Math 242 Exam #1 Key (Fall 2010)

1. $\overrightarrow{PQ} = \langle 3-(-4), -5-1\rangle = \langle 7,-6\rangle = 7\mathbf{i} - 6\mathbf{j}$

2. Define $\mathbf{v} = 5\cdot\langle -7,9\rangle/|\langle -7,9\rangle| = 5\cdot\langle -7,9\rangle/\sqrt{130} = \langle -35/\sqrt{130}, 45/\sqrt{130}\rangle$.

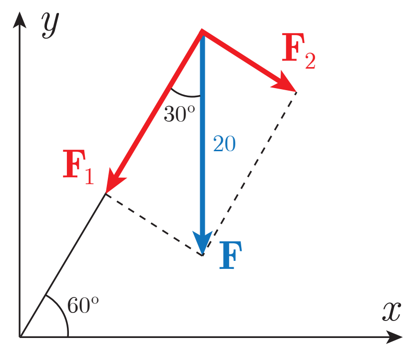
3. The velocity vector of the plane in still air is $\mathbf{p} = \langle -320, 0\rangle$, and the velocity vector of the wind is $\mathbf{w} = \langle -40/\sqrt{2}, -40/\sqrt{2}\rangle$. When the two meet, the resultant velocity of the plane is given by $\mathbf{v} = \mathbf{p} + \mathbf{w} = \langle -348.284, -28.284\rangle$. Then the speed of the plane is $|\mathbf{v}| = \sqrt{348.284^2 + 28.284^2} = 349.43$ mi/hr, and the direction of the plane is $\theta = \tan^{-1}\left(\frac{28.284}{348.284}\right) = 4.64^\circ$ south of west.

4. Complete the square for each variable to get the equation into what turns out to be the "center-radius form" for the equation of a sphere:

$$(x^2-6x) + (y^2+6y) + (z^2-8z) = 2$$
$$(x^2-6x+9) + (y^2+6y+9) + (z^2-8z+16) = 2+9+9+16$$
$$(x-3)^2 + (y+3)^2 + (z-4)^2 = 36$$

So the set of points that satisfy the equation is a sphere centered at $(3,-3,4)$ with radius 6.

5.



The unit vector that points in the direction of $\mathbf{F}_1$ is $\hat{\mathbf{F}}_1 = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$, and the unit vector in the direction of $\mathbf{F}_2$ is $\hat{\mathbf{F}}_2 = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$. It remains to ascertain the magnitudes of these two component vectors of $\mathbf{F}$. This is simple trigonometry: $\cos\frac{\pi}{3} = |\mathbf{F}_2|/20 \Rightarrow |\mathbf{F}_2| = 10$, and $\sin\frac{\pi}{3} = |\mathbf{F}_1|/20 \Rightarrow |\mathbf{F}_1| = 10\sqrt{3}$. Hence:

$$\mathbf{F}_1 = |\mathbf{F}_1|\cdot\hat{\mathbf{F}}_1 = 10\sqrt{3}\left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle = \left\langle -5\sqrt{3}, -15\right\rangle$$
$$\mathbf{F}_2 = |\mathbf{F}_2|\cdot\hat{\mathbf{F}}_2 = 10\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle = \left\langle 5\sqrt{3}, -5\right\rangle$$

6. The variables a and b must be such that $\langle 1,a,b\rangle\cdot\langle 4,-8,2\rangle = 4-8a+2b = 0$, which implies that $b = 4a-2$. Thus the set of all vectors of the form $\langle 1,a,b\rangle$ that are orthogonal to $\langle 4,-8,2\rangle$ is $\{\langle 1,a,4a-2\rangle \mid a\in\mathbb{R}\}$.

7. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 6 \\ 1 & 2 & -1 \end{vmatrix} = (4\mathbf{i}+6\mathbf{j}+6\mathbf{k}) - (-4\mathbf{k}+12\mathbf{i}-3\mathbf{j}) = -8\mathbf{i}+9\mathbf{j}+10\mathbf{k} = \langle -8,9,10\rangle$

8. Let $P=(1,0,2)$ and $Q=(3,-2,3)$. By definition the line will consist of all points $R\in\mathbb{R}^3$ such that $\overrightarrow{PR}$ is parallel to $\overrightarrow{PQ} = \langle 2,-2,1\rangle$. Using the formula $\mathbf{r}(t) = \langle x_0+at, y_0+bt, z_0+ct\rangle$ as developed in the notes (and textbook) with $(x_0,y_0,z_0) = (1,0,2)$ and $\langle a,b,c\rangle = \langle 2,-2,1\rangle$ gives: $\mathbf{r}(t) = \langle 1+2t, -2t, 2+t\rangle$.

