Math 242 Exam #1 Key (Fall 2012)

1 Two such vectors are

$$\frac{6\overrightarrow{qp}}{|\overrightarrow{qp}|} = \frac{6\langle -7, 8\rangle}{\sqrt{(-7)^2 + 8^2}} = \frac{6}{\sqrt{113}}\langle -7, 8\rangle$$

and

$$-\frac{6\overrightarrow{qp}}{|\overrightarrow{qp}|} = -\frac{6}{\sqrt{113}}\langle -7, 8\rangle$$

2 We have $\mathbf{F}_{PC} = \langle 200, 0\rangle$ and $\mathbf{F}_{GD} = \langle 0, -150\rangle$, and so the net force on the ATM is

$$\mathbf{F} = \mathbf{F}_{PC} + \mathbf{F}_{GD} = \langle 200, -150\rangle.$$

The magnitude of the force is

$$|\mathbf{F}| = \sqrt{200^2 + 150^2} = 250 \text{ N},$$

and the direction of the force is

$$\tan^{-1}(150/200) \approx 36.9^\circ \text{ south of east}$$

3 Canoe's velocity vector (relative to water) is $\mathbf{u} = \langle -8, 0\rangle$, while the water's velocity vector is

$$\mathbf{v} = \left\langle -3/\sqrt{2}, 3/\sqrt{2} \right\rangle.$$

The velocity of the canoe relative to the shore is thus

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = \left\langle -8 - 3/\sqrt{2}, 3/\sqrt{2} \right\rangle.$$

The speed is thus $|\mathbf{w}| = 10.34$ km/h, and the direction is

$$\theta = \tan^{-1}\left(\frac{3/\sqrt{2}}{8 + 3/\sqrt{2}}\right) = \tan^{-1}(0.2326) = 11.8^\circ$$

north of west.

4 Midpoint is at $(-2, 2, 5)$, which is the center of the sphere. Letting $|p - q|$ denote the distance between p and q, radius of the sphere is

$$r = \frac{1}{2}|p - q| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2} = \frac{1}{2}\sqrt{32} = 2\sqrt{2}.$$

Equation of the sphere is thus

$$(x + 2)^2 + (y - 2)^2 + (z - 5)^2 = 8.$$

5 First we have

$$\mathbf{u} + 3\mathbf{v} = \langle -1, 1, 0\rangle + \langle 6, -12, 3\rangle = \langle 5, -11, 3\rangle,$$

and so

$$|\mathbf{u} + 3\mathbf{v}| = \sqrt{5^2 + 11^2 + 3^2} = \sqrt{155}.$$

6 $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = -\dfrac{50}{107}\langle -9, 5, 1\rangle.$

7 The cross product of the two vectors will do:

$$\langle 6, -2, 4\rangle \times \langle 1, 2, 3\rangle = -14\mathbf{i} - 14\mathbf{j} + 14\mathbf{k} = -14\langle 1, 1, -1\rangle.$$

8 We have $y(t) = -t + 4$, so $y(t) = -2$ implies that $-t + 4 = -2$, or $t = 6$. So the line intersects the plane at the point

$$\mathbf{r}(6) = \big\langle 2(6) + 1, -6 + 4, 6 - 6\big\rangle = \langle 13, -2, 0\rangle.$$

9 Let $\mathbf{v} = \langle 3 - 1, -3 - 0, 3 - 1\rangle = \langle 2, -3, 2\rangle$ and $\mathbf{r}_0 = \langle 1, 0, 1\rangle$. Then an equation (i.e. parameterization) for the line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 1, 0, 1\rangle + \langle 2t, -3t, 2t\rangle,$$

or

$$\mathbf{r}(t) = \langle 1 + 2t, -3t, 1 + 2t\rangle, \quad -\infty < t < \infty.$$

10 From $\mathbf{r}'(t) = \langle 1, 0, -2/t^2\rangle$ we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1 + 4/t^4}}\left\langle 1, 0, -\frac{2}{t^2}\right\rangle.$$

Now we obtain

$$\mathbf{T}(2) = \frac{1}{\sqrt{1 + 4/2^4}}\left\langle 1, 0, -\frac{2}{2^2}\right\rangle = \left\langle \frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right\rangle$$

