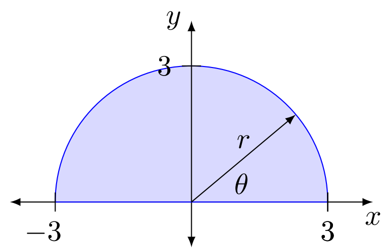
Math 242 Exam #4 Key (Fall 2011)

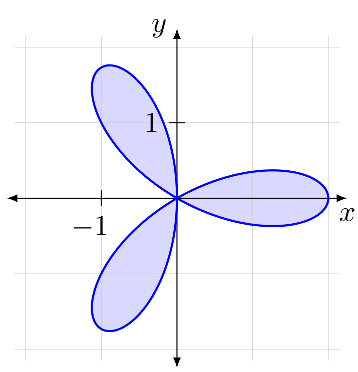
1. In the first quadrant $y = x^2$ and $y = 8 - x^2$ intersect at $(2, 4)$, so $\iint_R (x+y)dA = \int_0^2 \int_{x^2}^{8-x^2} (x+y)dydx =$ $\int_0^2 \left[xy + \frac{1}{2}y^2\right]_{x^2}^{8-x^2} dx = \int_0^2 \left[x(8-x^2) + \frac{1}{2}(8-x^2)^2 - x^3 - \frac{1}{2}x^4\right] dx = \int_0^2 \left(32 + 8x - 8x^2 - 2x^3\right) dx = \frac{152}{3}.$

2. $\int_0^{1/4} \int_0^{\sqrt{x}} y\cos(16\pi x^2)dydx = \int_0^{1/4} \left[\frac{y^2}{2}\cos(16\pi x^2)\right]_0^{\sqrt{x}} = \int_0^{1/4} \frac{x\cos(16\pi x^2)}{2} dx.$ Let $u = 16\pi x^2$ to get $\int_0^{\pi} \frac{\cos u}{x} \cdot \frac{1}{32\pi} du = \frac{1}{64\pi} \int_0^{\pi} \cos u \, du = \frac{1}{64\pi}[\sin u]_0^{\pi} = 0.$

3. Sketch is below. We see $R = \{(r, \theta) : 0 \le r \le 3 \text{ and } 0 \le \theta \le \pi\}$. Converting to polar coordinates gives: $\iint_R 2xy \, dA = \int_0^{\pi} \int_0^3 2(r\cos\theta)(r\sin\theta)r \, drd\theta = \int_0^{\pi} \int_0^3 2r^3 \cos\theta \sin\theta \, drd\theta = \int_0^{\pi} \cos\theta\sin\theta \left[\frac{1}{2}r^4\right]_0^3 d\theta =$ $\frac{81}{2} \int_0^{\pi} \cos\theta \sin\theta \, d\theta = \frac{81}{4} \int_0^{\pi} \sin(2\theta) \, d\theta = 0.$



4. By definition area is given by $A = \int_0^{\pi} \int_0^{2\cos 3\theta} r \, drd\theta = \int_0^{\pi} \left[\frac{1}{2}r^2\right]_0^{2\cos 3\theta} d\theta = 2\int_0^{\pi} \cos^2 3\theta \, d\theta = \int_0^{\pi} \frac{1 + \cos 6\theta}{2} d\theta$ $= \int_0^{\pi} (1 + \cos 6\theta) d\theta = \left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi} = \pi$, where along the way we make use of the old trigonometric identity $\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$. Note a critical thing: the entire curve is traced out exactly once as θ ranges from 0 to π, so if you integrate with respect to θ from 0 to 2π you will get the area times 2!



5. $\int_{-3}^{3} \int_{-2}^{2} \int_{-1}^{1} (xy + xz + yz) \, dxdydz = \int_{-3}^{3} \int_{-2}^{2} 2yz \, dydz = \int_{-3}^{3} (4z - 4z) \, dz = \int_{-3}^{3} (0) \, dz = 0$

6. $V = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}\int_{0}^{3-y} dzdydx = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}(3-y)\,dydx = \int_{-\sqrt{3}}^{\sqrt{3}}\left[3y - \frac{1}{2}y^2\right]_{x^2}^{3} dx = \int_{-\sqrt{3}}^{\sqrt{3}}\left(\frac{9}{2} - 3x^2 + \frac{1}{2}x^4\right) dx = \left[\frac{9}{2}x - x^3 + \frac{1}{10}x^5\right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{24\sqrt{3}}{5}$.

7. We have $\sqrt{x^2+y^2} \le z \le 4$, where $z = \sqrt{x^2+y^2}$ has the yz-trace $z = \sqrt{y^2} = |y|$ which should remind us that $z = \sqrt{x^2+y^2}$ must be a cone with tip at $(0,0,0)$ and opening upward along the positive z-axis. Let $\mathcal{D}$ be the region enclosed by the plane $z = 4$ above and the cone $z = \sqrt{x^2+y^2}$ below. The intersection of the cone with the plane is at points (x,y,z) where $\sqrt{x^2+y^2} = 4$, and so the projection of $\mathcal{D}$ onto the xy-plane is $\mathcal{R} = \{(x,y) : -\sqrt{16-x^2} \le y \le \sqrt{16-x^2}, -4 \le x \le 4\}$ — the closed disc centered at the origin with radius 4. Now notice that $\mathcal{R}$ is precisely the region the points (x,y) range over according to our limits of integration. Hence the region we're integrating over is $\mathcal{D}$. In cylindrical coordinates we have $\mathcal{D} = \{(r,\theta,z) : 0 \le r \le 4, 0 \le \theta \le 2\pi, r \le z \le 4\}$, and so

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4} dzdydx = \int_{0}^{2\pi}\int_{0}^{4}\int_{r}^{4} r\,dzdrd\theta = \frac{64}{3}\pi$$

8. $\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2\sin\varphi\,d\rho d\varphi d\theta = \int_0^{\pi}\int_0^{\pi/6}\sin\varphi\left[\frac{1}{3}\rho^3\right]_{2\sec\varphi}^{4} d\varphi d\theta = \int_0^{\pi}\int_0^{\pi/6}\left(\frac{64}{3}\sin\varphi - \frac{8}{3}\sin\varphi\sec^3\varphi\right) d\varphi d\theta$ $= \int_0^{\pi}\left(\frac{64}{3}\int_0^{\pi/6}\sin\varphi d\varphi - \frac{8}{3}\int_0^{\pi/6}\frac{\sin\varphi}{\cos^3\theta}d\varphi\right) d\theta$, and since $\frac{\sin\varphi}{\cos^3\varphi} = \frac{1}{2}(\sec^2\varphi)'$ (if this isn't clear then employ a substitution) we get $\int_0^{\pi}\left(-\frac{64}{3}\left[\cos\varphi\right]_0^{\pi/6} - \frac{8}{3}\int_0^{\pi/6}\frac{1}{2}\left(\sec^2\varphi\right)' d\varphi\right) d\theta = \int_0^{\pi}\left(\frac{64}{3} - \frac{32\sqrt{3}}{3} - \frac{4}{3}\left[\sec^2\varphi\right]_0^{\pi/6}\right) d\theta = \int_0^{\pi}\left(\frac{188 - 96\sqrt{3}}{9}\right) d\theta = \frac{188 - 96\sqrt{3}}{9}\pi$.

9. As shown in the Chapter 14 notes: $\mathcal{V}(D) = \iiint_D dV = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{4\cos\varphi} \rho^2\sin\varphi\,d\rho d\varphi d\theta$