MATH 242 EXAM #2 KEY (FALL 2011)

1. For $p(1,1,0)$, $q(-2,1,1)$, $r(1,2,3)$, we have $\overrightarrow{pq} = \langle -3,0,1\rangle$ and $\overrightarrow{pr} = \langle 0,1,3\rangle$. Now,

$$\mathbf{n} = \overrightarrow{pq} \times \overrightarrow{pr} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = -\mathbf{i} + 9\mathbf{j} - 3\mathbf{k} = \langle -1, 9, -3\rangle,$$

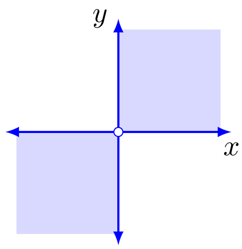
so the equation of the plane is given by $\mathbf{n} \cdot \langle x-1, y-1, z\rangle = 0$, or $x - 9y + 3z = -8$.

2. We have planes $P : x + 2y - 3z = 1$ and $Q : x + y + z = 2$. Now, the intersection of P and the plane $z = 0$ is the set of points on the line $\ell_0 : x + 2y = 1$, and the intersection of Q and $z = 0$ is the line $\ell_0' : x + y = 2$. So the point that is an element of $\ell_0 \cap \ell_0'$ must be a point in $P \cap Q$. We find this point by finding the solution to the system $x + 2y = 1$, $x + y = 2$, which is $(3, -1)$. Thus $(3, -1, 0) \in P \cap Q$ (since we're on the plane $z = 0$).

Next, the intersection of P and the plane $z = 1$ is the line $\ell_1 : x + 2y = 4$, and the intersection of Q and $z = 1$ is the line $\ell_1' : x + y = 1$. Again, a point in $\ell_1 \cap \ell_1'$ is a point in $P \cap Q$. The system $x + 2y = 4$, $x + y = 1$ has solution $(-2, 3)$, and thus $(-2, 3, 1) \in P \cap Q$ (recall we're now on the plane where z is 1).

So the line of intersection for P and Q contains points $r_0(3, -1, 0)$ and $r_1(-2, 3, 1)$. Let $\mathbf{v} = \overrightarrow{r_0 r_1} = \langle -5, 4, 1\rangle$. An equation for the line is thus $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 3, -1, 0\rangle + t\langle -5, 4, 1\rangle$.

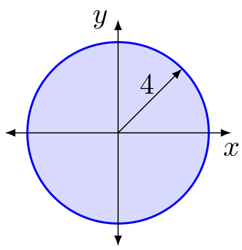
3.



We must have $(x, y) \neq (0, 0)$ to avoid division by zero, and also (x, y) must be such that $xy \geq 0$ to avoid square roots of negatives. The domain of f must therefore consist of Quadrants I and III, and all points on the coordinate axes *except* for the origin:

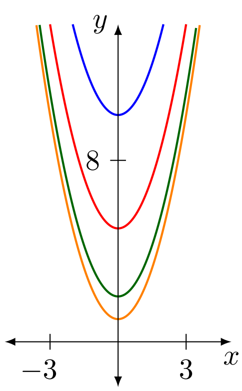
$$\text{Dom}(f) = \{(x, y) : x, y \geq 0 \ \text{ or } \ x, y < 0\} - \{(0, 0)\}$$

4.



We need $16 - x^2 - y^2 \geq 0$, so $\text{Dom}(\varphi) = \{(x, y) : x^2 + y^2 \leq 16\}$ (pictured to the left), and $\text{Ran}(f) = \{z : 0 \leq z \leq 4\} = [0, 4]$.

5.



The level curve $z = 0$ is in orange, $z = 1$ in green, $z = 2$ in red, and $z = 3$ in blue.

6. $\lim_{(x,y)\to(1,2)} \frac{\sqrt{y}-\sqrt{x+1}}{y-x-1} \cdot \frac{\sqrt{y}+\sqrt{x+1}}{\sqrt{y}+\sqrt{x+1}} = \lim_{(x,y)\to(1,2)} \frac{y-x-1}{(y-x-1)\left(\sqrt{y}+\sqrt{x+1}\right)} = \lim_{(x,y)\to(1,2)} \frac{1}{\sqrt{y}+\sqrt{x+1}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$

7. First approach $(0,0)$ on the path $(x(t),y(t)) = (t,0)$ (i.e. the x-axis), so limit becomes: $\lim_{t\to 0} \frac{y(t)}{\sqrt{x^2(t)+y^2(t)}} = \lim_{t\to 0} \frac{0}{\sqrt{t^2+0}} = 0$. Next, approach $(0,0)$ on the path $(x(t),y(t)) = (t,t/2)$ for $t>0$ (i.e. the part of the line $y=x/2$ in Quadrant I), so limit becomes: $\lim_{t\to 0^+} \frac{t/2}{\sqrt{t^2-(t/2)^2}} = \lim_{t\to 0^+} \frac{t/2}{(t/2)\sqrt{3}} = \frac{1}{\sqrt{3}}$. The limits don't agree, so the original limit cannot exist.

8. $\lim_{(x,y)\to(0,1)} \frac{2y\sin x}{x(y+6)} = \lim_{y\to 1} \frac{2y}{y+6} \cdot \lim_{x\to 0} \frac{\sin x}{x} = \frac{2}{7}\cdot 1 = \frac{2}{7}$. (The proposition in section 13.3 of my notes can be used to fully justify this.)

9a. $g_x(x,y) = \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2}$ and $g_y(x,y) = \frac{2xy}{x^2+y^2}$

9b. $h_x(x,y,z) = -\sin(x+2y+3z)$, $h_y(x,y,z) = -2\sin(x+2y+3z)$, $h_z(x,y,z) = -3\sin(x+2y+3z)$.

10a. Along the path $y=x$ the limit becomes $\lim_{(x,x)\to(0,0)} -\frac{x\cdot x}{x^2+x^2} = \lim_{(x,x)\to(0,0)} -\frac{1}{2} = -\frac{1}{2}$, which implies that $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$ and therefore f is not continuous at $(0,0)$.

10b. By an established theorem in the textbook (and notes), since f is not continuous at $(0,0)$ it cannot be differentiable at $(0,0)$.

10c. $f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}(0) = 0$. Thus, even though f is not differentiable at $(0,0)$, it can have partial derivatives at $(0,0)$.

11. Here $w(t) = f(x,y)$ with $f(x,y) = \cos(2x)\sin(3y)$, $x = x(t) = t/2$ and $y = y(t) = t^4$. By Chain Rule 1 in notes,

$$\begin{aligned} w'(t) &= f_x(x,y)x'(t) + f_y(x,y)y'(t) = -\sin(2x)\sin(3y) + 12t^3\cos(2x)\cos(3y) \\ &= -\sin(t)\sin(3t^4) + 12t^3\cos(t)\cos(3t^4). \end{aligned}$$

12. Here $z(s,t) = f(x,y)$ with $f(x,y) = xy - 2x + 3y$, $x = x(s,t) = \sin s$ and $y = y(s,t) = \tan t$. By Chain Rule 2 in notes,

$$z_s(s,t) = f_x(x,y)x_s(s,t) + f_y(x,y)y_s(s,t) = (y-2)\cos s + (x+3)(0) = (\tan t - 2)\cos s,$$

and

$$z_t(s,t) = f_x(x,y)x_t(s,t) + f_y(x,y)y_t(s,t) = (y-2)(0) + (x+3)\sec^2 t = (\sin s + 3)\sec^2 t$$