Calculus

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1

Foundations

1.1 – Logic

There are two **truth values**: **true** and **false**. A **statement** is a word string (i.e. a combination of words) that has a truth value deriving from prior knowledge. Thus "Two plus five is seven" is a statement, but the Vulcan expression "Live long and prosper" is not a statement. There are gray areas. Is the Klingon expression "Today is a good day to die" a statement or not? As Obi-Wan would say, "It depends on your point of view." Fortunately in our study of calculus such gray areas will not arise.

In mathematics prior knowledge consists of lemmas, theorems, and corollaries. A **lemma** is usually a technical result that facilitates the proof of a theorem, whereas a **corollary** is a result that follows immediately from a theorem. In these notes a **theorem** is a relatively important result, while a less important result is called a **proposition**. It will be sometimes convenient to refer to lemmas, theorems, corollaries, and propositions collectively as "theorems" so as to avoid excessive verbiage.

There is one fundamental problem. Each theorem in mathematics is by definition a true statement, so its truth must derive from prior knowledge. This prior knowledge we would reasonably surmise is comprised of one or more other theorems. But these other theorems in turn must derive their truth from still other theorems, and so on. The problem is one of an "infinite regress." In order for mathematics to advance there must be some means of halting this infinitely regressing chain of theorems, and the solution is simple even if (for many mathematicians) unsatisfactory: allow the chain of theorems to regress only so far as is necessary in order for the prior knowledge from which they derive their truth to be credibly declared a "self-evident" truth. In mathematics self-evident truths and basic assumptions are called **axioms** or **postulates**.

Axioms often include undefined terms. For example, in plane geometry one axiom is "A straight line may be drawn between any two points." The terms "line" and "point" are undefined. This is not a serious problem: every instance of the words "line" and "point" could be replaced by "sillygon" and "shimsong" in the English translation of Euclid's *Elements*, and the truth value of not a single theorem would be affected.

In formal logic statements are often represented by symbols such as P and Q. Thus P is rather akin to a variable such as x in elementary algebra, but whereas x in algebra usually represents a numerical value, in logic P is regarded as having truth values. In fact it is more accurate to say that, in formal logic, P is a variable that assumes truth values; but since only statements have truth values, it is still fine to think of P as standing for a statement.

Two statements P and Q are said to be **equivalent** if they always have the same truth value; that is, Q is true whenever P is true, and Q is false whenever P is false. For instance the statements "0 is less than 1" and "1 is greater than 0" are equivalent. The symbol $\Leftrightarrow$ stands for "is equivalent to," so that $P \Leftrightarrow Q$ is read as "P is equivalent to Q." (We will dispense with putting quotes around statements that are expressed entirely symbolically.) In mathematics logically equivalent statements are entirely interchangeable.

The relation[1] $\Leftrightarrow$ has a **transitive** property. If $P_1, P_2, \dots, P_n$ are statements such that $P_i \Leftrightarrow P_{i+1}$ for all $1 \le i \le n-1$, then $P_1 \Leftrightarrow P_n$.

Example 1.1. In algebra, the operation of "solving" an equation featuring a variable x is actually one of establishing a chain of equivalencies that link the original equation to one or more simpler equations in which the x is isolated. What this accomplishes, of course, is to determine precisely what value(s) x may assume in order for the original equation to become a true statement.[2] Consider the equation $2x - 3 = 7$. We have

$$2x - 3 = 7 \quad \Leftrightarrow \quad 2x = 10 \quad \Leftrightarrow \quad x = 5;$$

that is, $2x - 3 = 7$ is equivalent to $x = 5$, or in other words $2x - 3 = 7$ can only ever be true if $x = 5$. ■

A statement P **implies** another statement Q, written $P \Rightarrow Q$, if Q is true whenever P is true; that is, if P is known to be true, then it can be concluded that Q is also true. We note that if $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true, then $P \Leftrightarrow Q$ is true; also, if $P \Leftrightarrow Q$ is true, then $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true. The relation $\Rightarrow$ has the same transitive property that $\Leftrightarrow$ possesses.

Example 1.2. Let P be the statement $x = 3$, and Q the statement $x^2 = 9$. Then clearly P implies Q (i.e. $P \Rightarrow Q$). But Q does not imply P. We have

$$Q \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad \sqrt{x^2} = \sqrt{9} \quad \Rightarrow \quad |x| = 3 \quad \Rightarrow \quad x = \pm 3,$$

where the statement $x = \pm 3$ is equivalent to the statement "$x = 3$ or $x = -3$," which certainly does not necessarily imply that $x = 3$, which is statement P. We conclude that $x = 3$ and $x^2 = 9$ are not equivalent statements. In particular if $x = -3$ then P is false, but since $x^2 = (-3)^2 = 9$ we see that Q is true. ■

Simple statements can be combined to make more complex statements. For instance, if P and Q are statements, then "P or Q" is another statement. And this is important: the statement "P or Q" is false only when *both* P and Q are false. Thus if P is the statement "1 is less than 2" and Q is the statement "1 is equal to 2," then the statement "P or Q," which stands for "1 is less than 2 or 1 is equal to 2," is true by virtue of the fact that P is true (even though Q is false). Of course, we have notation at hand to write "P or Q" as $1 \le 2$.

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[1] The idea of a mathematical relation is defined more formally in §1.4, but for now it's enough to say that a relation is some rule that "relates" objects to one another, such as $>$ relates 9 to 8 in the statement $9 > 8$.

[2] Truth is a really big deal in mathematics.

Another common compound statement is "P and Q," and it's important to remember that it is only true if *both* P and Q are true. If either P is false or Q is false, then "P and Q" is false. For instance, "1 is less than 2 and 1 is equal to 2" is false.

Many calculus theorems are if-then statements: "If P, then Q." One important thing to remember about "If P, then Q" is that it is only false if P is true and Q is false. Curiously, if P is false then "If P, then Q" is true regardless of whether Q is true or false. Another important thing about "If P, then Q" is that "If not P, then not Q" is an equivalent statement.

1.2 – Set Theory

A **set** is a collection of objects. An object in a set is called an **element** of the set. If S is a set and a is an element of S, we write $a \in S$. If a is not an element we write $a \notin S$. Two sets A and B are said to be **equal**, written $A = B$, if they contain precisely the same elements. The **empty set**, denoted by $\varnothing$ or $\{\ \}$, is the set that contains no elements.

A set is **well-defined** if, given any object a, it can be determined, without ambiguity, whether a is an element of S or not.

Example 1.3. If S is said to be the set of whole numbers greater than 3 and less than 9, then S is well-defined since it is clear that the elements of S must be 4, 5, 6, 7, and 8.

If, on the other hand, S is said to be the set of "astronomically humongous" whole numbers, then while it would be reasonably clear that, say, 1, 10, and 50 must not be elements of S, it would not at all be certain that 10,000, 1,000,000, or 50,000,000 are elements. Thus S would not be well-defined.

If S is said to be the set of "the ten best books ever written," should we consider it to be well-defined? ■

One common way of defining a set is **roster notation**, which is a notation which lists some or all of the set's elements between braces. Thus, if S is the set of whole numbers greater than 3 and less than 9, we can write

$$S = \{4, 5, 6, 7, 8\}.$$

The set of whole numbers from 1 to 100 can be presented in roster notation by

$$\{1, 2, 3, \ldots, 99, 100\}.$$

The set of whole numbers greater than or equal to 1 is known as the set of **natural numbers** (or **counting numbers**) and is given the special symbol $\mathbb{R}$. In roster notation we can write

$$\mathbb{R} = \{1, 2, 3, 4, \ldots\}.$$

Also of great importance is the set of **integers**, symbol $\mathbb{Z}$, which consists of the whole numbers and their negatives, and so

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The set of **rational numbers**, symbol $\mathbb{Q}$, consists of all numbers that are expressible as a ratio of integers. Thus if r is a rational number, then there must exist $p, q \in \mathbb{Z}$ such that $r = p/q$, with the one caveat being that we must have $q \neq 0$ (otherwise no number results at all). It is not practical to present $\mathbb{Q}$ in roster notation since a complete listing of rational numbers cannot be given which preserves their order. Starting in the middle at 0 as was done for the integers, the question arises: what is the "next higher" rational number? There is none! Before 1 there is 1/2, before 1/2 there is 1/3, before 1/3 there is 1/4, and so on *ad infinitum.*

A more robust way of defining sets is with **set-builder notation**. The general form is

$$\{x : p(x)\} \quad \text{or} \quad \{x \mid p(x)\},$$

which is read as "the set of all x such that $p(x)$," with $p(x)$ being some logical statement whose truth value depends on the value of x. The understanding is that if $p(x)$ is true (truth value 1), then x is an element of the set; and if $p(x)$ is false (truth value 0), then x is not an element of the set. For example we may let $p(x)$ be the statement "x is an integer, and x is greater than 3 and less than 10," in which case $\{x : p(x)\}$ may be written as

$$\{x : x \in \mathbb{Z} \text{ and } 3 < x < 10\},$$

which is read as: "The set of all x such that $x \in \mathbb{Z}$ and $3 < x < 10$." Thus the set consists of integers that are greater than 3 and less than 10, which is to say

$$\{x : x \in \mathbb{Z} \text{ and } 3 < x < 10\} = \{4, 5, 6, 7, 8, 9\}.$$

Returning to the rational numbers, we can write

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

In the next section we will define the set $\mathbb{R}$ of real numbers, which includes all the rational numbers.

If all the elements of a set A are also elements of a set B, then we say A is a **subset** of B and write $A \subseteq B$. More precisely,

$$A \subseteq B \quad \Leftrightarrow \quad \forall x(x \in A \to x \in B).$$

Thus $\mathbb{R} \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{Q}$, for instance. An equivalent way of writing $A \subseteq B$ is $B \supseteq A$, which can be read as "A is a subset of B" or "B is a superset of A." It should be clear that $A = B$ if and only if $A \subseteq B$ and $A \supseteq B$, just as in algebra $x = y$ if and only if $x \leq y$ and $x \geq y$.

We now define a variety of set operations that will be indispensable throughout the book. First there is the notion of the **union** of two sets A and B, defined to be the set $A \cup B$ given by

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

where $A \cup B$ is read as "A union B." The **intersection** of A and B is the set $A \cap B$ given by

$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$

where $A \cap B$ is read as "A intersect B." Thus if $A = \{1, 4, 8, 9\}$ and $B = \{1, 2, 3, 4, 5\}$, then

$$A \cup B = \{1, 4, 8, 9\} \cup \{1, 2, 3, 4, 5\} = \{1, 2, 3, 4, 5, 8, 9\},$$

and

$$A \cap B = \{1, 4, 8, 9\} \cap \{1, 2, 3, 4, 5\} = \{1, 4\}.$$

The **complement** of a set A, written A^c, is defined to be the set of all objects in a set $U \supseteq A$ that are not elements of A. More precisely,

$$A^c = \{x : x \in U \text{ and } x \notin A\}.$$

The set U is called the **universe of discourse** and usually contains all objects that are relevant to an analysis. In calculus we usually have $U = \mathbb{R}$, the set of real numbers discussed in Section 1.3.

Example 1.4. Let $A = \{3, 4, 5, 6, \ldots\}$. If $U = \mathbb{R}$, then

$$A^c = \{x : x \in \mathbb{R} \text{ and } x \notin A\} = \{x : x = 1, 2, 3, \ldots \text{ and } x \neq 3, 4, 5, \ldots\} = \{1, 2\}.$$

If $U = \mathbb{Z}$, then

$$A^c = \{x : x \in \mathbb{Z} \text{ and } x \notin A\} = \{\ldots, -2, -1, 0, 1, 2\} = \{2, 1, 0, -1, -2, \ldots\}.$$

Finally, if $U = A$ then $A^c = \varnothing$. ■

Definition 1.5. *Let S be a set. An **order** on S is a relation among its elements, denoted by $<$, that has the following properties:*

1. *For all $a, b \in S$ exactly one of the following statements must be true: $a < b$, $a = b$, $b < a$.*
2. *For all $a, b, c \in S$, if $a < b$ and $b < c$, then $a < c$.*

*A set S with an order $<$ is called an **ordered set** and may be denoted by $(S, <)$.*

The relation $<$ may be pronounced as "is less than," although all the symbol really represents is this abstract setting is a general rule for ordering (i.e. arranging) the elements of S in some fashion, so that given any two elements a and b in S, we can determine which element comes "first". Thus $<$ could also be read as "comes before" or "is the predecessor to" or "precedes". We take $a > b$ to be equivalent to $b < a$, and define $a \leq b$ and $a \geq b$ to signify the statements "$a < b$ or $a = b$" and "$a > b$ or $a = b$," respectively. (More about relations in general will be discussed in §1.4.)

