# Math 242 Exam #3 Key (Fall 2011)

**1a.** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle -9x^2, 2\rangle$

**1b.** Direction of steepest ascent is $\dfrac{\nabla f(1,2)}{|\nabla f(1,2)|} = \dfrac{\langle -9,2\rangle}{\sqrt{(-9)^2+2^2}} = \dfrac{1}{\sqrt{85}}\langle -9,2\rangle$, and direction of steepest descent is $-\dfrac{1}{\sqrt{85}}\langle -9,2\rangle$.

**1c.** Let $C_0$ be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$. Then for any $t$ the tangent vector to $C_0$ at the point $(x(t), y(t))$, which is $\mathbf{r}'(t)$, must be in the direction of $-\nabla f(x,y) = \langle 9x^2(t), -2\rangle$. Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle 9x^2(t), -2\rangle,$$

from which we obtain the differential equations $x' = 9x^2$ and $y' = -2$. The first equation can be solved by the Method of Separation of Variables:

$$\frac{dx}{dt} = 9x^2 \;\Rightarrow\; \frac{1}{9x^2}\,dx = dt \;\Rightarrow\; \int \frac{1}{9x^2}\,dx = \int dt \;\Rightarrow\; -\frac{1}{9x} = t + K_1 \;\Rightarrow\; x(t) = -\frac{1}{9t+K_1},$$

with arbitrary constant $K_1$. The equation $y' = -1$ easily gives $y(t) = -2t + K_2$ for arbitrary constant $K_2$. Since $C$ is given to start at $(1,2,3)$, we must have $C_0$ start at $(1,2)$; that is, $\mathbf{r}(0) = \langle x(0), y(0)\rangle = \langle 1,2\rangle$. From $-1/(9\cdot 0 + K_1) = x(0) = 1$ we obtain $K_1 = -1$, and from $-2(0) + K_2 = y(0) = 2$ we obtain $K_2 = 2$. Therefore an equation for $C_0$ is

$$\mathbf{r}(t) = \left\langle \frac{1}{1-9t},\; 2-2t\right\rangle, \quad t \geq 0.$$

**2.** We have $\nabla f(x,y) = \left\langle \dfrac{2x}{4+x^2+y^2}, \dfrac{2y}{4+x^2+y^2}\right\rangle$, so $\nabla f(-1,2) = \left\langle -\dfrac{2}{9}, \dfrac{4}{9}\right\rangle$. The unit vector in the direction of $\langle 2,1\rangle$ is $\mathbf{u} = \left\langle \dfrac{2}{\sqrt5}, \dfrac{1}{\sqrt5}\right\rangle$. Now, $D_{\mathbf{u}}f(-1,2) = \nabla f(-1,2)\cdot\mathbf{u} = \left\langle -\dfrac{2}{9}, \dfrac{4}{9}\right\rangle \cdot \left\langle \dfrac{2}{\sqrt5}, \dfrac{1}{\sqrt5}\right\rangle = -\dfrac{4}{9\sqrt5} + \dfrac{4}{9\sqrt5} = 0.$

**3.** This is really just a Calculus 1 problem. From $f(x,y) = 12$ we get the equation $16 - \dfrac{x^2}{4} - \dfrac{y^2}{16} = 12$, or $4x^2 + y^2 = 64$. The equation (implicitly) defines $y$ as a function of $x$ in a neighborhood of $x = 2\sqrt3$, so we apply implicit differentiation with respect to $x$: $(4x^2+y^2)' = (64)' \;\Rightarrow\; 8x + 2y\cdot y' = 0 \;\Rightarrow\; y'(x) = -\dfrac{4x}{y}$. Now, at the point $(2\sqrt3, 4)$ the slope of the tangent line to the curve $f(x,y) = 12$ is $y'(2\sqrt3) = -\dfrac{4(2\sqrt3)}{4} = -2\sqrt3$.

**4a.** First, $f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, and $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$. Using $z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$ with $(x_0,y_0) = (3,-4)$, we get $z = \dfrac{3}{5}(x-3) - \dfrac{4}{5}(y+4) + 5$, which simplifies to $3x - 4y - 5z = 0$.

**4b.** The tangent plane serves as a linearization $L$ of function $f$ in a neighborhood of $(3,-4)$, so that $L(x,y) \approx f(x,y)$ for $(x,y)$ near $(3,-4)$. From (4a) we have $L(x,y) = \dfrac{3}{5}x - \dfrac{4}{5}y$, and so $f(3.06,-3.92) \approx L(3.06,-3.92) = 4.972$.

**5.** $f_x(x,y) = x^2+2y$, $f_y(x,y) = -y^2+2x$, $f_{xx}(x,y) = 2x$, $f_{yy}(x,y) = -2y$, $f_{xy}(x,y) = 2$, and $\Phi(x,y) = (2x)(-2y)-2^2 = -4xy-4$. Setting $f_x(x,y) = f_y(x,y) = 0$, we get the system of equations $x^2+2y = 0$ & $-y^2+2x = 0$. From the first equation we get $y = -x^2/2$, which when substituted into the second equation gives $-\frac{1}{4}x^4 + 2x = 0$. This becomes $x^4 - 8x = 0$ and then $x(x^3-8) = 0$, which has (real) solutions $x = 0, 2$. From this we obtain the critical points $(0,0)$, $(2,-2)$.

