Math 242 Exam #4 Key (Fall 2010)

1. $\displaystyle\int_{-3}^{4}\int_{-2}^{2}\int_{-1}^{1}(xy+xz+yz)\,dxdydz=\int_{-3}^{4}\int_{-2}^{2}2yz\,dydz=\int_{-3}^{4}(4z-4z)dz=0$

2. The hyperboloid intersects the sphere at points (x,y,z) where $1+x^2+y^2=19-x^2-y^2$, or $x^2+y^2=9$. Thus the region $\mathcal{D}$ enclosed by the two surfaces projects onto the xy-plane as the region $\mathcal{R}=\{(x,y):x^2+y^2\le 9\}$, which is a circle centered at the origin with radius 3. The hyperboloid, moreover, is the "lower" boundary of $\mathcal{D}$ while the sphere is the "upper" boundary. We thus have volume V given by

$$V=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}}dzdydx=\int_0^{2\pi}\int_0^3\int_{\sqrt{1+r^2}}^{\sqrt{19-r^2}}r\,dzdrd\theta=\int_0^{2\pi}\int_0^3\left(\sqrt{19-r^2}-\sqrt{1+r^2}\right)r\,drd\theta$$
$$=-\frac{1}{3}\int_0^{2\pi}\left(10^{3/2}-19^{3/2}+10^{3/2}-1\right)d\theta=\frac{2\pi}{3}\left(1+19\sqrt{19}-20\sqrt{10}\right)$$

3. $\displaystyle\int_1^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln(2y)}e^{x+y^2-z}dxdydz=\int_1^{\ln 8}\int_1^{\sqrt{z}}ye^{y^2-z}\,dydz=\frac{1}{2}\int_1^{\ln 8}\left(1-e^{1-z}\right)dz=\frac{1}{2}\ln 8+\frac{e}{16}-1$

4. We have $\sqrt{x^2+y^2}\le z\le 4$, where $z=\sqrt{x^2+y^2}$ has the yz-trace $z=\sqrt{y^2}=|y|$ which should remind us that $z=\sqrt{x^2+y^2}$ must be a cone with tip at $(0,0,0)$ and opening upward along the positive z-axis. Let $\mathcal{D}$ be the region enclosed by the plane $z=4$ above and the cone $z=\sqrt{x^2+y^2}$ below. The intersection of the cone with the plane is at points (x,y,z) where $\sqrt{x^2+y^2}=4$, and so the projection of $\mathcal{D}$ onto the xy-plane is $\mathcal{R}=\{(x,y):-\sqrt{16-x^2}\le y\le\sqrt{16-x^2},-4\le x\le 4\}$ — the closed disc centered at the origin with radius 4. Now notice that $\mathcal{R}$ is precisely the region the points (x,y) range over according to our limits of integration. Hence the region we're integrating over is $\mathcal{D}$. In cylindrical coordinates we have $\mathcal{D}=\{(r,\theta,z):0\le r\le 4,0\le\theta\le 2\pi,r\le z\le 4\}$, and so

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4}dzdydx=\int_0^{2\pi}\int_0^4\int_r^4 r\,dzdrd\theta=\frac{64}{3}\pi$$

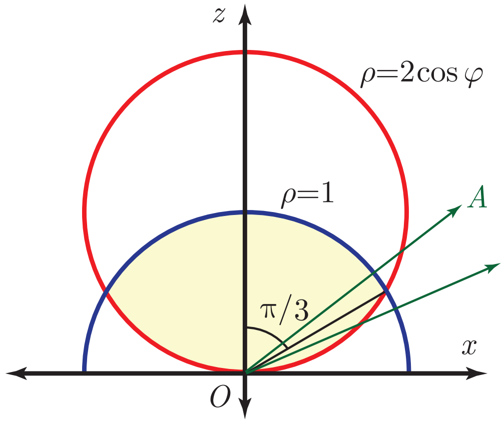
5. $\displaystyle m=\int_0^{2\pi}\int_0^6\int_0^{6-r}(7-z)r\,dzdrd\theta=\int_0^{2\pi}\int_0^6(24r-r^2-\frac{1}{2}r^3)drd\theta=\int_0^{2\pi}198\,d\theta=396\pi$

6. First find where the two surfaces intersect, which will be where $2\cos\varphi=1$. Solving gives $\varphi=\pi/3$. So, as is evident in the figure below which shows a "slice" of the situation on the xz-plane, for $0\le\varphi\le\pi/3$ we find that $0\le\rho\le 1$ (see ray $\overrightarrow{OA}$ in the figure), but for $\pi/3<\rho\le\pi/2$ we have $0\le\rho\le 2\cos\varphi$ (see ray $\overrightarrow{OB}$). In any event we always have $0\le\theta\le 2\pi$. To find the volume of our enclosed region we find the volumes of two subregions: volume V_1 for the subregion with $0\le\varphi\le\pi/3$, and volume V_2 for the subregion with $\pi/3<\varphi\le\pi/2$...

$$V_1=\int_0^{2\pi}\int_0^{\pi/3}\int_0^1\rho^2\sin\varphi\,d\rho d\varphi d\theta=\int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin\varphi\,d\varphi d\theta=\frac{\pi}{3}$$

$$V_2=\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,d\rho d\varphi d\theta=\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\frac{8}{3}\sin\varphi\cos^3\varphi\,d\varphi d\theta=\int_0^{2\pi}\frac{8}{192}\,d\theta=\frac{\pi}{12}$$

Hence the total volume of the enclosed region is $\pi/3+\pi/12=5\pi/12$.



7. Mass is $m = \iint_R \rho(x,y)dA = \int_0^2 \int_0^4 \left(1 + \frac{x}{2}\right) dxdy = \int_0^2 8\,dy = 16$. Coordinates for the center mass are $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{1}{m} \iint_R x\rho(x,y)dA = \frac{1}{16} \int_0^2 \int_0^4 \left(x + \frac{x^2}{2}\right) dxdy = \frac{1}{16} \int_0^2 \frac{56}{3} dy = \frac{7}{3}$ and $\bar{y} = \frac{1}{m} \iint_R y\rho(x,y)dA = \frac{1}{16} \int_0^2 \int_0^4 \left(y + \frac{xy}{2}\right) dxdy = \frac{1}{16} \int_0^2 8y\,dy = 1$. Thus center of mass is located at $\left(\frac{7}{3}, 1\right)$.

8. $\varphi(x,y) = x + y^2$, so $\mathbf{F}(x,y) = \nabla\varphi(x,y) = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} \right\rangle = \langle 1, 2y \rangle$

9. An equipotential curve for φ has the general form $\varphi(x,y) = C$, or $x + y^2 = C$. The given curve contains $(1,1)$, which means C must be such that $C = 1 + 1^2 = 2$ and therefore $x + y^2 = 2$ is the equation for the curve. Solving for x gives $x = 2 - y^2$, so the curve can be parameterized by $\mathbf{r}(t) = \langle 2 - t^2, t \rangle$, $-\infty < t < \infty$. If the point (x,y) lies on the curve, then at this point we have $t = y$ and so $\mathbf{r}'(y) = \langle -2y, 1 \rangle$ is the tangent vector to the curve. Now, at this point, we have

$$\mathbf{F}(x,y) \cdot \langle -2y, 1 \rangle = \langle 1, 2y \rangle \cdot \langle -2y, 1 \rangle = 2y - 2y = 0,$$

which shows that the vector field $\mathbf{F}$ is orthogonal to the curve at this point. Since the point (x,y) is arbitrary, we've shown that $\mathbf{F}$ is orthogonal to *all* points of the curve.