Math 242 Exam #5 Key (Fall 2011)

1. Totally trivial: $\mathbf{F}(x,y) = \nabla\varphi(x,y) = \langle \varphi_x(x,y), \varphi_y(x,y)\rangle = \langle \cos x \sin y, \sin x \cos y\rangle$.

2. Get a direction vector for C that is consistent with its orientation: $\mathbf{v} = \langle 3,6,3\rangle - \langle 1,4,1\rangle = \langle 2,2,2\rangle$. Now, a parameterization for C is $\mathbf{r}(t) = \langle 1,4,1\rangle + t\mathbf{v} = \langle 1+2t, 4+2t, 1+2t\rangle$, for $t \in [0,1]$. Letting $f(x,y,z) = xy/z$ and noting that $|\mathbf{r}'(t)| = |\langle 2,2,2\rangle| = \sqrt{12}$, we obtain

$$\int_C \frac{xy}{z} = \int_C f = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt = \int_0^1 f(1+2t, 4+2t, 1+2t)\sqrt{12}\,dt$$
$$= 2\sqrt{3}\int_0^1 \frac{(1+2t)(4+2t)}{1+2t}\,dt = 2\sqrt{3}\int_0^1 (4+2t)\,dt = 2\sqrt{3}[4t+t^2]_0^1 = 10\sqrt{3}$$

3. The flux of $\mathbf{F}$ across C given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $t \in [a,b]$, is defined to be $\int_C \mathbf{F}\cdot\mathbf{n}$, and in the notes (and book), it is found that $\int_C \mathbf{F}\cdot\mathbf{n} = \int_a^b \big[f(\mathbf{r}(t))y'(t) - g(\mathbf{r}(t))x'(t)\big]\,dt$. Here we have $f(x,y) = y-x$, $g(x,y) = x$, $x(t) = 2\cos t$ and $y(t) = 2\sin t$. Thus, $\int_C \mathbf{F}\cdot\mathbf{n} = \int_0^{2\pi} [f(2\cos t, 2\sin t)(2\cos t) - g(2\cos t, 2\sin t)(-2\sin t)]\,dt = \int_0^{2\pi} [(2\sin t - 2\cos t)(2\cos t) - (2\cos t)(-2\sin t)]\,dt = 4\int_0^{2\pi} 2\cos t\sin t\,dt - 4\int_0^{2\pi}\cos^2 t\,dt = \int_0^{2\pi}\sin(2t)\,dt - 4\int_0^{2\pi}\frac{1+\cos(2t)}{2}\,dt = 4\left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} - 2\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} = 4\cdot 0 - 2\cdot 2\pi = -4\pi$.

4. For brevity let $\mathbf{x} = \langle x,y,z\rangle$. We have $\mathbf{F} = \langle f,g,h\rangle$ with $f(\mathbf{x}) = \frac{1}{y}$, $g(\mathbf{x}) = -\frac{x}{y^2}$, and $h(\mathbf{x}) = 2z-1$. Since $f_y(\mathbf{x}) = -\frac{1}{y^2} = g_x(\mathbf{x})$, $f_z(\mathbf{x}) = 0 = h_x(\mathbf{x})$, and $g_z(\mathbf{x}) = 0 = h_y(\mathbf{x})$, it follows that $\mathbf{F}$ is conservative.

Now we go about finding φ. From $\nabla\varphi = \mathbf{F}$ we have $\varphi_x(\mathbf{x}) = 1/y$, $\varphi_y(\mathbf{x}) = -x/y^2$, and $\varphi_z(\mathbf{x}) = 2z-1$. Hence $\varphi(\mathbf{x}) = \int \varphi_x(\mathbf{x})\,dx = x/y + c(y,z)$. Differentiating this with respect to y gives $\varphi_y(\mathbf{x}) = -x/y^2 + c_y(y,z)$, which, when compared to $\varphi_y(\mathbf{x}) = -x/y^2$, informs us that $c_y(y,z) = 0$ and therefore $c(y,z) = c(z)$ (that is, the function c must not be a function of y).