Certainly if S happens to be the set of integers $\mathbb{Z}$, say, we would expect $<$ to represent the familiar "less than" relation. It should be clear that $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all ordered sets.

Definition 1.6. *Let $(S, <)$ be an ordered set, and suppose $A \subseteq S$. If there exists some $\alpha \in S$ such that $x \leq \alpha$ for all $x \in A$, then α is an **upper bound** for A and A is **bounded above**. If there exists some $\beta \in S$ such that $x \geq \beta$ for all $x \in A$, then β is a **lower bound** for A and A is **bounded below**.*

Definition 1.7. *Suppose $(S, <)$ is an ordered set, $A \subseteq S$, and A is bounded above. Suppose $\alpha \in S$ has the following properties:*

1. *α is an upper bound for A.*
2. *If γ is an upper bound for A, then $\gamma \geq \alpha$.*

*Then α is the **least upper bound** of A, also called the **supremum** of A, and we write*

$$\alpha = \sup(A).$$

Definition 1.8. *Suppose $(S, <)$ is an ordered set, $A \subseteq S$, and A is bounded below. Suppose $\beta \in S$ has the following properties:*

1. *β is a lower bound for A.*
2. *If γ is a lower bound for A, then $\gamma \leq \alpha$.*

*Then β is the **greatest lower bound** of A, also called the **infimum** of A, and we write*

$$\beta = \inf(A).$$

Example 1.9. Consider the set $A \subseteq \mathbb{Q}$ given by

$$A = \left\{x : x \in \mathbb{Q} \text{ and } x \leq \tfrac{1}{2}\right\},$$

which is bounded above but not bounded below. Certainly $3 \in \mathbb{Q}$ is an upper bound for A, but it is not the least upper bound since smaller numbers such as 2, 1, and $\frac{3}{5}$ are also upper bounds. Clearly the least upper bound is $\frac{1}{2}$, and so we write $\sup(A) = \frac{1}{2}$. In fact if we remove $\frac{1}{2}$ from A to obtain the set

$$A' = \left\{x : x \in \mathbb{Q} \text{ and } x < \tfrac{1}{2}\right\},$$

the least upper bound remains unchanged: $\sup(A') = \frac{1}{2}$. What this shows is that the least upper bound of a set *need not be an element of the set.*

Now consider the set $B \subseteq \mathbb{Q}$ given by

$$B = \left\{x : x \in \mathbb{Q} \text{ and } x \leq \sqrt{2}\right\},$$

the set of all rational numbers not larger than $\sqrt{2}$. This set is also bounded above. One upper bound is $\frac{3}{2}$, since

$$\tfrac{3}{2} = 1.5 > 1.41421356 \approx \sqrt{2},$$

but clearly $\frac{3}{2}$ is not the least upper bound. Successively smaller upper bounds for B are:

$$1.42, 1.419, 1.418, 1.417, 1.416, 1.415, 1.4149, 1.4148, 1.4147, 1.4146, ...$$

All of these upper bounds are elements of $\mathbb{Q}$, but none of them is the least upper bound. The least upper bound for B is $\sqrt{2}$, however $\sqrt{2}$ is *not* an element of $\mathbb{Q}$. There exists no $\alpha \in \mathbb{Q}$ such that $\alpha = \sup(B)$! ■

1.3 – Real Numbers

Given a set S, a **binary operation** on S is a rule for combining two elements of S in order to obtain another element of S. The basic operations of addition, subtraction, multiplication, and division encountered in arithmetic are examples of binary operations. So, for instance, the addition operation $+$ takes any two elements of $\mathbb{Z}$ (i.e. any two integers m and n) and returns another element of $\mathbb{Z}$ (i.e. an integer $m+n$).

Definition 1.10. *The* ***real number system*** $(\mathbb{R}, +, \cdot, >)$ *consists of a set* $\mathbb{R}$ *of objects called* ***real numbers****, together with binary operations* $+$ *and* $\cdot$ *on* $\mathbb{R}$*, and a relation* $>$*, that are subject to the following axioms:*

F1. $a+b=b+a$ *for all* $a,b \in \mathbb{R}$.
F2. $a+(b+c)=(a+b)+c$ *for all* $a,b,c \in \mathbb{R}$.
F3. *There exists some* $0 \in \mathbb{R}$ *such that* $a+0=a$ *for all* $a \in \mathbb{R}$.
F4. *For all* $a \in \mathbb{R}$ *there exists some* $b \in \mathbb{R}$ *such that* $a+b=0$.
F5. $a \cdot b = b \cdot a$ *for all* $a,b \in \mathbb{R}$.
F6. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ *for all* $a,b,c \in \mathbb{R}$.
F7. *There exists some* $1 \in \mathbb{R}$ *such that* $1 \neq 0$ *and* $a \cdot 1 = a$ *for all* $a \in \mathbb{R}$.
F8. *For all* $a \in \mathbb{R}$ *such that* $a \neq 0$ *there exists some* $b \in \mathbb{R}$ *such that* $a \cdot b = 1$.
F9. $a \cdot (b+c) = a \cdot b + a \cdot c$ *for all* $a,b,c \in \mathbb{R}$.
O1. *For all* $a \in \mathbb{R}$ *exactly one of the following statements must be true:* $a > 0$, $a = 0$, $0 > a$.
O2. *For all* $a,b \in \mathbb{R}$, *if* $a,b > 0$, *then* $a+b > 0$.
O3. *For all* $a,b \in \mathbb{R}$, *if* $a,b > 0$, *then* $a \cdot b > 0$.
O4. *For all* $a,b \in \mathbb{R}$, *if* $a > b$, *then* $a+c > b+c$ *for all* $c \in \mathbb{R}$.
C1. *For any* $S \subseteq \mathbb{R}$ *such that* $S \neq \varnothing$, *if* S *has an upper bound in* $\mathbb{R}$, *then* S *has a least upper bound in* $\mathbb{R}$.

The element b in Axiom F4 is called the **additive inverse** of a, which we usually denote by the symbol $-a$ so that (together with Axiom F1) we have

$$(-a)+a = a+(-a) = 0.$$

The binary operations $+$ and $\cdot$ are naturally called **addition** and **multiplication**, and as a notational convenience we define a **subtraction** operation $-$ by

$$a - b = a + (-b),$$

where of course $-b$ denotes the additive inverse of b.

Any set of objects that satisfies Axioms F1 to F9 is called a **field**, and so for that reason these axioms are called the **field axioms** of the real number system. Axioms O1 to O4 are the **order axioms**, which together with the field axioms make the real number system an **ordered field**. Finally, Axiom C1 is the **Completeness Axiom**, which as an isolated statement defines the property known as **completeness**. The Completeness Axiom warrants closer examination, but first we verify that the relation $>$ in Definition 1.10 is in fact an order for the elements of $\mathbb{R}$.

Proposition 1.11. $\mathbb{R}$ *is an ordered set.*

Proof. Let $a, b \in \mathbb{R}$. By Axiom F4 there exists some $-b \in \mathbb{R}$ such that $b + (-b) = 0$, and thus

$$a + (-b) \in \mathbb{R}$$

since $+$ is given to be a binary operation on $\mathbb{R}$. By Axiom O1 exactly one of

$$a + (-b) > 0, \quad a + (-b) = 0, \quad 0 > a + (-b)$$

is true, and so by Axiom O4 it follows that exactly one of

$$(a + (-b)) + b > 0 + b, \quad (a + (-b)) + b = 0 + b, \quad 0 + b > (a + (-b)) + b$$

is true. Now we employ Axioms F2 and F3 to obtain

$$a + ((-b) + b) > b, \quad a + ((-b) + b) = b, \quad b > a + ((-b) + b),$$

whence

$$a + 0 > b, \quad a + 0 = b, \quad b > a + 0.$$

Therefore, using Axiom F3, we conclude that exactly one of

$$a > b, \quad a = b, \quad b > a$$

is true, which shows that the first property given in Definition 1.5 holds.

For the second property, let $a, b, c \in \mathbb{R}$, and suppose $c > b$ and $b > a$. By Axiom F4 there exists some $-a, -b \in \mathbb{R}$ such that $a + (-a) = 0$ and $b + (-b) = 0$, and so by Axiom O4

$$c + (-b) > b + (-b) \quad \text{and} \quad b + (-a) > a + (-a),$$

whence

$$c - b > 0 \quad \text{and} \quad b - a > 0.$$

Now, by Axiom O2,

$$(c - b) + (b - a) > 0, \tag{1.1}$$

where

$$\begin{aligned}
(c - b) + (b - a) &= (c + (-b)) + (b + (-a)) \\
&= ((c + (-b)) + b) + (-a) && \text{(Axiom F2)} \\
&= (b + (-b + c)) + (-a) && \text{(Axiom F1)} \\
&= ((b + (-b)) + c) + (-a) && \text{(Axiom F2)} \\
&= (0 + c) + (-a) \\
&= (c + 0) + (-a) && \text{(Axiom F1)} \\
&= c + (-a) && \text{(Axiom F3)}
\end{aligned}$$

Putting this result into (1.1) yields

$$c + (-a) > 0.$$

By Axiom O4 we may add a to both sides to obtain

$$(c + (-a)) + a > 0 + a,$$

whereupon application of Axioms F1 and F2 on the left, and F3 on the right, yields

$$c > a$$

as desired. ■

The **rational number system** $(\mathbb{Q}, +, \cdot, >)$ consists of the set of rational numbers defined in the previous section, along with the usual arithmetic operations of addition and multiplication, and also the familiar order relation $>$ known as "greater than". It can be verified, though it is tedious, that $(\mathbb{Q}, +, \cdot, >)$ satisfies all the field and order axioms in Definition 1.10. The only axiom $(\mathbb{Q}, +, \cdot, >)$ fails to satisfy is Axiom C1, the Completeness Axiom. In fact we discovered this in Example 1.9, when we considered the set $B \subseteq \mathbb{Q}$ given by

$$B = \left\{ x : x \in \mathbb{Q} \text{ and } x \leq \sqrt{2} \right\}.$$

B has an upper bound in $\mathbb{Q}$ (for instance 2), yet it does not have a *least* upper bound in $\mathbb{Q}$. This violates the completeness property given by Axiom C1, and so the rational number system is set apart from the real number system.

One might wonder how we "know" the set of objects $\mathbb{R}$ presented in Definition 1.10 really is the set of real numbers encountered in elementary algebra courses? The numbers 0 and 1 appear to be mentioned, but where is $\sqrt{2}$ among the axioms? Where is π? It turns out that, just as the rational numbers can be constructed from the integers, so too can the real numbers of elementary algebra be constructed from the rational numbers. The precise way in which this is done is beyond the scope of this book, but suffice it to say that once the feat is accomplished there is no doubt that the end result is a set of *numerical* quantities that satisfy all the axioms in Definition 1.10. But again, how do we "know" the set of objects given by Definition 1.10 are the same as the *numerical* quantities that are constructed from the rational numbers? In fact we *don't* know. So why are the objects in the definition above being called real numbers? The reason stems from a basic truth: mathematics does not so much study objects themselves, but rather relationships between objects.

The reality of the situation is as follows. It can be proven that any two sets of objects, say $\mathbb{R}_1$ and $\mathbb{R}_2$, that satisfy all the axioms of Definition 1.10 are **isomorphic**. This essentially means that the sets of objects are identical in every way except for the symbols used to denote them. In $\mathbb{R}_1$ the square root of two and pi may be represented by the symbols $\sqrt{2}$ and π, while in $\mathbb{R}_2$ they may be represented by x and y. But this makes no substantive difference! The symbols x and y have as much claim to be the numbers square root of two and pi as the symbols $\sqrt{2}$ and π. Symbols represent things, and have no innate properties that hold interest for the mathematician.

1.4 – Relations and Functions

Recall that for an ordered pair (x, y), x is called the **first coordinate** and y the **second coordinate**.

Suppose X and Y are sets. Then the **Cartesian product** of X and Y, written $X \times Y$, is defined as

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\},$$

which is to say that $X \times Y$ is the set of all ordered pairs (x, y) such that x (the first-coordinate value) is an element of X and y (the second-coordinate value) is an element of Y. Thus, if $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$, then

$$X \times Y = \{(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4), (c, 1), (c, 2), (c, 3), (c, 4)\}$$

and

$$Y \times X = \{(1, a), (2, a), (3, a), (4, a), (1, b), (2, b), (3, b), (4, b), (1, c), (2, c), (3, c), (4, c)\}.$$

Notice that in general $X \times Y \neq Y \times X$.

