Math 242 Exam #1 Key (Fall 2011)

1. $\overrightarrow{pq} = \langle -2-3, -7-(-5)\rangle = \langle -5,-2\rangle = -5\mathbf{i} - 2\mathbf{j}$.

2. Let $\mathbf{F}_h$ and $\mathbf{F}_v$ denote the horizontal and vertical components of $\mathbf{F}$, respectively. We have $|\mathbf{F}_h| = 200\cos(22°) = 185.44$ N, and $|\mathbf{F}_v| = 200\sin(22°) = 74.92$ N; and so $\mathbf{F}_h = \langle 185.4, 0\rangle$ and $\mathbf{F}_v = \langle 0, 74.9\rangle$.

3. Canoe's velocity vector (relative to water) is $\mathbf{u} = \langle -7, 0\rangle$, while the water's velocity vector is $\mathbf{v} = \langle -3/\sqrt{2}, 3/\sqrt{2}\rangle$. The velocity of the canoe relative to the shore is thus $\mathbf{w} = \mathbf{u}+\mathbf{v} = \langle -7-3/\sqrt{2}, 3/\sqrt{2}\rangle$. The speed is thus $|\mathbf{w}| = 9.36$ km/h, and the direction is $\theta = \tan^{-1}\left(\dfrac{3/\sqrt{2}}{7+3/\sqrt{2}}\right) = \tan^{-1}(0.2326) = 13.1°$ north of west.

4. Midpoint is at $(-2,2,5)$, are is the center of the sphere. Letting $|p-q|$ denote the distance between p and q, radius of the sphere is $r = \frac{1}{2}|p-q| = \frac{1}{2}\sqrt{4^2+0^2+4^2} = \frac{1}{2}\sqrt{32} = 2\sqrt{2}$. Equation of the sphere is thus

$$(x+2)^2 + (y-2)^2 + (z-5)^2 = 8.$$

5a. $|\mathbf{v}| = \sqrt{5^2+(-2)^2+4^2} = \sqrt{45} = 3\sqrt{5}$.

5b. $\mathbf{v}/|\mathbf{v}| = \langle 5,-2,4\rangle/(3\sqrt{5}) = \left\langle \dfrac{5}{3\sqrt{5}}, -\dfrac{2}{3\sqrt{5}}, \dfrac{4}{3\sqrt{5}}\right\rangle$ and $-\mathbf{v}/|\mathbf{v}| = \left\langle -\dfrac{5}{3\sqrt{5}}, \dfrac{2}{3\sqrt{5}}, -\dfrac{4}{3\sqrt{5}}\right\rangle$.

6. $\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \dfrac{20+30}{4+36}\langle 2,6\rangle = \frac{5}{4}\langle 2,6\rangle = \left\langle \dfrac{5}{2}, \dfrac{15}{2}\right\rangle$.

7. Let $\mathbf{p} = \operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{7}{3}\langle 1,1,1\rangle = \langle \frac{7}{3}, \frac{7}{3}, \frac{7}{3}\rangle$, and let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle 4,3,0\rangle - \langle \frac{7}{3}, \frac{7}{3}, \frac{7}{3}\rangle = \langle \frac{5}{3}, \frac{2}{3}, -\frac{7}{3}\rangle$. Note that $\mathbf{n}\cdot\mathbf{v} = 0$ (which shows $\mathbf{n}\perp\mathbf{v}$), and $\mathbf{p}+\mathbf{n} = \mathbf{v}$.

8. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k} = -2\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} = \langle -2,-4,-4\rangle$.

9. Let $\mathbf{v} = \langle 3-1, -3-0, 3-1\rangle = \langle 2,-3,2\rangle$ and $\mathbf{r}_0 = \langle 1,0,1\rangle$. Then an equation (i.e. parameterization) for the line is $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 1,0,1\rangle + \langle 2t,-3t,2t\rangle$, or $\mathbf{r}(t) = \langle 1+2t, -3t, 1+2t\rangle$, $-\infty < t < \infty$.

10. $\operatorname{Dom}(\mathbf{r}) = \{t \,:\, -3 \le t \le 3 \text{ and } t \ge 0 \text{ and } t \ne 1\} = \{t \,:\, t\in[0,3] \text{ and } t\ne 1\} = [0,1)\cup(1,3]$.

11. $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \cos t, -\sin t, \dfrac{1}{2\sqrt{t}}\right\rangle$ and $|\mathbf{v}(t)| = \sqrt{\cos^2 t + \sin^2 t + \dfrac{1}{4t}} = \sqrt{1+\dfrac{1}{4t}}$, and so $\mathbf{T}(9) = \dfrac{\mathbf{v}(9)}{|\mathbf{v}(9)|} = \dfrac{6}{\sqrt{37}}\left\langle \cos 9, -\sin 9, \dfrac{1}{6}\right\rangle \approx \langle -0.899, -0.407, 0.164\rangle$.

12. $\mathbf{r}(r) = \int \mathbf{r}'(t)\,dt = \left\langle \frac{2}{3}t^{3/2} + C_1, \frac{1}{\pi}\sin(\pi t) + C_2, 4\ln|t| + C_3 \right\rangle$. Now,

$$\mathbf{r}(1) = \langle 2,3,4\rangle = \left\langle \frac{2}{3}1^{3/2} + C_1, \frac{1}{\pi}\sin(\pi) + C_2, 4\ln|1| + C_3 \right\rangle = \langle 2/3 + C_1, C_2, C_3\rangle,$$

so $2/3 + C_1 = 2$, $C_2 = 3$, and $C_3 = 4$, giving $\mathbf{r}(t) = \left\langle \frac{2}{3}t^{3/2} + \frac{4}{3}, \frac{1}{\pi}\sin(\pi t) + 3, 4\ln|t| + 4 \right\rangle$.

13. $\mathcal{L}(C) = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 |\langle t, \sqrt{2t+1}\rangle|\,dt = \int_0^2 \sqrt{t^2+2t+1}\,dt = \int_0^2 \sqrt{(t+1)^2}\,dt = \int_0^2 (t+1)\,dt = \left[\frac{1}{2}t^2 + t\right]_0^2 = 4$. A happy ending.

14. $\mathbf{v}(t) = \langle 2, 4\cos t, -4\sin t\rangle \Rightarrow |\mathbf{v}(t)| = \sqrt{4 + 16\cos^2 t + 16\sin^2 t} = \sqrt{20} = 2\sqrt{5}$, so $\mathbf{T}(t) = \mathbf{v}(t)/|\mathbf{v}(t)| = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}}\sin t \right\rangle \Rightarrow \mathbf{T}'(t) = \left\langle 0, -\frac{2}{\sqrt{5}}\sin t, -\frac{2}{\sqrt{5}}\cos t \right\rangle \Rightarrow |\mathbf{T}'(t)| = \frac{2}{\sqrt{5}}$. Finally we obtain

$$\kappa = |\mathbf{T}'(t)|/|\mathbf{v}(t)| = \frac{1}{5}.$$