At this point we have $\varphi(\mathbf{x}) = x/y + c(z)$. This implies that $\varphi_z(\mathbf{x}) = c'(z)$, which, when compared to $\varphi_z(\mathbf{x}) = 2z-1$ above, gives $c'(z) = 2z-1$. So $c(z) = z^2 - z + c$, where c is an arbitrary constant. Choosing c be zero, we obtain $\varphi(\mathbf{x}) = x/y + z^2 - z$.

5. The hypotheses of the Fundamental Theorem for Line Integrals are satisfied, so $\int_C \nabla\varphi\cdot d\mathbf{r} = \varphi(\mathbf{r}(\pi)) - \varphi(\mathbf{r}(0)) = \varphi(0,-1,1) - \varphi(0,1,0) = (0-1+1) - (0+1+0) = -1$.

6. We have $\mathbf{F} = \langle f,g\rangle$ with $f(x,y) = 2xy$ and $g(x,y) = x^2 - y^2$, and since R is connected and simply connected, and ∂R is simple, closed and piecewise-smooth, by Green's Theorem

$$\oint_{\partial R} \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA = \iint_R (2x-2x)\,dA = 0.$$

7. We have $\mathbf{F} = \langle f, g\rangle$ with $f(x,y)=0$ and $g(x,y)=xy$, and since R is connected and simply connected, and ∂R is simple, closed and piecewise-smooth, by Green's Theorem

$$\oint_{\partial R} \mathbf{F}\cdot\mathbf{n} = \iint_R (f_x + g_y)\,dA = \iint_R x\,dA = \int_0^2\int_0^{-2x+4} x\,dydx = \int_0^2 [xy]_0^{-2x+4}\,dx$$
$$= \int_0^2 (-2x^2+4x)\,dx = \left[-\frac{2}{3}x^3 + 2x^2\right]_0^2 = \frac{8}{3}.$$

8. $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \langle D_x, D_y, D_z\rangle \cdot \langle yz\sin x, xz\cos y, xy\cos z\rangle = D_x(yz\sin x) + D_y(xz\cos y) + D_z(xy\cos z) = yz\cos x - xz\sin y - xy\sin z$

9. Totally trivial:

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ D_x & D_y & D_z\\ 0 & z^2-y^2 & yz\end{vmatrix} = \begin{vmatrix} D_y & D_z\\ z^2-y^2 & yz\end{vmatrix}\mathbf{i} - \begin{vmatrix} D_x & D_z\\ 0 & yz\end{vmatrix}\mathbf{j} + \begin{vmatrix} D_x & D_y\\ 0 & z^2-y^2\end{vmatrix}\mathbf{k}$$
$$= \left[D_y(yz) - D_z(z^2-y^2)\right]\mathbf{i} - \left[D_x(yz) - D_z(0)\right]\mathbf{j} + \left[D_x(z^2-y^2) - D_y(0)\right]\mathbf{k}$$
$$= -z\mathbf{i} = \langle -z, 0, 0\rangle.$$

10. First find $\operatorname{curl}\mathbf{F}$:

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ D_x & D_y & D_z\\ y & -2z & -x\end{vmatrix} = \begin{vmatrix} D_y & D_z\\ -2z & -x\end{vmatrix}\mathbf{i} - \begin{vmatrix} D_x & D_z\\ y & -x\end{vmatrix}\mathbf{j} + \begin{vmatrix} D_x & D_y\\ y & -2z\end{vmatrix}\mathbf{k}$$
$$= [D_y(-x) - D_z(-2z)]\,\mathbf{i} - [D_x(-x) - D_z(y)]\,\mathbf{j} + [D_x(-2z) - D_y(y)]\,\mathbf{k} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Now, $(\operatorname{curl}\mathbf{F})\cdot\mathbf{v} = 0 \quad\Rightarrow\quad \langle 2,1,-1\rangle\cdot\langle a,b,c\rangle = 2a+b-c = 0 \quad\Rightarrow\quad c = 2a+b$, so the set of vectors $\{\langle a, b, 2a+b\rangle : a,b\in\mathbb{R}\}$ will satisfy the condition.