In particular if $[a, b]$ and $[c, d]$ are closed intervals in $\mathbb{R}$, we have

$$[a, b] \times [c, d] = \big\{(x, y) : x \in [a, b] \text{ and } y \in [c, d]\big\} = \big\{(x, y) : a \leq x \leq b \text{ and } c \leq y \leq d\big\},$$

which forms a rectangle in the xy-plane as shown in Figure 1.

Definition 1.12. *A* ***relation*** *R is an ordered triple (X, Y, Γ), where X and Y are arbitrary sets, and $\Gamma \subseteq X \times Y$ is a set of ordered pairs called the* ***graph*** *of R. We may write $R = (X, Y, \Gamma)$ and say R is a* ***relation from*** *X* ***into*** *Y. The set X is the* ***domain*** *of R, written* $\mathrm{Dom}(R)$*, and Y is the* ***co-domain*** *of R. The* ***range*** *of R is the set* $\mathrm{Ran}(R) \subseteq Y$ *given by*

$$\mathrm{Ran}(R) = \{y : \exists x \in X((x, y) \in \Gamma)\}.$$

Two relations $R_1 = (X_1, Y_1, \Gamma_1)$ and $R_2 = (X_2, Y_2, \Gamma_2)$ are ***equal*** *if and only if $X_1 = X_2$, $Y_1 = Y_2$, and $\Gamma_1 = \Gamma_2$, in which case we write $R_1 = R_2$.*

Observe that the range of a relation R is just the set of second-coordinate values of the ordered pairs belonging to the graph Γ of R. The definition for a relation may seem rather arcane, but the idea behind the ordered triple (X, Y, Γ) is simple: There is a set of objects X,

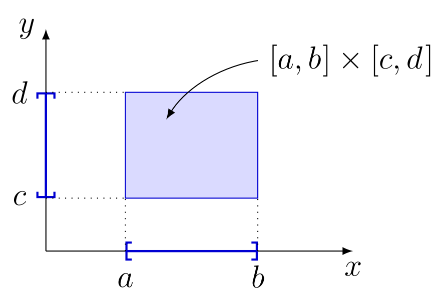


Figure 1. The Cartesian product of two closed intervals in $\mathbb{R}$.

another set of objects Y, and then Γ is some *rule* that relates (i.e. pairs) objects in X to objects in Y in some fashion.

Example 1.13. One kind of relation we have already encountered is the order relation introduced in Definition 1.5, denoted by $<$, which establishes an order among the elements of a set S so as to create an ordered set $(S, <)$. In terms of Definition 1.12 the relation $<$ is a triple (S, S, Γ), where $\Gamma \subseteq S \times S$ consists precisely of all ordered pairs (s_1, s_2) of elements of S for which $s_1 < s_2$ is true. Put another way, $(s_1, s_2) \in \Gamma$ if and only if $s_1 < s_2$. ■

Some authors identify a relation $R = (X, Y, \Gamma)$ with is graph Γ, and may even say that a relation is simply any set of ordered pairs. This makes sense if we are studying relations that all have a fixed domain X and co-domain Y, in which case the only thing that distinguishes one relation from another are their graphs (which must be subsets of $X \times Y$). We do not take this approach because we will be adopting a convention that will result in variation among the domains of the relations we encounter. Before establishing the convention we'll consider an example that will help to make clear why a convention above and beyond Definition 1.12 is even necessary.

Example 1.14. An inequality from algebra such as

$$2x - 5y \leq 10 \tag{1.2}$$

can be used to define a relation once the allowed sets of values that the variables x and y may assume are specified. If x and y are declared to be real-valued variables, then we may take the inequality (1.2) to define a relation R from $\mathbb{R}$ into $\mathbb{R}$ with graph

$$\Gamma_R = \{(x, y) : x, y \in \mathbb{R} \text{ and } 2x - 5y \leq 10\}.$$

On the other hand we may wish to only allow x to take on integer values, in which case (1.2) defines a relation R' from $\mathbb{Z}$ into $\mathbb{R}$ with graph

$$\Gamma_{R'} = \{(x, y) : x \in \mathbb{Z},\ y \in \mathbb{R} \text{ and } 2x - 5y \leq 10\}.$$

It's important to recognize that $R = (\mathbb{R}, \mathbb{R}, \Gamma_R)$ and $R' = (\mathbb{Z}, \mathbb{R}, \Gamma_{R'})$ are not the same relation since the domain of R is $\mathbb{R}$ and the domain of R' is $\mathbb{Z}$. Indeed the graphs of R and R' are different. For instance $(\frac{1}{2}, 0)$ is an element of Γ_R but not an element of $\Gamma_{R'}$, since $\frac{1}{2} \notin \mathbb{Z}$.

Now define a relation R'' from $\mathbb{C}$ into $\mathbb{R}$ with graph

$$\Gamma_{R''} = \{(x, y) : x \in \mathbb{C},\ y \in \mathbb{R} \text{ and } 2x - 5y \leq 10\}.$$

It turns out that $\Gamma_{R''} = \Gamma_R$, since no complex number that is not a real number can be substituted for x to satisfy (1.2), for the simple reason that there is no such thing as "bigger" or "smaller" nonreal complex numbers, and no nonreal complex number is "bigger" or "smaller" than any real number. Nevertheless $R = (\mathbb{R}, \mathbb{R}, \Gamma_R)$ and $R'' = (\mathbb{C}, \mathbb{R}, \Gamma_{R''})$ are not equal since the given domains of the relations are not equal! ■

In the study of calculus relations are typically given as algebraic equations or inequalities involving two variables x and y. However, in Example 1.14 we saw how the inequality (1.2) could be used to define three different relations. Which one should we choose? This is largely a question of choosing a sensible domain, and in calculus the domain will always be a subset of $\mathbb{R}$. Certainly this disqualifies R'' in Example 1.14, but R and R' are still in the running.

Let $S(x,y)$ be an algebraic statement concerning variable objects x and y. Then $S(x,y)$ will be taken to define a relation $(X,\mathbb{R},\Gamma)$ for which

$$\Gamma = \{(x,y) \in \mathbb{R}\times\mathbb{R} : S(x,y) \text{ is true}\}. \tag{1.3}$$

and

$$X = \{x : x \in \mathbb{R} \text{ and there exists some } y \in \mathbb{R} \text{ such that } S(x,y) \text{ is true}\} \tag{1.4}$$

In the case of the inequality (1.2) the statement $S(x,y)$ is

"$2x - 5y$ is less than or equal to 10."

Now, given any $x \in \mathbb{R}$ we can certainly manage to find at least one $y \in \mathbb{R}$ such that $2x - 5y$ is less than or equal to 10; indeed, all we need to do is choose any value for y for which

$$y \geq \frac{2x-10}{5}.$$

Thus, for any $x \in \mathbb{R}$ some real number y can be found which will make $S(x,y)$ true, and so by (1.4) we take (1.2) to define a relation with domain $X = \mathbb{R}$. As for the graph, we have by (1.3)

$$\Gamma = \{(x,y) \in \mathbb{R}\times\mathbb{R} : 2x - 5y \leq 10 \text{ is true}\}.$$

Therefore (1.2) is taken to define the relation $(\mathbb{R},\mathbb{R},\Gamma)$, which is in fact equal to the relation R in Example 1.14 since $\Gamma = \Gamma_R$.

The foregoing deliberations can ultimately be distilled down to a succinct convention—the convention promised after Definition 1.12 which will generally apply to every relation we encounter that is not a function (defined below), and to every function for which we are not interested in finding an inverse function (defined in §7.1). There will be occasions when we will wish to waive the convention, either by restricting the domain or the co-domain of a relation to some smaller subset, but at those times it will be explicitly stated that the convention is not in effect.

Convention. *Any set of ordered pairs $\Gamma \subseteq \mathbb{R}\times\mathbb{R}$ will be understood to define a relation $R = (X,\mathbb{R},\Gamma)$ with $X = \{x \in \mathbb{R} : (x,y) \in \Gamma$ for some $y \in \mathbb{R}\}$.*

The advantage of wording the convention this way is that is makes no mention of some statement $S(x,y)$ involving variables x and y. Thus we can consider completely arbitrary sets of ordered pairs for which no associated statement such as $2x - 5y \leq 10$ is apparent.

For example, consider the following sets of ordered pairs:

$$\Gamma_1 = \{(a,2),(a,4),(b,3),(c,1)\},$$
$$\Gamma_2 = \{(a,2),(b,3),(b,4)\},$$
$$\Gamma_3 = \{(a,1),(a,2),(a,3)\}.$$

We assume that $a,b,c \in \mathbb{R}$. By the Convention stated above we take these sets of ordered pairs to define the relations

$$R_1 = (\{a,b,c\},\mathbb{R},\Gamma_1), \quad R_2 = (\{a,b\},\mathbb{R},\Gamma_2), \quad \text{and} \quad R_3 = (\{a\},\mathbb{R},\Gamma_3).$$

Thus the co-domain of every relation is $\mathbb{R}$. The domains and ranges of the relations vary, however:

$$\mathrm{Dom}(R_1) = \{a, b, c\}, \quad \mathrm{Dom}(R_2) = \{a, b\}, \quad \text{and} \quad \mathrm{Dom}(R_3) = \{a\},$$

and

$$\mathrm{Ran}(R_1) = \{1, 2, 3, 4\}, \quad \mathrm{Ran}(R_2) = \{2, 3, 4\}, \quad \text{and} \quad \mathrm{Ran}(R_3) = \{1, 2, 3\}.$$

Definition 1.15. *A relation $R = (X, Y, \Gamma)$ is a **function** if for each $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in \Gamma$.*

Note in the definition that

$$\{x : (x, y) \in \Gamma \text{ for some } y \in \mathbb{R}\} = X,$$

which is in agreement with the convention stated above. As with relations in general we will take the codomain Y of a function to be $\mathbb{R}$ unless otherwise specified.

If (X, Y, Γ) is a function it is typical to give it the symbol f, though this by no means is a requirement. The notation $f : X \to Y$ is frequently used to denote any function f with domain X and codomain Y (leaving the set $\Gamma \subseteq X \times Y$ unspecified), which may be read as "f maps X into Y" or "f is a function from X into Y".

Definition 1.16. *Suppose $f = (X, Y, \Gamma)$ is a function. If $(x, y) \in \Gamma$, then in **function notation** we write $f(x) = y$ to indicate that y is the unique element of Y that is paired with $x \in X$ by f. For any $A \subseteq X$ the set*

$$f(A) = \{y \in Y : f(x) = y \text{ for some } x \in A\}.$$

*is the **image of** A **under** f.*

More compactly we may write

$$f(A) = \{f(x) : x \in A\},$$

which cuts out y as the "middleman." Observe that, of necessity, $f(X) = \mathrm{Ran}(f)$. Using function notation we can restate Definition 1.15 of a function as follows.

Definition 1.17. *A relation f from X into Y is a function $f : X \to Y$ if for every $x \in X$ there is exactly one $y \in Y$ such that $f(x) = y$.*

The utility of function notation cannot be overstated. First and perhaps foremost, function notation facilitates the common interpretation of a function as being a "machine" that receives an x value as input and returns a y value as output. Thus if $(2, 5) \in f$, we can think of f as taking in 2 and putting out 5, and write $f(2) = 5$.

$$2 \longrightarrow \boxed{\;f\;} \longrightarrow 5$$

We'll make frequent references to this interpretation later on.

The domain of a function f is just the set of all the first-coordinate values of all the ordered pairs that it contains, and the range of f is the set of all the second-coordinate values.

It's important to remember that the domain X and co-domain Y of a function f are integral parts of the function's definition. Quite often in mathematics a function f is defined by use of a "rule" of some kind, such as an algebraic equation or a verbal description, which expresses a relationship between x and y that must be satisfied in order for (x, y) to be eligible to be an element of f. Indeed in many textbooks one will find a statement to the effect that a function *is* a rule. In any event it is commonly left to the reader to choose an appropriate domain and co-domain in which to apply a given rule so as to obtain a well-defined function, but the context of the discussion usually makes the choice clear.

