12.2.65. Prove that the midpoint of the line segment joining $p(x_1, y_1, z_1)$ & $q(x_2, y_2, z_3)$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

The problem here, as I see it, is that "midpoint" has not been rigorously defined, and even worse the precise definition of a "line segment" has not been developed either. And yet we are asked here to produce a rigorous proof of some result hinging on these notions. This is a bother. The **line segment** $\overline{pq}$ in $\mathbb{R}^3$ is commonly defined as the following set of points:

$$\overline{pq} = \{(x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1),\ z_1 + t(z_2 - z_1)) \mid 0 \le t \le 1\}. \tag{1}$$

There are nicer ways to write (1) using vector notation, but that is best deferred to, say, section 12.5.

The **midpoint** m_{pq} of the segment $\overline{pq}$ is defined to be the point in $\overline{pq}$ that is "equidistant" from p and q, meaning the distance between m_{pq} and p equals the distance between m_{pq} and q: $D(m_{pq}, p) = D(m_{pq}, q)$. What we have to prove here is $m = \left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2), \frac{1}{2}(z_1+z_2)\right)$ fits this definition.

That $m \in \overline{pq}$ is easy to verify: look at the element of $\overline{pq}$ that we get when we let $t = 1/2$:

$$\left(x_1 + \tfrac{1}{2}(x_2 - x_1),\ y_1 + \tfrac{1}{2}(y_2 - y_1),\ z_1 + \tfrac{1}{2}(z_2 - z_1)\right)$$

This is precisely $\left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2), \frac{1}{2}(z_1+z_2)\right) = m$. Done.

Now, to establish equidistance, just take the direct route:

$$\begin{aligned}
D(m,p) &= \sqrt{\left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \left(y_1 - \frac{y_1+y_2}{2}\right)^2 + \left(z_1 - \frac{z_1+z_2}{2}\right)^2} \\
&= \tfrac{1}{2}\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2} \\
&= \tfrac{1}{2}D(p,q)
\end{aligned}$$

$$\begin{aligned}
D(m,q) &= \sqrt{\left(x_2 - \frac{x_1+x_2}{2}\right)^2 + \left(y_2 - \frac{y_1+y_2}{2}\right)^2 + \left(z_2 - \frac{z_1+z_2}{2}\right)^2} \\
&= \tfrac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \\
&= \tfrac{1}{2}D(p,q)
\end{aligned}$$

Therefore $D(m,p) = D(m,q)$, and so $m = m_{pq}$. $\blacksquare$