3)\sec^2 t = (\sin s + 3)\sec^2 t.$$

11a $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle -9x^2, 2\rangle$

11b Direction of steepest ascent is

$$\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle -9, 2\rangle}{\sqrt{(-9)^2 + 2^2}} = \frac{1}{\sqrt{85}}\langle -9, 2\rangle,$$

and direction of steepest descent is

$$-\frac{1}{\sqrt{85}}\langle -9, 2\rangle.$$

11c Let C_0 be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$. Then for any t the tangent vector to C_0 at the point $(x(t), y(t))$, which is $\mathbf{r}'(t)$, must be in the direction of $-\nabla f(x,y) = \langle 9x^2(t), -2\rangle$. Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle 9x^2(t), -2\rangle,$$

from which we obtain the differential equations $x' = 9x^2$ and $y' = -2$. The first equation can be solved by the Method of Separation of Variables:

$$\frac{dx}{dt} = 9x^2 \quad\Rightarrow\quad \frac{dx}{9x^2} = dt \quad\Rightarrow\quad \int \frac{1}{9x^2}\,dx = \int dt \quad\Rightarrow\quad -\frac{1}{9x} = t + K \quad\Rightarrow\quad x(t) = -\frac{1}{9t+K},$$

with arbitrary constant K. The equation $y' = -1$ easily gives $y(t) = -2t + K'$ for arbitrary constant K'. Since C is given to start at $(1,2,3)$, we must have C_0 start at $(1,2)$; that is, $\mathbf{r}(0) = \langle x(0), y(0)\rangle = \langle 1, 2\rangle$. From $-1/(9\cdot 0 + K) = x(0) = 1$ we obtain $K = -1$, and from $-2(0) + K' = y(0) = 2$ we obtain $K' = 2$. Therefore an equation for C_0 is

$$\mathbf{r}(t) = \left\langle \frac{1}{1-9t},\ 2-2t \right\rangle, \quad t \geq 0.$$

12 First get the unit vector in the direction of $\langle 1, \sqrt{3}\rangle$:

$$\mathbf{u} = \frac{\langle 1, \sqrt{3}\rangle}{2} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle.$$

Now,

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y)\cdot \mathbf{u} = \langle e^x \sin y, e^x \cos y\rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{e^x \sin y}{2} + \frac{\sqrt{3}e^x \cos y}{2},$$

and so

$$D_{\mathbf{u}} f(0, \pi/4) = \frac{e^0 \sin(\pi/4)}{2} + \frac{\sqrt{3}e^0 \cos(\pi/4)}{2} = \frac{1/\sqrt{2}}{2} + \frac{\sqrt{3}\cdot 1/\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$