An example of a rule is to put numbers in correspondence with their square roots. That is, we can declare f to be a function that consists of ordered pairs having the property that the second coordinate, y, is the square root of the first coordinate, x, so that $(x, y) \in f$ implies that $(x, y) = (x, \sqrt{x})$. Algebraically the rule can be expressed by the equation $y = \sqrt{x}$, or even better we can employ function notation and write the rule as $f(x) = \sqrt{x}$. However, as stated this rule by itself does not *fully* determine a function because a domain and co-domain were not specified! The foremost question here is: do we want to permit square roots of negative numbers? If the answer is yes, then we can let $\mathrm{Dom}(f) = \mathbb{R}$ and obtain a function $f : \mathbb{R} \to \mathbb{C}$ which, for instance, contains ordered pairs like $(-4, \sqrt{-4}) = (-4, 2i)$ and $(-13, \sqrt{-13}) = (-13, i\sqrt{13})$.[3] If the answer is no, then we might let $\mathrm{Dom}(f) = [0, \infty)$ (i.e. we only permit $x \geq 0$) and obtain a function $f : [0, \infty) \to \mathbb{R}$ which does *not* contain pairs like $(-4, 2i)$ or $(-13, i\sqrt{13})$.

For the purposes of this course, whenever we're given a rule like $f(x) = \sqrt{x}$ from which to construct a function f, *we will always define the domain of f to be the largest set $X \subseteq \mathbb{R}$ such that $f(X) \subseteq \mathbb{R}$.* That is, given a rule f, we will take $\mathrm{Dom}(f)$ to be the set of all real numbers x such that the associated y value (where $y = f(x)$) is also a real number. Do you like set-builder notation? Then here you are:

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : f(x) \in \mathbb{R}\}. \tag{1.5}$$

It then follows that, for the range of f, we have

$$\mathrm{Ran}(f) = \{f(x) : x \in \mathrm{Dom}(f)\}. \tag{1.6}$$

Following this convention, a rule like $f(x) = \sqrt{x}$ *does* fully determine a function since we know to let $\mathrm{Dom}(f) = [0, \infty)$, and so we can safely adopt the viewpoint that a function *is* a rule. We've now encountered three interpretations of a function: a function is a set of ordered pairs, a function is a machine, and a function is a rule. In light of the machine interpretation and Definition 1.17, a function is a machine that returns no more than one output for each input.

Example 1.18. Find the domain and range of the function f given by the rule $f(x) = x^2 + 5$.

Solution. The function f will contain ordered pairs (x, y) such that $y = f(x)$, where in this case we're given $f(x) = \sqrt[4]{81 - x^2}$. Following our convention (1.5), we start with

$$\mathrm{Dom}(f) = \left\{x \in \mathbb{R} : \sqrt[4]{81 - x^2} \in \mathbb{R}\right\}$$

and ask: what can x be in order that $\sqrt[4]{81 - x^2}$ is a real number? For fourth roots, like with square roots, we need the radicand to be nonnegative: $81 - x^2 \geq 0$, so we need $-9 \leq x \leq 9$ and conclude that $\mathrm{Dom}(f) = [-9, 9]$.

[3] Recall that $\mathbb{C}$ is the set of complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

As for the range, by (1.6) $\mathrm{Ran}(f)$ is the set of all the values the expression $f(x) = \sqrt[4]{81 - x^2}$ can assume for $x \in [-9, 9]$. Notice that $f(x) = 0$ when $x = -9$ or $x = 9$, and peaks at a value of $f(0) = \sqrt[4]{81} = 3$ when $x = 0$. Thus, for $-9 \leq x \leq 9$ we find that $0 \leq f(x) \leq 3$, so $\mathrm{Ran}(f) = [0, 3]$.

To complete the ordered pair $(1, y)$, we note that $x = 1$ and therefore

$$y = f(1) = \sqrt[4]{81 - 1^2} = \sqrt[4]{80} = 2\sqrt[4]{5},$$

and so the pair must be $(1, 2\sqrt[4]{5})$.

Turning to the pair $(x, 2)$, we must find x such that $y = f(x) = 2$, or $\sqrt[4]{81 - x^2} = 2$. This equation implies $81 - x^2 = 2^4$, or $x^2 = 65$ and thus $x = \pm\sqrt{65}$. Therefore two possible ordered pairs result: $(-\sqrt{65}, 2)$ and $(\sqrt{65}, 2)$. ■

Example 1.19. Find the domain and range of the function f given by the rule $f(x) = \sqrt[4]{81 - x^2}$, then complete the ordered pairs $(1, y)$ and $(x, 2)$ so that they belong to f.

Solution. The function f will contain ordered pairs (x, y) such that $y = f(x)$, where here we're given $f(x) = \sqrt[4]{81 - x^2}$. Once again by (1.5), we start with

$$\mathrm{Dom}(f) = \left\{x \in \mathbb{R} : \sqrt[4]{81 - x^2} \in \mathbb{R}\right\}$$

and ask: what can x be in order that $\sqrt[4]{81 - x^2}$ is a real number? For fourth roots, like with square roots, we need the radicand to be nonnegative: $81 - x^2 \geq 0$, so we need $-9 \leq x \leq 9$ and conclude that $\mathrm{Dom}(f) = [-9, 9]$.

As for the range, by (1.6) $\mathrm{Ran}(f)$ is the set of all the values the expression $f(x) = \sqrt[4]{81 - x^2}$ can assume for $x \in [-9, 9]$. Notice that $f(x) = 0$ when $x = -9$ or $x = 9$, and peaks at a value of $f(0) = \sqrt[4]{81} = 3$ when $x = 0$. Thus, for $-9 \leq x \leq 9$ we find that $0 \leq f(x) \leq 3$, so $\mathrm{Ran}(f) = [0, 3]$.

To complete the ordered pair $(1, y)$, we note that $x = 1$ and therefore

$$y = f(1) = \sqrt[4]{81 - 1^2} = \sqrt[4]{80} = 2\sqrt[4]{5},$$

and so the pair must be $(1, 2\sqrt[4]{5})$.

Turning to the pair $(x, 2)$, we must find x such that $y = f(x) = 2$, or $\sqrt[4]{81 - x^2} = 2$. This equation implies $81 - x^2 = 2^4$, or $x^2 = 65$ and thus $x = \pm\sqrt{65}$. Therefore two possible ordered pairs result: $(-\sqrt{65}, 2)$ and $(\sqrt{65}, 2)$. ■

Section 1.4 Exercises

1. Let $f(x) = x^3 + 2x^2 - 3$.
   (a) Find $f(0)$, $f(3)$, and $f(-3)$
   (b) Find $f(1/a)$ and $f(-x)$
2. Let $g(t) = t - \dfrac{1}{t}$.
   (a) Find $g(1)$, $g(-1)$, and $g(3/2)$
   (b) Find $g(a+3)$ and $g(1/a)$

For #3 – 8, find the domain and range of the function.

3. $f(x) = x^2 + 1$ for $-4 \leq x \leq 3$
4. $f(x) = 2 - 3x$ for $-3 \leq x \leq 7$
5. $f(x) = \sqrt{7 - 4x}$
6. $f(x) = \sqrt{x^2 - 25}$
7. $f(x) = |x - 8| - 5$
8. $f(x) = 6/x$

For #9 – 22, find the domain of the function (*not* the range).

9. $f(x) = \sqrt[3]{x} + 13$
10. $g(x) = \sqrt{x} + 13$
11. $h(x) = \sqrt[4]{x - 6} - 9$
12. $j(x) = \sqrt{2x^2 + 5x - 3}$
13. $k(x) = \dfrac{x - 4}{x + 5}$
14. $\ell(x) = \dfrac{1}{x^2 + 6x - 27}$
15. $p(x) = \dfrac{x - 4}{x^2 - 16}$
16. $q(x) = \dfrac{x^2}{\sqrt{5 - x}}$
17. $r(x) = \dfrac{81}{\sqrt{x^2 + 2x - 3}} + 3x$
18. $s(x) = \sqrt[6]{\dfrac{x + 1}{x - 4}}$
19. $u(x) = \sqrt{\dfrac{x^2 - 3x}{x + 2}}$
20. $v(x) = \sqrt{x - 4} + \sqrt{x + 8}$
21. $w(x) = \sqrt{x - 4} + \sqrt{12 - x}$
22. $z(x) = \sqrt[3]{x - 2} + \sqrt[4]{x^2 - 9} + \sqrt[6]{2x - 1}$

1.5 – Function Combinations and Compositions

New functions can be built from old ones in many ways. Typically the old functions are common, simple functions that are put together to construct a more complex function that models some observed phenomenon. How such constructions are accomplished in the context of real-valued functions we make precise in this section, starting with the notions of taking sums, differences, products, and quotients of functions whose domains have nonempty intersections.

Definition 1.20. *Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be functions such that $X \cap Y \neq \varnothing$. Define functions $f + g, f - g, fg : X \cap Y \to \mathbb{R}$ by*

$$(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x), \quad (fg)(x) = f(x)g(x)$$

for all $x \in X \cap Y$.

Let $Z = \{x \in X \cap Y : g(x) \neq 0\}$, and define $f/g : Z \to \mathbb{R}$ by

$$(f/g)(x) = f(x)/g(x).$$

In addition to adding, subtracting, multiplying, and dividing functions, we also introduce function exponentiation.

Definition 1.21. *Let $f : X \to \mathbb{R}$ be a function. For any $n \in \mathbb{R}$ we define $f^n : X \to \mathbb{R}$ by*

$$f^n(x) = [f(x)]^n.$$

The arithmetic operations for functions given by Definitions 1.20 and 1.21 are collectively called function **combinations**. Another function combination that is perhaps the most important operation of all is function **composition**.

Definition 1.22. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. The **composition of** g **with** f is the function $g \circ f : X \to Z$ given by*

$$(g \circ f)(x) = g(f(x))$$

for all $x \in X$.

Generally if f and g are functions such that $\mathrm{Ran}(f) \subseteq \mathrm{Dom}(g)$, then it is possible to construct the composition $g \circ f$ and we will have

$$\mathrm{Dom}(g \circ f) = \mathrm{Dom}(f).$$

If $\mathrm{Ran}(f) \subseteq \mathrm{Dom}(g)$ is not the case, but it is still true that $\mathrm{Ran}(f) \cap \mathrm{Dom}(g) \neq \varnothing$, then it is still possible to construct $g \circ f$ by defining

$$\mathrm{Dom}(g \circ f) = \big\{x : x \in \mathrm{Dom}(f) \text{ and } f(x) \in \mathrm{Dom}(g)\big\}. \tag{1.7}$$

What we are doing in adopting this convention (which we shall strictly adhere to throughout this text) is restricting the domain of g to $\mathrm{Ran}(f) \cap \mathrm{Dom}(g)$.

If $\mathrm{Ran}(f) \cap \mathrm{Dom}(g) \neq \varnothing$, then $\mathrm{Dom}(g \circ f) \neq \varnothing$ also and we will say $g \circ f$ **exists**. If $\mathrm{Dom}(g \circ f) = \varnothing$ then $g \circ f$ does not exist.

Example 1.23. Let $f(x) = \sqrt{36 - x^2}$ and $g(x) = \sqrt{9x - 5}$. Find $g \circ f$ and its domain.

Solution. We have $\mathrm{Dom}(f) = [-6, 6]$ and $\mathrm{Dom}(g) = [5/9, \infty)$, and also

$$\mathrm{Ran}(f) = \{f(x) : x \in \mathrm{Dom}(f)\} = \left\{\sqrt{36 - x^2} : -6 \leq x \leq 6\right\} = [0, 6].$$

Since

$$\mathrm{Ran}(f) \cap \mathrm{Dom}(g) = [0, 6] \cap [5/9, \infty) \neq \varnothing,$$

it follows that $\mathrm{Dom}(g \circ f) \neq \varnothing$ and the function $g \circ f$ exists.

To find $g \circ f$ means to find an algebraic expression in x for $(g \circ f)(x)$. This is straightforward: assuming $x \in \mathrm{Dom}(g \circ f)$, we have

$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{36 - x^2}\right) = \sqrt{9\sqrt{36 - x^2} - 5}.$$

To find the domain of $g \circ f$, by (1.7) we have

$$\begin{aligned} \mathrm{Dom}(g \circ f) &= \{x : x \in \mathrm{Dom}(f) \ \&\ f(x) \in \mathrm{Dom}(g)\} \\ &= \left\{x : -6 \leq x \leq 6 \text{ and } \sqrt{36 - x^2} \geq \tfrac{5}{9}\right\} \\ &= \left\{x : -6 \leq x \leq 6 \text{ and } x^2 \leq \tfrac{2891}{81}\right\} \\ &= \left\{x : -6 \leq x \leq 6 \text{ and } -\tfrac{7\sqrt{59}}{9} \leq x \leq \tfrac{7\sqrt{59}}{9}\right\} \\ &= \left[-\tfrac{7\sqrt{59}}{9}, \tfrac{7\sqrt{59}}{9}\right], \end{aligned}$$

since $\frac{7\sqrt{59}}{9} \approx 5.97 < 6$. ∎

The composition of any number of functions may be effected, as the following more general definition makes precise.

