Math 242
Fall 2015 Name:
Exam 5

1. $\boxed{\text{10 pts.}}$ Determine whether the vector field

$$\mathbf{F}(x,y) = \langle ye^x + \sin y, e^x + x\cos y\rangle$$

is conservative. If it is, determine a potential function for $\mathbf{F}$.

2. $\boxed{\text{10 pts.}}$ Evaluate

$$\int_C \nabla(xyz)\cdot d\mathbf{r},$$

where C is the curve having parametrization $\mathbf{r}(t) = \langle \cos t, \sin t, t/\pi\rangle$ for $0 \le t \le \pi$.

3. $\boxed{\text{10 pts.}}$ Use Green's Theorem to evaluate

$$\oint_C (2y^2 - 2x^2y)\,dx + (x^3 + xy)\,dy,$$

where C is the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$, $(-1,1)$ and counterclockwise (i.e. positive) orientation.

4. $\boxed{\text{10 pts. each}}$ Let $\mathbf{F}(x,y,z) = \langle 2xy + z^4, x^2, 4xz^3\rangle$.
   (a) Find the divergence of $\mathbf{F}$.
   (b) Find the curl of $\mathbf{F}$.

5. Consider the surface Σ that is the part of the paraboloid $z = 2x^2 + 2y^2$ for $0 \le z \le 8$.
   (a) $\boxed{\text{5 pts.}}$ Give a parametric description of Σ in the form

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle.$$

   (b) $\boxed{\text{10 pts.}}$ Use $\mathbf{r}(u,v)$ to find the area of Σ by evaluating the appropriate surface integral.

6. $\boxed{\text{15 pts.}}$ Let $\mathbf{F}(x,y,z) = \langle x^2 - z^2, y, 2xz\rangle$, and let C be the boundary of the part of the plane $z = 4 - x - y$ in the first octant. Assuming C has a counterclockwise (i.e. positive) orientation, evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ by evaluating the surface integral in Stokes' Theorem using an appropriate choice of surface Σ.

$$\iint_\Sigma f = \iint_R f(\mathbf{r}(u,v))\|(\mathbf{r}_u \times \mathbf{r}_v)(u,v)\|\, dA.$$

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{n}(u,v)\|(\mathbf{r}_u \times \mathbf{r}_v)(u,v)\|\, dA,$$

Green's Theorem—Circulation Form. Let $C \subset \mathbb{R}^2$ be a simple closed piecewise-smooth curve that bounds a region R, so that $C = \partial R$. If f and g have continuous first partial derivatives on an open region containing R, then

$$\oint_{\partial R} f\, dx + g\, dy = \iint_R (g_x - f_y)\, dA.$$

Green's Theorem—Flux Form. Let $C \subset \mathbb{R}^2$ be a simple closed piecewise-smooth curve that bounds a region R, so that $C = \partial R$. If f and g have continuous first partial derivatives on an open region containing R, then

$$\oint_{\partial R} f\, dy - g\, dx = \iint_R (f_x + g_y)\, dA.$$

Stokes' Theorem. Let $\Sigma \subset \mathbb{R}^3$ be a piecewise-smooth orientable surface with piecewise-smooth simple closed boundary $\partial\Sigma$. Let $\mathbf{n}$ be an orientation for Σ, and let $\partial\Sigma$ have orientation consistent with $\mathbf{n}$. If $\mathbf{F} = \langle f, g, h \rangle$ is a vector field such that f, g, and h have continuous first partial derivatives on an open set containing Σ, then

$$\oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} = \iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$