Now, $\Phi(0,0) = -4 < 0$ implies that $f$ has a saddle point at $(0,0)$; and $\Phi(2,-2) = 12 > 0$ and $f_{xx}(2,-2) = 4 > 0$ implies that $f$ has a local minimum at $(2,-2)$.

**6.** $f_x(x,y) = 4x$, $f_y(x,y) = 2y$, $f_{xx}(x,y) = 4$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = 0$, $\Phi(x,y) = 8$. Setting $f_x(x,y) = f_y(x,y) = 0$ yields the system $4x = 0$ & $2y = 0$, which gives $(0,0)$ as the only critical point. (Note that $(0,0) \in R$.) Since $f_{xx}(0,0) = 4 > 0$ and $\Phi(0,0) = 8 > 0$, $f$ has a local minimum at $(0,0)$.

Along the top side of $R$ we have $y = 2$, which yields the function $f_1(x) = 2x^2+8$ for $x \in [-1,1]$. Using the Closed Interval Method of Calculus 1 on $f_1$ in $[-1,1]$, we find the global maximum of $f_1$ occurs at $x = \pm 1$ (corresponding to points $(\pm 1, 2)$ for $f$), and the global minimum occurs at $x = 0$ (corresponding to $(0,2)$ for $f$).

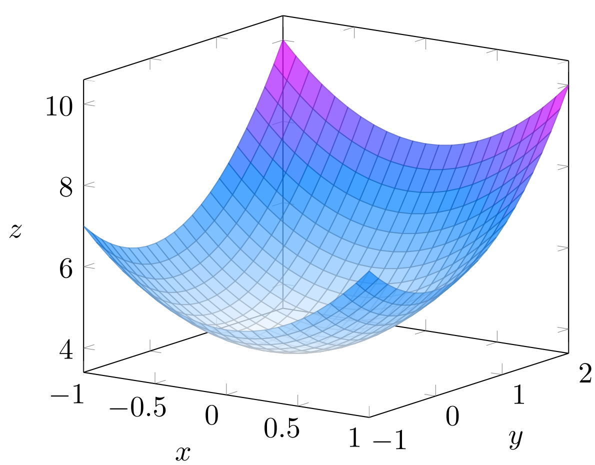
Along the bottom of $R$ we have $y = -1$, which yields the function $f_2(x) = 2x^2+5$ for $x \in [-1,1]$. We find the global maximum of $f_2$ occurs at $x = \pm 1$, and the global minimum occurs at $x = 0$.

Along the left side of $R$ we have $x = -1$, which yields the function $f_3(y) = y^2+6$ for $y \in [-1,2]$. The global maximum of $f_3$ occurs at $y = 2$, and the global minimum occurs at $y = 0$.

Along the right side of $R$ we have $x = 1$, which yields the function $f_4(y) = y^2+6$ for $y \in [-1,2]$. The global maximum of $f_4$ occurs at $y = 2$, and the global minimum occurs at $y = 0$.

Any point in $\mathbb{R}^2$ that corresponds to a point where any of the functions $f_i$ has an extremum is a point where $f$ itself has an extremum. Thus to find the global extrema of $f$ we evaluate $f$ at all these points as well as all critical points. We have: $f(\pm 1, 2) = 10$, $f(\pm 1, -1) = 7$, $f(\pm 1, 0) = 6$, $f(0,2) = 8$, $f(0,-1) = 5$, and $f(0,0) = 4$.

Therefore $f$ has a global minimum at $(0,0)$, and a global maximum at $(\pm 1, 2)$.




**7.** $\iint_R e^{x+2y}\,dA = \int_1^{\ln 3}\int_0^{\ln 2} e^{x+2y}\,dxdy = \int_1^{\ln 3} e^{2y}\left(\int_0^{\ln 2} e^x\,dx\right)dy = \int_1^{\ln 3} e^{2y}\,[e^x]_0^{\ln 2}\,dy = \int_1^{\ln 3} e^{2y}\,dy = \frac{1}{2}\left[e^{2y}\right]_1^{\ln 3} = \frac{1}{2}(9-e^2) = \frac{9-e^2}{2}$.

**8.** $\iint_R y^3 \sin(xy^2)\, dA = \int_0^{\sqrt{\pi/2}} \int_0^1 y^3 \sin(xy^2)\, dx dy = \int_0^{\sqrt{\pi/2}} \left[ -\frac{y^3}{y^2} \cos(xy^2) \right]_0^1 dy = \int_0^{\sqrt{\pi/2}} -y(\cos y^2 - 1)\, dy =$
$\int_0^{\sqrt{\pi/2}} y\, dy - \int_0^{\sqrt{\pi/2}} y \cos(y^2)\, dy = \frac{\pi}{4} - \frac{1}{2} \int_0^{\sqrt{\pi/2}} \left[\sin(y^2)\right]' dy = \frac{\pi}{4} - \frac{1}{2} \left[\sin(y^2)\right]_0^{\sqrt{\pi/2}} = \frac{\pi}{4} - \frac{1}{2}$