Definition 1.24. *Let $f_i : X_i \to X_{i+1}$ be functions for $i = 1, 2, \ldots, n$. We define the **composition of** $f_1, \ldots, f_n$ to be the function*

$$f_n \circ \cdots \circ f_2 \circ f_1 : X_1 \to X_{n+1}$$

given by

$$(f_n \circ \cdots \circ f_2 \circ f_1)(x) = f_n(\cdots f_2(f_1(x)) \cdots)$$

for all $x \in X_1$.

In particular if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$, then

$$h \circ g \circ f : X \to W$$

is given by

$$(h \circ g \circ f)(x) = h(g(f(x)))$$

for all $x \in X$. If it is not the case that $\mathrm{Ran}(f) \subseteq \mathrm{Dom}(g)$ or $\mathrm{Ran}(g) \subseteq \mathrm{Dom}(h)$, then we adopt the convention that

$$\mathrm{Dom}(h \circ g \circ f) = \left\{x \in X : x \in \mathrm{Dom}(f),\ f(x) \in \mathrm{Dom}(g), \text{ and } g(f(x)) \in \mathrm{Dom}(h)\right\} \tag{1.8}$$

Example 1.25. Let $f(x) = \sqrt{10-x}$, $g(x) = 12/(x-9)$, and $h(x) = x/(x+6)$. Find $h \circ g \circ f$ and its domain.

Solution. For any x in the domain of $h \circ g \circ f$,

$$(h \circ g \circ f)(x) = h(g(f(x))) = h\big(g\big(\sqrt{10-x}\,\big)\big) = h\left(\frac{12}{\sqrt{10-x}-9}\right)$$

$$= \frac{\dfrac{12}{\sqrt{10-x}-9}}{\dfrac{12}{\sqrt{10-x}-9}+6} = \frac{12}{12+6(\sqrt{10-x}-9)} = \frac{12}{6\sqrt{10-x}-42}.$$

To actually find the domain of $h \circ g \circ f$ we first obtain the domains of f, g, and h:

$$\mathrm{Dom}(f) = (-\infty, 10],$$

$$\mathrm{Dom}(g) = (-\infty, 9) \cup (9, \infty),$$

$$\mathrm{Dom}(h) = (-\infty, -6) \cup (-6, \infty).$$

By (1.8) we have

$$\mathrm{Dom}(h \circ g \circ f) = \big\{x \in X : x \leq 10,\ f(x) \neq 9, \text{ and } g(f(x)) \neq -6\big\}, \tag{1.9}$$

where $f(x) \neq 9$ is equivalent to $f(x) \in (-\infty, 9) \cup (9, \infty)$, and $g(f(x)) \neq -6$ is equivalent to $g(f(x)) \in (-\infty, -6) \cup (-6, \infty)$. Now,

$$f(x) \neq 9 \quad \Leftrightarrow \quad \sqrt{10-x} \neq 9 \quad \Leftrightarrow \quad 10-x \neq 81 \quad \Leftrightarrow \quad x \neq -71, \tag{1.10}$$

and

$$g(f(x)) \neq -6 \quad \Leftrightarrow \quad \frac{12}{\sqrt{10-x}-9} \neq -6 \quad \Leftrightarrow \quad 12 \neq -6\big(\sqrt{10-x}-9\big)$$

$$\Leftrightarrow \quad \sqrt{10-x} \neq 7 \quad \Leftrightarrow \quad 10-x \neq 49 \quad \Leftrightarrow \quad x \neq -39. \tag{1.11}$$

Putting the results of (1.10) and (1.11) into (1.9) yields

$$\mathrm{Dom}(h \circ g \circ f) = \big\{x \in X : x \leq 10,\ x \neq -71, \text{ and } x \neq -39\big\}$$

$$= (-\infty, -71) \cup (-71, -39) \cup (-39, 10]$$

as the domain. ∎

Section 1.5 Exercises

For #1 – 6, find $f+g$, $f-g$, fg, f/g, and their domains.

1. $f(x)=\sqrt{1-x}, \quad g(x)=\dfrac{1}{x-2}$
2. $f(x)=\sqrt{10+x}, \quad g(x)=\sqrt{50-x}$
3. $f(x)=\sqrt{9-x^2}, \quad g(x)=\sqrt{x^2-1}$
4. $f(x)=\dfrac{2}{x+2}, \quad g(x)=\dfrac{x}{x+2}$
5. $f(x)=\dfrac{1}{\sqrt{2x-3}}, \quad g(x)=3x^2-8$
6. $f(x)=\sqrt[6]{3-x}, \quad g(x)=\sqrt[4]{x-5}$

For #7 – 12, find $f\circ g$, $g\circ f$, $f\circ f$, $g\circ g$, and their domains.

7. $f(x)=3x^2-7, \quad g(x)=x+5$
8. $f(x)=\sqrt{x-3}, \quad g(x)=x^2$
9. $f(x)=\dfrac{1}{x-1}, \quad g(x)=\dfrac{x-1}{x+1}$
10. $f(x)=\sqrt[3]{x}, \quad g(x)=1-\sqrt{x}$
11. $f(x)=\sqrt{x^2-4}, \quad g(x)=\sqrt{2-x}$
12. $f(x)=\dfrac{1}{\sqrt[4]{x}}, \quad g(x)=x^2-4x$

For #13 – 14, find $f\circ g\circ h$ and its domain.

13. $f(x)=\sqrt{x-2}, \quad g(x)=\sqrt[4]{x-1}, \quad h(x)=\sqrt[3]{x+3}$
14. $f(x)=\sqrt{2x}, \quad g(x)=\dfrac{x}{x-1}, \quad h(x)=\sqrt[5]{x}$

For #15 – 18, find simple functions that do the job of the complex function.

15. $H(x)=(x-8)^4$. Find functions f and g so that $f\circ g=H$.
16. $L(x)=\dfrac{1}{5x-3}$. Find functions f and g so that $f\circ g=L$.
17. $\Phi(x)=\sqrt[3]{\sqrt{x}-1}$. Find functions f, g and h so that $f\circ g\circ h=\Phi$.
18. $W(x)=\dfrac{9}{(4-\sqrt{x})^2}$. Find functions f, g and h so that $f\circ g\circ h=W$.

1.6 – Mathematical Proofs: Tools and Techniques

Throughout these notes there are many proofs, and a fair number of the proofs juggle absolute values and inequalities. One of the simplest inequalities involving absolute value is

$$-|t| \leq t \leq |t| \tag{1.12}$$

for any $t \in \mathbb{R}$. Indeed if $t \geq 0$ then $t = |t|$, and if $t < 0$ then $t = -(-t) = -|t|$, so in fact $|t| = \pm|t|$ depending on whether t is nonnegative or negative.

Another inequality featuring absolute value, arguably the most useful of them all, is the **Triangle Inequality**. In words it amounts to a simple fact: the sum of the lengths of the two shortest sides of a triangle is less than the length of the longest side. Algebraically this fact is rendered as follows.

Proposition 1.26 (**Triangle Inequality**). *For any* $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|.$$

Proof. Let $x, y \in \mathbb{R}$. With (1.12) we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$, and adding these results yields

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Therefore $|x + y| \leq |x| + |y|$. ∎

A slight variant of the Triangle Inequality is

$$|x - y| \leq |x| + |y|$$

for any $x, y \in \mathbb{R}$. This follows immediately from the proposition above and the fact that $|-t| = |t|$ for any $t \in \mathbb{R}$:

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|.$$

A more significant variant is the following.

Proposition 1.27 (**Reverse Triangle Inequality**). *For any* $x, y \in \mathbb{R}$,

$$\big||x| - |y|\big| \leq |x - y|.$$

Proof. Let $x, y \in \mathbb{R}$. Using the Triangle Inequality, we obtain the inequalities

$$|x| = |(x - y) + y| \leq |x - y| + |y|$$

and

$$|y| = |(y - x) + x| \leq |y - x| + |x| = |x - y| + |x|,$$

which become

$$|x| - |y| \leq |x - y| \quad \text{and} \quad |x| - |y| \geq -|x - y|,$$

respectively, so that

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

and therefore $||x| - |y|| \leq |x - y|$. ∎

Combining our findings, we have

$$|x| - |y| \leq \big||x| - |y|\big| \leq |x - y| \leq |x| + |y|.$$

The Triangle Inequality has the following generalization: for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|,$$

or in sigma notation

$$\left|\sum_{k=1}^{n} x_k\right| \leq \sum_{k=1}^{n} |x_k|.$$

2

Limits and Continuity

2.1 – Neighborhoods and Limit Points

Given a point $c \in \mathbb{R}$, a **neighborhood** of c is any open interval $I \subseteq \mathbb{R}$ that contains c. Thus, for any $a < c$ and $b > c$, the interval (a, b) is a neighborhood of c. Other neighborhoods of c are $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$. In fact, $\mathbb{R}$ itself qualifies as a neighborhood for any real number, albeit an uninteresting one.

A common way to construct a neighborhood for a real number c is to designate some $\gamma > 0$, usually regarded as being quite small, and consider the interval $(c - \gamma, c + \gamma)$. This is the open interval with **center** c and **radius** γ, which we will frequently denote by the symbol $B_\gamma(c)$ and call the γ-neighborhood of c; that is,

$$B_\gamma(c) = (c - \gamma, c + \gamma)$$

for any $\gamma > 0$.

A **deleted neighborhood** of $c \in \mathbb{R}$ is a neighborhood of c with c removed. Thus, if (a, b) is a neighborhood of c for any $-\infty \leq a < c$ and $c < b \leq \infty$, then $(a, c) \cup (c, b)$ is a deleted neighborhood of c, where of course $(a, c) \cup (c, b)$ is the set of all real numbers between a and b except for c:

$$(a, c) \cup (c, b) = \{x : a < x < b \text{ and } x \neq c\}.$$

Removing c from a γ-neighborhood $(c - \gamma, c + \gamma)$ of c produces what we shall call a deleted γ-neighborhood of c:

$$(c - \gamma, c) \cup (c, c + \gamma) = \{x : c - \gamma < x < c + \gamma \text{ and } x \neq c\}.$$

It will be convenient to use the significantly more compact symbol $B'_\gamma(c)$ to denote the deleted γ-neighborhood of c:

$$B'_\gamma(c) = (c - \gamma, c) \cup (c, c + \gamma).$$

Observe that $x \in B'_\gamma(c)$ if and only if $0 < |x - c| < \gamma$.

Definition 2.1. *A point $x \in \mathbb{R}$ is a **limit point** of a set $S \subseteq \mathbb{R}$ if*

$$B'_\gamma(x) \cap S \neq \varnothing$$

for all $\gamma > 0$.

For example, a is a limit point of the interval (a, b). This is because no matter how small $\gamma > 0$ is, there are points in (a, b) that lie between a and $a + \gamma$, and hence

$$\begin{aligned} B'_\gamma(a) \cap (a,b) &= \big[(a-\gamma, a) \cup (a, a+\gamma)\big] \cap (a,b) \\ &= \big[(a-\gamma, a) \cap (a,b)\big] \cup \big[(a, a+\gamma) \cap (a,b)\big] \\ &= (a, a+\gamma) \cap (a,b) \neq \varnothing \end{aligned}$$

for all $\gamma > 0$. Similarly, b is a limit point since for any $\gamma > 0$, no matter how small, there are points in (a, b) that lie between $b - \gamma$ and b, and so

$$B'_\gamma(b) \cap (a,b) \neq \varnothing$$

for all $\gamma > 0$. In fact, every $x \in (a, b)$ is a limit points of (a, b). Put another way, the set of limit points of the open interval (a, b) is the closed interval $[a, b]$.

Example 2.2. Recall the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, otherwise known as the set of positive integers. Let S be the set consisting of the reciprocals of the natural numbers:

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

This set has at least one limit point: the number 0. To see this, note that for any $\gamma > 0$ we can find a sufficiently large natural number N so that $1/N < \gamma$, and hence $1/N \in B'_\gamma(0) \cap S$. That is, $B'_\gamma(0) \cap S \neq \varnothing$ for all $\gamma > 0$, and therefore 0 is a limit point of S.

