Math 242 Exam #2 Key (Fall 2010)

1. $\mathbf{T} = \dfrac{\langle 2, 4\cos t, -4\sin t\rangle}{\sqrt{4+16\cos^2 t+16\sin^2 t}} = \dfrac{\langle 2, 4\cos t, -4\sin t\rangle}{\sqrt{20}} = \dfrac{1}{\sqrt{5}}\langle 1, 2\cos t, -2\sin t\rangle$; $\kappa = \dfrac{1}{|\mathbf{r}'(t)|}\cdot\left|\dfrac{d\mathbf{T}}{dt}\right| = 0.1\,|\langle 0, -2\sin t, -2\cos t\rangle| = 0.2$; and $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \langle 0, -\sin t, -\cos t\rangle$.

2. We have $p = (1,2,-3)$ and $\mathbf{n} = \langle -2, 5, -1\rangle$. The plane P is the set of all points $q = (x,y,z)$ such that $\overrightarrow{pq}\cdot\mathbf{n} = 0$, which gives $\langle x-1, y-2, z+3\rangle\cdot\langle -2,5,-1\rangle = 0$, or $-2x+5y-z = 11$.

3. To find an equation for our line L it suffices to find two points that lie on it. Setting $z=0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane: $x+2y=1$ & $x+y=1$; this system has solution $x=1, y=0$, so $(1,0,0)$ is a point lying on both planes. Setting $z=1$ gives equations of lines in which the planes intersect the $z=1$ plane: $x+2y-1=1$ & $x+y+1=1$; this system has solution $x=-2, y=2$, so $(-2,2,1)$ is a point lying on both planes. So, $p_0=(1,0,0)$ and $p_1=(-2,2,1)$ lie on the line of intersection L for the planes. The direction of L is then $\mathbf{v} = \overrightarrow{p_0p_1} = \langle -3,2,1\rangle$, and by definition L is the set of all points $q=(x,y,z)$ for which $\overrightarrow{p_0q}$ is parallel to $\mathbf{v}$—meaning $\overrightarrow{p_0q} = t\mathbf{v}$ for some $t\in\mathbb{R}$. From this we obtain $\langle x-1, y, z\rangle = t\langle -3,2,1\rangle$, or $\langle x(t), y(t), z(t)\rangle = \langle -3t+1, 2t, t\rangle$ for $-\infty<t<\infty$. Letting $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then gives us $\mathbf{r}(t) = \langle -3t+1, 2t, t\rangle$, $-\infty<t<\infty$.

4a. Dom $g = \{(x,y)\mid y<x^2\}$ (all points in $\mathbb{R}^2$ that lie below the parabola $y=x^2$).

4b. Dom $h = \{(x,y)\mid y\le \frac{1}{2}x+2\}$ (all points on or below the line $y=\frac{1}{2}x+2$).

5a. Direct substitution can be employed since the point $(1,\ln 2, 3)$ lies in the domain of a function that is a combination of polynomial and exponential functions: $3e^{1\cdot\ln 2} = 3\cdot 2 = 6$.

5b. $(2,2)$ lies on the boundary of the function's domain, and the definition of limit requires that (x,y) approach $(2,2)$ while staying in the domain of the function $f(x,y) = \dfrac{y^2-4}{xy-2x}$, which is $\{(x,y)\mid xy\ne 2x\}$. For $xy\ne 2x$ we have $\dfrac{y^2-4}{xy-2x} = \dfrac{y+2}{x}$, so $\displaystyle\lim_{(x,y)\to(2,2)}\frac{y^2-4}{xy-2x} = \lim_{(x,y)\to(2,2)}\frac{y+2}{x} = 2$.

6. Along the path $y=x$ we get $\displaystyle\lim_{(x,y)\to(0,0)}\frac{y^3+x^3}{xy^2} = \lim_{(x,x)\to(0,0)}\frac{x^3+x^3}{x\cdot x^2} = \lim_{(x,x)\to(0,0)}(2) = 2$; but along the path $y=2x$ we get $\displaystyle\lim_{(x,y)\to(0,0)}\frac{y^3+x^3}{xy^2} = \lim_{(x,2x)\to(0,0)}\frac{(2x)^3+x^3}{x\cdot(2x)^2} = \lim_{(x,x)\to(0,0)}\frac{9x^3}{4x^3} = \frac{9}{4}$. Since the limit approaches two different values depending on the path taken, it can not exist.

7a. $f_x(x,y) = y^3\sec^2 xy$ and $f_y(x,y) = xy^2\sec^2 xy + 2y\tan xy$.

7b. $\rho_u = \dfrac{1}{v+2w}$, $\rho_v = -\dfrac{u}{(v+2w)^2}$, and $\rho_w = -\dfrac{2u}{(v+2w)^2}$.

8a. Along the path $y=x$ the limit becomes $\displaystyle\lim_{(x,x)\to(0,0)} -\frac{x\cdot x}{x^2+x^2} = \lim_{(x,x)\to(0,0)} -\frac{1}{2} = -\frac{1}{2}$, which implies that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \ne f(0,0) = 0$ and therefore f is not continuous at $(0,0)$.

8b. By an established theorem in the textbook, since f is not continuous at $(0,0)$ it cannot be differentiable at $(0,0)$.

8c. $f_x(0,0) = \lim_{h \to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0}(0) = 0$. Thus, even though f is not differentiable at $(0,0)$, it can have partial derivatives at $(0,0)$.

9. $z_s = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = \cos x \cos 2y - 2 \sin x \sin 2y$, and similarly $z_t = \cos x \cos 2y + 2 \sin x \sin 2y$.

10. Here $F(x,y) = ye^{xy} - 2$ is a function that is differentiable on its domain, so by an established theorem $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{y^2 e^{xy}}{xye^{xy} + e^{xy}} = -\dfrac{y^2}{xy+1}$.