Math 242 Exam #3 Key (Fall 2010)

1. The direction needs to be a unit vector: $\hat{\mathbf{u}} = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$. Also $f_x(x,y) = -\frac{y}{(x-y)^2}$ and $f_y(x,y) = \frac{x}{(x-y)^2}$. So $D_{\hat{\mathbf{u}}}f(4,1) = \nabla f(4,1)\cdot\hat{\mathbf{u}} = \langle f_x(4,1), f_y(4,1)\rangle \cdot \langle u_1, u_2\rangle = \left\langle -\frac{1}{9}, \frac{4}{9}\right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle = \frac{1}{\sqrt{5}}$.

2a. $\nabla f(x,y) = \langle 12x+4y, 4x-6y\rangle$, so direction of steepest ascent is $\nabla f(6,-1) = \langle 68, 30\rangle$, which as a unit vector is $\frac{1}{\sqrt{5524}}\langle 68,30\rangle = \frac{1}{\sqrt{1381}}\langle 34,15\rangle$. Direction of steepest descent is in the opposite direction: $-\frac{1}{\sqrt{1381}}\langle 34,15\rangle$.

2b. A direction of no change is orthogonal to a direction of maximum change, so one possibility is $\langle -30, 68\rangle$.

3. Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ be the path C. At each point $(x(t), y(t))$ on C the tangent vector $\mathbf{r}'(t)$ must point in the direction of steepest descent, which is $-\nabla f(x,y)$. Hence we can define $\mathbf{r}(t)$ so that $\mathbf{r}'(t) = -\nabla f(x,y)$, which gives $\langle x'(t), y'(t)\rangle = -\langle -2x, -4y\rangle = \langle 2x, 4y\rangle$. This tells us that $\frac{dy}{dx} = \frac{2y}{x}$, whence $\int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$. Solving this leads to $\ln|y| = 2\ln|x| + C$ and then $y = Kx^2$. Since $(1,1)$ is a point that lies on C (i.e. $y=1$ when $x=1$), we obtain $K=1$ and so $y=x^2$ is the equation for C. (Equivalently we can define C by letting $x(t) = t$ and $y(t) = [x(t)]^2 = t^2$, then $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle t, t^2\rangle$).

4. Here $F(x,y,z) = z - x^2e^{x-y}$, and by definition the plane tangent to the surface $F(x,y,z) = 0$ at $(2,2,4)$ has equation $\nabla F(2,2,4)\cdot\langle x-2, y-2, z-4\rangle = 0$, which gives $\langle -8,4,1\rangle\cdot\langle x-2,y-2,z-4\rangle = 0$, or $8x-4y-z=4$.

5. $f_x(x,y) = 4x^3-4y$ and $f_y(x,y) = 4y-4x$, so $f_x(x,y) = f_y(x,y) = 0$ only at the points $(0,0)$, $(1,1)$, and $(-1,-1)$. Since $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 4$, and $f(x,y)_{xy}(x,y) = -4$, we find that: $D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = -16 < 0$ (so f has a saddle point at $(0,0)$), $D(1,1) = 12\cdot 4 - (-4)^2 = 32 > 0$ with $f_{xx}(1,1) = 12 > 0$ (so f has a local minimum value at $(1,1)$), and $D(-1,-1) = 32 > 0$ with $f_{xx}(-1,-1) = 12 > 0$ (so f has a local minimum value at $(-1,-1)$).

6. $\displaystyle\int_0^{\pi/2} [\sin xy]_0^1\,dx = \int_0^{\pi/2}\sin x\,dx = [-\cos x]_0^{\pi/2} = 1.$

7. $\displaystyle\int_{-1}^1\int_1^2 (x^2+xy)dxdy = \int_{-1}^1 \left[\frac{x^3}{3} + \frac{x^2y}{2}\right]_1^2 dy = \int_{-1}^1\left(\frac{7}{3}+\frac{3}{2}y\right)dy = \left[\frac{7}{3}y + \frac{3}{4}y^2\right]_{-1}^1 = \frac{14}{3}.$

8. $\displaystyle\int_{-1}^1\int_{-x-1}^{2x+2} y^2\,dydx = \int_{-1}^1\left[\frac{y^3}{3}\right]_{-x-1}^{2x+2}dx = \frac{1}{3}\int_{-1}^1(9x^3+27x^2+27x+9)dx = 3\int_{-1}^1(x^3+3x^2+3x+1)dx =$
$\displaystyle 3\left[\frac{1}{4}x^4 + x^3 + 32x^2 + x\right]_{-1}^1 = 12.$

9. $\int_0^{1/4}\int_0^{\sqrt{x}} y\cos(16\pi x^2)dydx = \int_0^{1/4}\left[\frac{y^2}{2}\cos(16\pi x^2)\right]_0^{\sqrt{x}} = \int_0^{1/4}\frac{x\cos(16\pi x^2)}{2}\,dx$. Let $u = 16\pi x^2$ to get $\int_0^{\pi}\frac{\cos u}{x}\cdot\frac{1}{32\pi}\,du = \frac{1}{64\pi}\int_0^{\pi}\cos u\,du = \frac{1}{64\pi}[\sin u]_0^{\pi} = 0$.

10. Area is given by $A = \int_0^{\pi}\int_0^{2\cos 3\theta} r\,drd\theta = \int_0^{\pi}\left[\frac{1}{2}r^2\right]_0^{2\cos 3\theta} d\theta = 2\int_0^{\pi}\cos^2 3\theta\,d\theta = \int_0^{\pi}\frac{1+\cos 6\theta}{2}\,d\theta = \int_0^{\pi}(1+\cos 6\theta)d\theta = \left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi} = \pi$, where along the way we make use of the old trigonometric identity $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$. Note a critical thing: the entire curve is traced out exactly once as θ ranges from 0 to π, so if you integrate with respect to θ from 0 to 2π you will get the area times 2!