Does S have any other limit points? Can 1 be a limit point? No, since the deleted neighborhood $B'_{0.1}(1) = (0.9, 1) \cup (1, 1.1)$ contains no elements of S. Now consider the number $1/n$ for some integer $n \geq 2$. We find that

$$\frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1},$$

where all the values in the inequality are elements of S, and no elements of S exist between $1/(n + 1)$ and $1/n$ or between $1/n$ and $1/(n - 1)$. Thus there exists a sufficiently small $\gamma > 0$ such that $B'_\gamma(1/n) \cap S = \varnothing$,[4] which shows that $1/n$ is not a limit point of S. Similar arguments can be made to show that no real number $x > 0$ can be a limit point of S, nor can any real $x < 0$ be a limit point. Therefore 0 is the only limit point of S. ■

[4] In fact we could choose to let γ be half the distance between $1/n$ and $1/(n + 1)$:

$$\gamma = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

2.2 – The Limit of a Function

Throughout this chapter we assume that any function f is real-valued, and has domain D that is a subset of the set of real numbers $\mathbb{R}$; that is, $f : D \to \mathbb{R}$ for some $D \subseteq \mathbb{R}$.

Definition 2.3. *Let f be a real-valued function, and let $c \in \mathbb{R}$ be a limit point of* $\mathrm{Dom}(f)$*. Given $L \in \mathbb{R}$, we say f has* ***limit*** *L at c, written*

$$\lim_{x\to c} f(x) = L,$$

if for each $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Remark. Throughout these notes, whenever we say that a limit $\lim_{x\to c} f(x)$ "exists," we mean there is some $L \in \mathbb{R}$ such that $\lim_{x\to c} f(x) = L$. Otherwise we say that the limit "does not exist."

Notation. A notational alternative to $\lim_{x\to c} f(x) = L$ is to write "$f(x) \to L$ as $x \to c$."

Example 2.4. Prove that

$$\lim_{x\to 4}(2x + 1) = 9.$$

Preliminary Analysis. By Definition 2.3 we must show that, for any $\epsilon > 0$, there is some $\delta > 0$ such that $0 < |x - 4| < \delta$ implies

$$\big|(2x + 1) - 9\big| < \epsilon, \tag{2.1}$$

or equivalently (simplifying the left-hand side),

$$|2x - 8| < \epsilon. \tag{2.2}$$

However, since $|2x - 8| = 2|x - 4|$, we can rewrite (2.2) as $|x - 4| < \epsilon/2$. Working backward, we see that $|x - 4| < \epsilon/2$ leads to (2.2), which in turn becomes (2.1). Thus, if we assume x is such that $0 < |x - 4| < \delta$ for $\delta = \epsilon/2$, then (2.1) necessarily follows. We now proceed with the formal proof with this choice for δ in mind. ∎

Proof. Let $\epsilon > 0$. Choose $\delta = \epsilon/2$ and suppose x is such that $0 < |x-4| < \delta$. Then $|x-4| < \epsilon/2$, and since

$$|x - 4| < \frac{\epsilon}{2} \quad \Rightarrow \quad |2x - 8| < \epsilon \quad \Rightarrow \quad \big|(2x + 1) - 9\big| < \epsilon,$$

the proof is done. ∎

Example 2.5. Prove that

$$\lim_{x\to 2}(x^2 + x - 12) = -6.$$

Preliminary Analysis. We must show that, for any $\epsilon > 0$, there's some $\delta > 0$ such that $0 < |x - 2| < \delta$ implies

$$\left|(x^2 + x - 12) - (-6)\right| < \epsilon,$$

or equivalently (simplifying the left-hand side),

$$|x - 2||x + 3| < \epsilon. \tag{2.3}$$

First consider what would happen if we just assumed $0 < |x - 2| < 1$ (so provisionally we're letting $\delta = 1$). Since

$$|x - 2| < 1 \;\;\Rightarrow\;\; -1 < x - 2 < 1 \;\;\Rightarrow\;\; 4 < x + 3 < 6,$$

the assumption $0 < |x - 2| < 1$ implies that $|x + 3| < 6$. Notice that $|x + 3|$ is one of the factors at left in (2.3). Of course, the other factor at left in (2.3) is $|x - 2|$, and if we were to make the assumption that $0 < |x - 2| < \epsilon/6$ in addition to $0 < |x - 2| < 1$, then we would obtain

$$\left|(x^2 + x - 12) - (-6)\right| = |x - 2||x + 3| < \frac{\epsilon}{6} \cdot 6 = \epsilon$$

as desired.

Thus we need δ chosen such that having $0 < |x - 2| < \delta$ implies $|x - 2|$ is less than both 1 and $\epsilon/6$ simultaneously. So what should δ be? The *smaller* of the two quantities 1 and $\epsilon/6$! We indicate this by writing $\delta = \min\{1, \epsilon/6\}$, which can be read as "$\delta$ is the minimum element of the set $\{1, \epsilon/6\}$." We are now ready to write a formal proof. ■

Proof. Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/6\}$ and suppose that $0 < |x-2| < \delta$. Then $|x-2| < \epsilon/6$ is immediate, and from $|x - 2| < 1$ we have $|x + 3| < 6$ as shown in the preliminary analysis. Now,

$$\left|(x^2 + x - 12) - (-6)\right| = |x - 2||x + 3| < \frac{\epsilon}{6} \cdot 6 = \epsilon,$$

and the proof is done. ■

Example 2.6. The proof that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

is relatively straightforward. Let $\epsilon > 0$, and choose $\delta = \epsilon$. Suppose x is such that $0 < |x| < \delta$. Then $|x| < \epsilon$ with $x \neq 0$, and we obtain

$$\left|x \sin\left(\frac{1}{x}\right)\right| = |x| \left|\sin\left(\frac{1}{x}\right)\right| \le |x| < \epsilon,$$

since $|\sin(1/x)| \le 1$. This finishes the proof. ■

Definition 2.3 is the definition of the limit of a real-valued function of a single real variable that we will use throughout these notes, and it is in agreement with the definition of limit found in more advanced texts on the subject of mathematical analysis. Unfortunately, Definition 2.3 is *not* consonant with the definition of limit found in most mainstream introductory calculus texts. In such texts the meaning of

$$\lim_{x\to c} f(x) = L$$

is, in fact, rather more complicated by virtue of having one more component: namely, a requirement that a deleted neighborhood of c be a subset of the domain of f. That is, introductory calculus texts state that there must exist some $\gamma > 0$ such that $B'_\gamma(c) \subseteq \mathrm{Dom}(f)$, otherwise $\lim_{x\to c} f(x)$ does not exist. Not only does this make for a more cumbersome and restrictive concept of limit than the one given here, it also often results in longer and more awkward proofs of otherwise simple theorems.

Fortunately, nearly every function of a single real variable that is encountered in a basic calculus course will have a domain that is either an interval of real numbers or a disjoint union of intervals. For such functions our definition of $\lim_{x\to c} f(x)$ and the definition given in mainstream texts will yield identical results for each $c \in \mathbb{R}$ for which *neither* of the following is true:

1. There exists $\gamma > 0$ such that $(c, c+\gamma) \subseteq \mathrm{Dom}(f)$ and $(c-\gamma, c) \cap \mathrm{Dom}(f) = \varnothing$.
2. There exists $\gamma > 0$ such that $(c-\gamma, c) \subseteq \mathrm{Dom}(f)$ and $(c, c+\gamma) \cap \mathrm{Dom}(f) = \varnothing$.

The occurrence of either Case (1) or Case (2) will automatically result in $\lim_{x\to c} f(x)$ not existing according to the definition of most calculus texts, but depending on the behavior of f the limit concept given by Definition 2.3 might exist. To address either Case (1) or (2), mainstream texts consider what is known as a **one-sided** limit. Specifically, to treat Case (1) or Case (2), texts employ respectively a right-hand or left-hand limit. We shall generally observe the same practice in these notes as well, not only in the interests of academic harmony, but also since there are situations in which the limit of Definition 2.3 fails to exists, and yet one or the other of the one-sided limits defined as follows does exist.

Definition 2.7 (Right-Hand Limit). *Let f be a real-valued function, and let $c \in \mathbb{R}$ be such that $\mathrm{Dom}(f) \cap (c, c+\gamma) \neq \varnothing$ for all $\gamma > 0$. Given $L \in \mathbb{R}$, we say f has **right-hand limit** L at c, written*

$$\lim_{x\to c^+} f(x) = L,$$

if for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$c < x < c + \delta \;\;\Rightarrow\;\; |f(x) - L| < \epsilon.$$

Definition 2.8 (Left-Hand Limit). *Let f be a real-valued function, and let $c \in \mathbb{R}$ be such that $\mathrm{Dom}(f) \cap (c-\gamma, c) \neq \varnothing$ for all $\gamma > 0$. Given $L \in \mathbb{R}$, we say f has **left-hand limit** L at c, written*

$$\lim_{x\to c^-} f(x) = L,$$

if for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$c - \delta < x < c \;\;\Rightarrow\;\; |f(x) - L| < \epsilon.$$

Notation. The limit $\lim_{x\to c^+} f(x) = L$ can be written as "$f(x) \to L$ as $x \to c^+$," and the limit $\lim_{x\to c^-} f(x) = L$ can be written as "$f(x) \to L$ as $x \to c^-$."

Theorem 2.9. *If*

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L \tag{2.4}$$

for some $L \in \mathbb{R}$, then

$$\lim_{x\to c} f(x) = L.$$

Proof. Suppose that (2.4) holds. Let $\epsilon > 0$. There exists some $\delta_1 > 0$ such that, for all $x \in \mathrm{Dom}(f)$,

$$c - \delta_1 < x < c \quad \Rightarrow \quad |f(x) - L| < \epsilon;$$

and also there exists some $\delta_2 > 0$ such that, for all $x \in \mathrm{Dom}(f)$,

$$c < x < c + \delta_2 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Suppose $x \in \mathrm{Dom}(f)$ is such that $0 < |x - c| < \delta$. Then either $c - \delta < x < c$ or $c < x < c + \delta$ must be the case, and since the former implies that $c - \delta_1 < x < c$, and the latter implies that $c < c < c + \delta_2$, it follows that $|f(x) - L| < \epsilon$. Therefore $\lim_{x\to c} f(x) = L$. ■

Remark. The converse of Theorem 2.9 is not true in general. That is, if $\lim_{x\to c} f(x) = L$, then it is not necessarily the case that both one-sided limits will equal L. For example we have $\lim_{x\to 0} \sqrt{x} = 0$, and while it is the case that $\lim_{x\to 0^+} \sqrt{x} = 0$ also, it turns out that $\lim_{x\to 0^-} \sqrt{x}$ does not exist! This is because the domain of $\sqrt{x}$ is $[0, \infty)$, and so the requirement in Definition 2.8 that $[0, \infty) \cap (-\gamma, 0) \neq \varnothing$ for all $\gamma > 0$ is not satisfied.

Example 2.10. Let f be a function with graph as depicted in Figure 2. Whereas $f(1) = 2$, from the graph we find that

$$\lim_{x\to 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 4,$$

and so since a limit can equal at most one real number it follows that $\lim_{x\to 1} f(x)$ does not exist. On the other hand we have $\lim_{x\to -1} f(x) = 3$ even though $f(-1)$ is undefined, and also $\lim_{x\to -2} f(x) = 1$ even though $f(-2) = 2$. Finally, we have $\lim_{x\to 3^-} f(x) = 1$ while $\lim_{x\to 3^+} f(x)$ does not exist. ■

Example 2.11. The limit

$$\lim_{x\to 0} \cos\left(\frac{1}{x}\right) \tag{2.5}$$

does not exist, which is to say it does not equal any real number. This will now be shown by constructing a proof by contradiction.

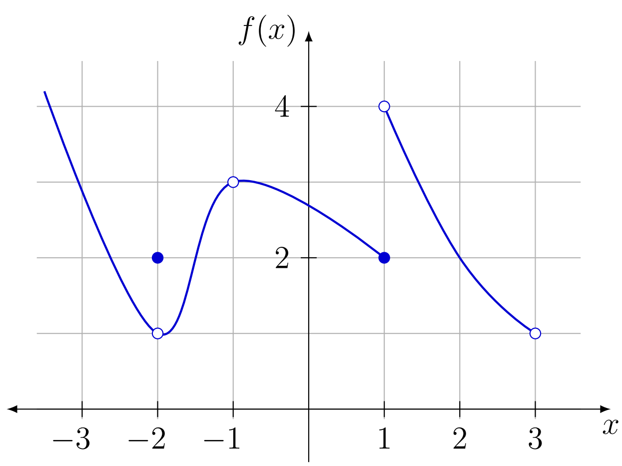


FIGURE 2.