9. From the parameterization of the curve we have $z(t) = 4 + 3t$, so we need only find t for which $z(t) = 16$: $4 + 3t = 16 \Rightarrow t = 4$. The point at which the plane intersects the curve is then $(4, 2(4), 4 + 3(4)) = (4, 8, 16)$ (just evaluate $\mathbf{r}(4)$).

10. First we find $\mathbf{r}'(t) = \left\langle \cos t, -\sin t, \dfrac{1}{2\sqrt{t}} \right\rangle$. Then $\mathbf{T}(t) = \hat{\mathbf{r}}'(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)| = \dfrac{\langle \cos t, -\sin t, 1/(2\sqrt{t})\rangle}{\sqrt{\cos^2 t + \sin^2 t + 1/(4t)}}$ $= \dfrac{\langle \cos t, -\sin t, 1/(2\sqrt{t})\rangle}{\sqrt{1 + 1/(4t)}} = \dfrac{\langle 2\sqrt{t}\cos t, -2\sqrt{t}\sin t, 1\rangle}{\sqrt{4t+1}} = \left\langle \dfrac{2\sqrt{t}\cos t}{\sqrt{4t+1}}, \dfrac{-2\sqrt{t}\sin t}{\sqrt{4t+1}}, \dfrac{1}{\sqrt{4t+1}} \right\rangle$, and so when $t = 9$ we obtain $\mathbf{T}(t) = \left\langle \dfrac{6\cos 9}{\sqrt{37}}, \dfrac{-6\sin 9}{\sqrt{37}}, \dfrac{1}{\sqrt{37}} \right\rangle = \langle -0.899, -0.407, 0.164\rangle$.

11. Integration gives us $\mathbf{r}(t) = \left\langle \frac{2}{3}t^{3/2} + A, \frac{1}{\pi}\sin \pi t + B, 4\ln t + C \right\rangle$, where A, B, and C are arbitrary constants. Now, $\mathbf{r}(1) = \langle 2, 3, 4\rangle$ and $\mathbf{r}(1) = \langle 2/3 + A, B, C\rangle$ taken together imply that $A = \frac{4}{3}$, $B = 3$, and $C = 4$. Therefore $\mathbf{r}(t) = \left\langle \frac{2}{3}t^{3/2} + \frac{4}{3}, \frac{1}{\pi}\sin \pi t + 3, 4\ln t + 4 \right\rangle$.

12. Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -90\sin 3t, 40\cos t\rangle$. Speed: $|\mathbf{v}(t)| = \sqrt{8100\sin^2 3t + 1600\cos^2 t}$. Acceleration: $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -270\cos 3t, -40\sin t\rangle$.

13. Arc length $L = \displaystyle\int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 \left|\left\langle t, 4(t+1)^{1/2}\right\rangle\right|\,dt = \int_0^2 \sqrt{t^2 + 16t + 16}\,dt = \int_0^2 \sqrt{(t+8)^2 - 48}\,dt =$

$$\underbrace{\int_8^{10} \sqrt{u^2 - 48}\,du}_{\text{substitution: } u = t+8} = \underbrace{\int_{\pi/6}^{\cos^{-1}(2\sqrt{3}/5)} \sqrt{48\sec^2\theta - 48}\cdot\sqrt{48}\sec\theta\tan\theta\,d\theta}_{\text{trigonometric substitution: let } u = \sqrt{48}\sec\theta} = \underbrace{48\int_{\pi/6}^{\cos^{-1}(2\sqrt{3}/5)} (\sec^3\theta - \sec\theta)\,d\theta}_{\text{use } \tan^2\theta = \sec^2\theta - 1} =$$

$10\sqrt{13} - 16 + 24\ln\left(\dfrac{6}{5+\sqrt{13}}\right)$, where along the way we use (or derive) $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$ and $\displaystyle\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$.