11 Let $\mathcal{I}$ denote the integral. By definition we obtain

$$\mathcal{I} = \left\langle \int_0^{\ln 2} e^{-t}\,dt,\ \int_0^{\ln 2} 2e^{2t}\,dt,\ \int_0^{\ln 2} -4e^t\,dt\right\rangle = \left\langle \big[-e^{-t}\big]_0^{\ln 2},\ \big[e^{2t}\big]_0^{\ln 2},\ \big[-4e^t\big]_0^{\ln 2}\right\rangle$$
$$= \left\langle \tfrac{1}{2}, 3, -4\right\rangle.$$

12 From $\mathbf{a}(t) = \langle 1, t\rangle$ we integrate to obtain $\mathbf{v}(t) = \langle t + a_1, t^2/2 + a_2\rangle$. Now,

$$\mathbf{v}(0) = \langle a_1, a_2\rangle = \langle 2, -1\rangle,$$

so we have $a_1 = 2$ and $a_2 = -1$, and obtain

$$\mathbf{v}(t) = \langle t + 2, t^2/2 - 1\rangle$$

for the velocity function. Integrating this function then yields

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t + b_1, \frac{1}{6}t^3 - t + b_2\right\rangle.$$

Now,

$$\mathbf{r}(0) = \langle b_1, b_2 \rangle = \langle 0, 8 \rangle,$$

so $b_1 = 0$ and $b_2 = 8$ and we obtain

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t, \frac{1}{6}t^3 - t + 8 \right\rangle.$$

13 Length of the curve is $\pi^2/8$. (This is in the book.)

14 The curvature is

$$\kappa(t) = \frac{4}{(1+16t^2)^{3/2}}.$$

(This also is in the book.)

15 The vector function $\mathbf{r}(t) = \langle t, e^t \rangle$, $t \in (-\infty, \infty)$, yields the same curve. We have

$$\mathbf{r}'(t) = \langle 1, e^t \rangle \quad \text{and} \quad |\mathbf{r}'(t)| = \sqrt{1+e^{2t}},$$

so that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+e^{2t}}}\langle 1, e^t \rangle = \left\langle \frac{1}{\sqrt{1+e^{2t}}}, \frac{e^t}{\sqrt{1+e^{2t}}} \right\rangle,$$

which yields

$$\mathbf{T}'(t) = \left\langle -\frac{e^{2t}}{(1+e^{2t})^{3/2}}, \frac{e^t}{(1+e^{2t})^{3/2}} \right\rangle = \frac{e^t}{(1+e^{2t})^{3/2}}\langle -e^t, 1 \rangle,$$

and thus

$$|\mathbf{T}'(t)| = \frac{e^t}{(1+e^{2t})^{3/2}}\sqrt{e^{2t}+1} = \frac{e^t}{1+e^{2t}}.$$

Finally we obtain

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{e^t}{1+e^{2t}} \cdot \frac{1}{\sqrt{1+e^{2t}}} = \frac{e^t}{(1+e^{2t})^{3/2}}$$

as the curvature of the curve at the point (t, e^t).

To find the value of t for which $\kappa(t)$ attains a maximum value, we first find $\kappa'(t)$:

$$\kappa'(t) = \frac{(1+e^{2t})^{3/2}e^t - e^t \cdot \frac{3}{2}(1+e^{2t})^{1/2} \cdot 2e^{2t}}{(1+e^{2t})^3} = \frac{e^t - 2e^{3t}}{(1+e^{2t})^{5/2}}.$$

Now we set $\kappa'(t) = 0$ to obtain

$$\frac{e^t - 2e^{3t}}{(1+e^{2t})^{5/2}} = 0,$$

and hence

$$e^t - 2e^{3t} = 0.$$

From this comes the equation $e^{2t} = 1/2$, which has solution

$$t = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{2}.$$

Thus the curve has maximum curvature at the point

$$\mathbf{r}\left(-\frac{\ln(2)}{2}\right) = \left\langle -\frac{\ln(2)}{2},\ e^{-\ln(2)/2} \right\rangle = \left\langle -\frac{\ln(2)}{2},\ \frac{1}{\sqrt{2}} \right\rangle.$$

The value of the maximum curvature is

$$\kappa\left(-\tfrac{1}{2}\ln(2)\right) = \frac{e^{-\ln(2)/2}}{\left[1 + e^{-\ln(2)}\right]^{3/2}} = \frac{1/\sqrt{2}}{(1+1/2)^{3/2}} = \frac{2\sqrt{3}}{9}.$$