Suppose that there is some $L \in \mathbb{R}$ such that $\cos(1/x) \to L$ as $x \to 0$. By definition this means that, for every $\epsilon > 0$, there exists some $\delta > 0$ such that, if $0 < |x| < \delta$, then

$$|\cos(1/x) - L| < \epsilon.$$

Thus, if we let $\epsilon = 1/2$, then there's some $\delta > 0$ such that $0 < |x| < \delta$ implies

$$|\cos(1/x) - L| < 1/2.$$

Now, let $n \in \mathbb{R}$ be sufficiently large so that $1/n < \delta$, in which case

$$0 < \frac{1}{2n\pi} < \delta \quad \text{and} \quad 0 < \frac{1}{(2n+1)\pi} < \delta,$$

and hence

$$\Big|\cos(2n\pi) - L\Big| < \frac{1}{2} \quad \text{and} \quad \Big|\cos\big((2n+1)\pi\big) - L\Big| < \frac{1}{2}.$$

Since $\cos(2n\pi) = \cos 0 = 1$ and

$$\cos\big((2n+1)\pi\big) = \cos(\pi) = -1,$$

we next obtain

$$|1 - L| < \frac{1}{2} \quad \text{and} \quad |-1 - L| < \frac{1}{2},$$

which we can write as $|L-1| < 1/2$ and $|L+1| < 1/2$. From $|L-1| < 1/2$ comes $1/2 < L < 3/2$, and from $|L+1| < 1/2$ comes $-3/2 < L < -1/2$. Both of these double inequalities must be satisfied simultaneously, which clearly is impossible no matter what real number L is. Thus there can be no $L \in \mathbb{R}$ such that $\cos(1/x) \to L$ as $x \to 0$.

Therefore the limit (2.5) does not exist. ■

Exercises

Use the definition of a limit, Definition 2.3, to prove the following limits.

1. $\lim_{x \to 8} 3x = 24$.

2. $\lim_{x \to 4}(5x - 3) = 17$.

3. $\lim_{x \to -2}(\frac{1}{2}x + 1) = 0$.

4. $\lim_{x \to 3}(17 - 7x) = -4$.

5. $\lim_{x \to -1} \dfrac{2x^2 - x - 3}{x + 1} = -5$.

6. $\lim_{x \to -3}(x^2 + 6x + 12) = 3$.

7. $\lim_{x \to 5}(x^2 - 3x + 1) = 11$.

8. $\lim_{x \to 6} \dfrac{2}{x} = \dfrac{1}{3}$.

2.3 – Properties of Limits

We now lay out and prove a theorem that gives some of the general properties of two-sided limits. The properties carry over without change to one-sided limits, with proofs that are much the same.

Theorem 2.12 (**Laws of Limits**). *Suppose $a, c \in \mathbb{R}$. If*

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M$$

for some $L, M \in \mathbb{R}$, then the following hold.

1. $\lim_{x\to c} a = a$
2. $\lim_{x\to c} af(x) = a \lim_{x\to c} f(x)$
3. $\lim_{x\to c} \big[f(x) \pm g(x)\big] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
4. $\lim_{x\to c} \big[f(x)g(x)\big] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
5. *Provided that $\lim_{x\to c} g(x) \neq 0$,*
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$
6. *For any integer $n > 0$,*
$$\lim_{x\to c} [f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n.$$
7. *For any integer $m > 0$,*
$$\lim_{x\to c} \sqrt[m]{f(x)} = \sqrt[m]{\lim_{x\to c} f(x)},$$
provided there exists $\gamma > 0$ such that $f(x) \geq 0$ for all $x \in B'_\gamma(c) \cap \mathrm{Dom}(f)$ if m is even.

Proof.
Proof of Law (1). Let $\epsilon > 0$. We can choose $\delta = 1$, and then, supposing that $0 < |x - c| < 1$, we find immediately that $|a - a| = 0 < \epsilon$.

Proof of Law (2). If $a = 0$, then

$$\lim_{x\to c} 0 \cdot f(x) = \lim_{x\to c} 0 = 0 = 0 \cdot \lim_{x\to c} f(x)$$

by Law (1). Assume that $a \neq 0$. Let $\epsilon > 0$. Since $\epsilon/|a| > 0$ and $\lim_{x\to c} f(x) = L$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon/|a|$. Now,

$$|a| \cdot |f(x) - L| < \frac{\epsilon}{|a|} \cdot |a|.$$

implies that $|af(x) - aL| < \epsilon$.

Proof of Law (3). Let $\epsilon > 0$. Then there is some $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies that $|f(x) - L| < \epsilon/2$, and there is some $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies that

$|g(x) - M| < \epsilon/2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Suppose that $0 < |x - c| < \delta$. Then, by the Triangle Inequality (see §1.6),

$$\begin{aligned} \big|(f(x) + g(x)) - (L + M)]\big| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and

$$\begin{aligned} \big|(f(x) - g(x)) - (L - M)]\big| &= |(f(x) - L) + (M - g(x))| \\ &\le |f(x) - L| + |M - g(x)| \\ &= |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Proof of Law (4). Let $\epsilon > 0$. There exists a $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < 1$, a $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies

$$|f(x) - L| < \frac{\epsilon/2}{|M| + 1},$$

and a $\delta_3 > 0$ such that $0 < |x - c| < \delta_3$ implies

$$|g(x) - M| < \frac{\epsilon/2}{|L| + 1}.$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Suppose that $0 < |x - c| < \delta$. From $|f(x) - L| < 1$ we obtain $|f(x)| < |L| + 1$, and thus

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + M(f(x) - L)| \\ &\le |f(x)| \cdot |g(x) - M| + |M| \cdot |f(x) - L| \\ &< (|L| + 1) \cdot \frac{\epsilon/2}{|L| + 1} + |M| \cdot \frac{\epsilon/2}{|M| + 1} \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Proof of Law (5). First it will be proved that $\lim_{x\to c} 1/g(x) = 1/M$ for $M \neq 0$. Let $\epsilon > 0$. Since $\lim_{x\to c} g(x) = M$ there exists some $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \quad\Rightarrow\quad |g(x) - M| < \frac{|M|}{2},$$

which in turn implies $|M| - |g(x)| < |M|/2$ and hence $|g(x)| > |M|/2 > 0$. Also there exists some $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \quad\Rightarrow\quad |g(x) - M| < \frac{M^2}{2}\epsilon.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Suppose that $0 < |x - c| < \delta$. Then

$$|g(x) - M| < \frac{\epsilon M^2}{2} = \epsilon|M| \cdot \frac{|M|}{2} < \epsilon|M| \cdot |g(x)|,$$

so, since $|g(x)| > 0$,

$$\left|\frac{g(x)-M}{M\cdot g(x)}\right| < \epsilon,$$

and finally

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon.$$

Therefore $\lim_{x\to c} 1/g(x) = 1/M$, and since $\lim_{x\to c} f(x) = L$, by Law (4) we obtain

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\left[f(x)\cdot\frac{1}{g(x)}\right] = L\cdot\frac{1}{M} = \frac{L}{M}.$$

Proof of Law (6). This law is easily proven using Law (4) and induction. The base case, when $n = 1$, is already given:

$$\lim_{x\to c}[f(x)]^1 = \lim_{x\to c} f(x) = L = L^1.$$

Now let n be an arbitrary positive integer and suppose that $\lim_{x\to c}[f(x)]^n = L^n$. Then

$$\lim_{x\to c}[f(x)]^{n+1} = \lim_{x\to c}[f(x)\cdot(f(x))^n] = \lim_{x\to c} f(x)\cdot\lim_{x\to c}[f(x)]^n = L\cdot L^n = L^{n+1}$$

by Law (4). ■

To prove Law (7) we need Proposition 2.41 below, along with the fact that $\lim_{x\to c}\sqrt[m]{x} = \sqrt[m]{c}$ for any integer $m > 0$, where $c \in (-\infty,\infty)$ if m is odd and $c \in (0,\infty)$ if m is even. The latter fact isn't proved until Chapter 7, so the definitive proof of Law (7) will have to wait until then.

Exercise. What is wrong with the following "proof" of Law (7)?

Let $m > 0$ be an integer, and assume $f(x) \geq 0$ for all x near c if m is even. Now, employing Law (6), we have

$$L = \lim_{x\to c} f(x) = \lim_{x\to c}\left[\sqrt[m]{f(x)}\right]^m = \left[\lim_{x\to c}\sqrt[m]{f(x)}\right]^m.$$

Now, taking mth roots throughout, we obtain

$$\sqrt[m]{L} = \sqrt[m]{\left[\lim_{x\to c}\sqrt[m]{f(x)}\right]^m} = \lim_{x\to c}\sqrt[m]{f(x)}$$

as desired. ■

Corollary 2.13. *Let* $\lim_{x\to c^+} f(x) = L$ *and* $\lim_{x\to c^+} g(x) = M$ *for some* $L, M \in \mathbb{R}$. *Then Laws (1)–(6) of Theorem 2.12 all hold with* $\lim_{x\to c}$ *replaced with* $\lim_{x\to c^+}$. *Law (7) changes as follows:*

$$\lim_{x\to c^+}\sqrt[m]{f(x)} = \sqrt[m]{L} = \sqrt[m]{\lim_{x\to c^+} f(x)}$$

for any integer $m > 0$, *provided there exists some* $\gamma > 0$ *such that* $f(x) \geq 0$ *for all* $x \in (c, c+\gamma)\cap\mathrm{Dom}(f)$ *if* m *is even.*

Corollary 2.14. *Let* $\lim_{x\to c^-} f(x) = L$ *and* $\lim_{x\to c^-} g(x) = M$ *for some* $L, M \in \mathbb{R}$*. Then Laws (1)–(6) of Theorem 2.12 all hold with* $\lim_{x\to c}$ *replaced with* $\lim_{x\to c^-}$*. Law (7) changes as follows:*

$$\lim_{x\to c^-} \sqrt[m]{f(x)} = \sqrt[m]{L} = \sqrt[m]{\lim_{x\to c^-} f(x)}$$

for any integer $m > 0$*, provided there exists some* $\gamma > 0$ *such that* $f(x) \geq 0$ *for all* $x \in (c-\gamma, c) \cap \operatorname{Dom}(f)$ *if* m *is even.*

Lemma 2.15. *For any* $c \in \mathbb{R}$, $\lim_{x\to c} x = c$.

Proof. Let $\epsilon > 0$. Choose $\delta = \epsilon$ and suppose that $0 < |x - c| < \delta$. Then we obtain $|x - c| < \epsilon$ and we're done. ∎

Proposition 2.16. *Suppose that* f *and* g *are polynomial functions. If* $c \in \mathbb{R}$*, then*

1. $\lim\limits_{x\to c} f(x) = f(c)$
2. $\lim\limits_{x\to c} f(x)/g(x) = f(c)/g(c)$ *if* $g(c) \neq 0$.

Proof. (1) Since f is a polynomial function there exists some integer $n \geq 0$ and real numbers $a_0, \ldots, a_n$ such that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for all $x \in \mathbb{R}$.

Let $c \in \mathbb{R}$ be arbitrary. Referencing the laws given by Theorem 2.12,

$$\begin{aligned}
f(c) &= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 \\
&= a_n \left(\lim_{x\to c} x\right)^n + a_{n-1}\left(\lim_{x\to c} x\right)^{n-1} + \cdots + a_1 \lim_{x\to c} x + a_0 && \text{(Lemma 2.15)} \\
&= a_n \lim_{x\to c} x^n + a_{n-1} \lim_{x\to c} x^{n-1} + \cdots + a_1 \lim_{x\to c} x + a_0 && \text{(Law 6)} \\
&= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \cdots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 && \text{(Laws 1,2)} \\
&= \lim_{x\to c} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) && \text{(Law 3)} \\
&= \lim_{x\to c} f(x).
\end{aligned}$$

This proves part (1) of the theorem.[5]

(2) Suppose that $g(c) \neq 0$. Since $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$ by part (1), we obtain

$$\frac{f(c)}{g(c)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} = \lim_{x\to c} \frac{f(x)}{g(x)},$$

where the second equality follows from Law 5. This proves part (2) of the theorem. ∎

[5] A common practice is to execute the steps in the reverse order, which is a flawed approach since we would possess no *a priori* knowledge of whether the limits we are working with actually exist, and so use of Theorem 2.12 would not be justified.

Proposition 2.16 enables us to evaluate polynomial and rational functions by "direct substitution," without having to craft an ϵ-δ argument as in the previous section. This greatly streamlines our labors, as the next example demonstrates when compared with the way the same limit was treated in Example 2.5

Example 2.17. Evaluate

$$\lim_{x\to 2}(x^2 + x - 12).$$

Solution. By Proposition 2.16 we substitute 2 for x to obtain

$$\lim_{x\to 2}(x^2 + x - 12) = 2^2 + 2 - 12 = -6,$$

and we're done. ■

The following theorem will enable us to evaluate some limits $\lim_{x\to c} f(x)$ when c is *not* in the domain of f.

Theorem 2.18. *Suppose* $\lim_{x\to c}\varphi(x) = L$ *for some* $c, L \in \mathbb{R}$. *If* c *is a limit point of* $\mathrm{Dom}(f)$, *and there exists some* $\gamma > 0$ *such that* $f(x) = \varphi(x)$ *for all* $x \in \mathrm{Dom}(f) \cap B'_\gamma(c)$, *then* $\lim_{x\to c} f(x) = L$.

Proof. Let $\epsilon > 0$. Since $\lim_{x\to c}\varphi(x) = L$, there is some $\delta_0 > 0$ such that, for any $x \in \mathrm{Dom}(\varphi)$,

$$0 < |x - c| < \delta_0 \quad\Rightarrow\quad |\varphi(x) - L| < \epsilon. \tag{2.6}$$

Choose $\delta = \min\{\delta_0, \gamma\}$, let $x \in \mathrm{Dom}(f)$, and suppose that $0 < |x - c| < \delta$. Then we have $0 < |x - c| < \delta_0$, and we also have $0 < |x - c| < \gamma$ so that

$$x \in \mathrm{Dom}(f) \cap B'_\gamma(c)$$

and hence $f(x) = \varphi(x)$. This shows that $x \in \mathrm{Dom}(\varphi)$, and since $0 < |x - c| < \delta_0$ also, it follows by (2.6) that $|\varphi(x) - L| < \epsilon$ and hence $|f(x) - L| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{x\to c} f(x) = L$. ■

In the statement of Theorem 2.18 it is understood that c is a limit point of the domain of φ, since otherwise the hypothesis $\lim_{x\to c}\varphi(x) = L$ makes no sense.

Example 2.19. Evaluate the limit

$$\lim_{x\to 9}\frac{2x^2 - 21x + 27}{x^2 - 9x}.$$

Solution. For convenience define

$$f(x) = \frac{2x^2 - 21x + 27}{x^2 - 9x}.$$

Now, for any $x \neq 0, 9$, we have

$$f(x) = \frac{2x^2 - 21x + 27}{x^2 - 9x} = \frac{(2x - 3)(x - 9)}{x(x - 9)} = \frac{2x - 3}{x}.$$

It's important to bear in mind that the last equality is justified only if x does not equal 0 or 9! If we define

$$g(x) = \frac{2x-3}{x},$$

then it is clear that $f(x) = g(x)$ for all $x \in (-\infty, 0) \cup (0, 9) \cup (9, \infty)$, and in particular for all $x \in (9-\gamma, 9) \cup (9, 9+\gamma)$ for sufficiently small $\gamma > 0$ (for instance we could let $\gamma = 1$). Since

$$\lim_{x\to 9} g(x) = \lim_{x\to 9} \frac{2x-3}{x} = \frac{2(9)-3}{9} = \frac{5}{3}$$

by Proposition 2.16, it follows that

$$\lim_{x\to 9} \frac{2x^2-21x+27}{x^2-9x} = \lim_{x\to 9} f(x) = \lim_{x\to 9} g(x) = \frac{5}{3}$$

by Theorem 2.18. ■

Example 2.20. Evaluate the limit

$$\lim_{x\to 0} \frac{\sqrt{2x^2+25}-5}{x^2}.$$

Solution. Letting

$$f(x) = \frac{\sqrt{2x^2+25}-5}{x^2},$$

observe that for any $x \neq 0$

$$\begin{aligned} f(x) &= \frac{\sqrt{2x^2+25}-5}{x^2} \cdot \frac{\sqrt{2x^2+25}+5}{\sqrt{2x^2+25}+5} = \frac{(2x^2+25)-25}{x^2(\sqrt{2x^2+25}+5)} \\ &= \frac{2x^2}{x^2(\sqrt{2x^2+25}+5)} = \frac{2}{\sqrt{2x^2+25}+5}. \end{aligned}$$

Thus if we let

$$g(x) = \frac{2}{\sqrt{2x^2+25}+5},$$

then $f(x) = g(x)$ for all $x \in (-\infty, 0) \cup (0, \infty)$, and so

$$\begin{aligned} \lim_{x\to 0} \frac{\sqrt{2x^2+25}-5}{x^2} &= \lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{2}{\sqrt{2x^2+25}+5} \\ &= \frac{\lim_{x\to 0}(2)}{\lim_{x\to 0}(\sqrt{2x^2+25}+5)} = \frac{2}{\lim_{x\to 0}\sqrt{2x^2+25}+\lim_{x\to 0}(5)} \\ &= \frac{2}{\sqrt{\lim_{x\to 0}(2x^2+25)}+5} = \frac{2}{\sqrt{2(0)^2+25}+5} = \frac{1}{5}, \end{aligned}$$

by Theorem 2.18, appropriate limit laws, and finally Proposition 2.16. ■

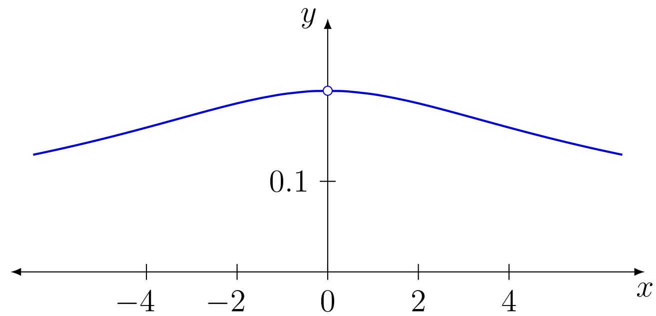


FIGURE 3.

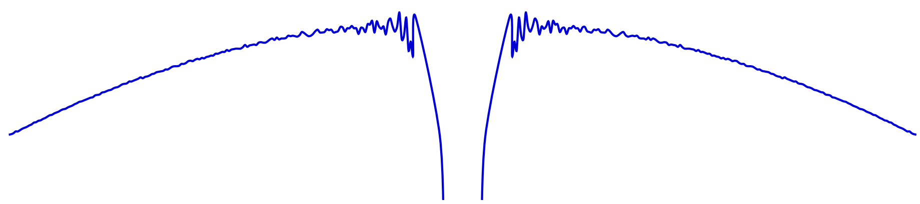


FIGURE 4.

In Figure 3 the graph of the function f in Example 2.20 looks much as it should. However, if we zoom in enough on the point $(0, \frac{1}{5})$ the graph will appear to become extremely erratic such as in Figure 4. The zig-zagging behavior is not a property of f, but rather an artifact of rounding errors! This phenomenon highlights one of the pitfalls of evaluating the limit of a function simply by examining the function's graph.

Sometimes the limit of a function is problematic to deal with directly, but it may be possible to bound the function between two other functions whose limits are straightforward and equal. When that's the case we can employ the following.

Theorem 2.21 (**Squeeze Theorem**). *Let $c \in \mathbb{R}$ be a limit point of* $\mathrm{Dom}(f)$. *Suppose there are functions φ and ψ, and some $\gamma > 0$, such that*

$$\varphi(x) \leq f(x) \leq \psi(x)$$

for all $x \in \mathrm{Dom}(f) \cap B'_\gamma(c)$. If

$$\lim_{x\to c} \varphi(x) = \lim_{x\to c} \psi(x) = L$$

for some $L \in \mathbb{R}$, then

$$\lim_{x\to c} f(x) = L.$$

Proof. Suppose that $\varphi(x), \psi(x) \to L \in \mathbb{R}$ as $x \to c$. Let $\epsilon > 0$. Since $\lim_{x\to c} \varphi(x) = L$, there exists some $\delta_1 > 0$ such that, for any $x \in \mathrm{Dom}(\varphi)$,

$$0 < |x - c| < \delta_1 \text{ implies } |\varphi(x) - L| < \epsilon.$$

Since $\lim_{x\to c} \psi(x) = L$, there exists some $\delta_2 > 0$ such that, for any $x \in \mathrm{Dom}(\psi)$,

$$0 < |x - c| < \delta_2 \text{ implies } |\psi(x) - L| < \epsilon.$$

Choose $\delta = \min\{\gamma, \delta_1, \delta_2\}$ and suppose $x \in \mathrm{Dom}(f)$ is such that $0 < |x - c| < \delta$; that is, $x \in \mathrm{Dom}(f) \cap B'_\delta(c)$. Since $B'_\delta(c) \subseteq B'_\gamma(c)$, it follows that $x \in \mathrm{Dom}(f) \cap B'_\gamma(c)$, and therefore $\varphi(x) \le f(x) \le \psi(x)$ holds. Moreover we have

$$0 < |x - c| < \delta_1 \quad \text{and} \quad 0 < |x - c| < \delta_2$$

since $\delta \le \delta_1, \delta_2$, and so

$$-\epsilon < \varphi(x) - L < \epsilon \quad \text{and} \quad -\epsilon < \psi(x) - L < \epsilon \tag{2.7}$$

both hold. From $\varphi(x) \le f(x) \le \psi(x)$ we obtain

$$\varphi(x) - L \le f(x) - L \le \psi(x) - L,$$

which together with (2.7) gives

$$-\epsilon < f(x) - L < \epsilon,$$

or equivalently $|f(x) - L| < \epsilon$.

We have now shown that, for any $x \in \mathrm{Dom}(f)$, $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$, and therefore

$$\lim_{x \to c} f(x) = L$$

as desired. ■

Example 2.22. Let $f(x) = x \sin(1/x)$ and consider the limit $\lim_{x \to 0} f(x)$. In Figure 5 we see that the graph of $\sin(1/x)$ becomes wildly oscillatory in the neighborhood of $x = 0$, and so we might expect that $\lim_{x \to 0} f(x)$ does not exist. But we shouldn't despair too quickly.

For all $x \in (-\infty, 0) \cup (0, \infty)$ we have $-1 \le \sin(1/x) \le 1$, and hence

$$-|x| \le x \sin\left(\frac{1}{x}\right) \le |x|.$$

Now, since $\lim_{x \to 0} |x| = 0$ and $\lim_{x \to 0}(-|x|) = -\lim_{x \to 0} |x| = 0$, by the Squeeze Theorem we conclude that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

as well. ■

Proposition 2.23. *Let $L \in \mathbb{R}$. Then*

$$\lim_{x \to c} f(x) = L \quad \textit{if and only if} \quad \lim_{h \to 0} f(c + h) = L.$$

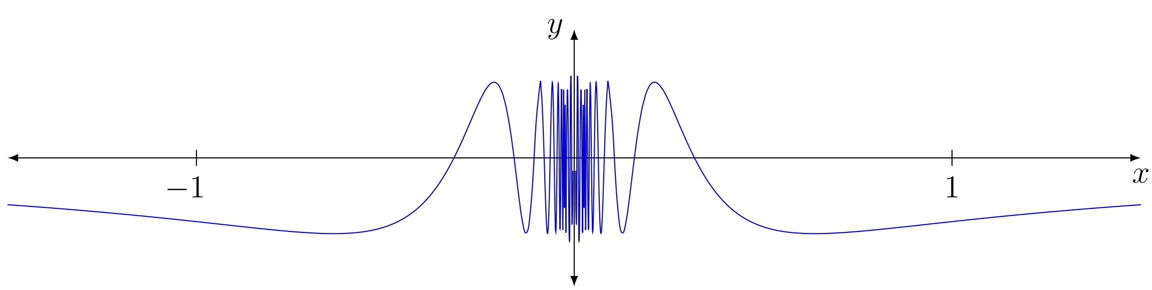


FIGURE 5. The graph of $\sin(1/x)$.

Proof. Suppose that $\lim_{x \to c} f(x) = L$. Let $\epsilon > 0$. Then there exists some $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$. Suppose that $0 < |h| < \delta$. Then we have

$$0 < |(c + h) - c| < \delta,$$

and hence $|f(c + h) - L| < \epsilon$. We have now shown that for every $\epsilon > 0$ there exists some $\delta > 0$ such that $0 < |h| < \delta$ implies $|f(c + h) - L| < \epsilon$, which is to say $\lim_{h \to 0} f(c + h) = L$.

For the converse, suppose that $\lim_{h \to 0} f(c+h) = L$. Let $\epsilon > 0$. Then there exists some $\delta > 0$ such that $0 < |h| < \delta$ implies that $|f(c + h) - L| < \epsilon$. Suppose that $0 < |x - c| < \delta$. Then we have

$$|f(c + (x - c)) - L| < \epsilon,$$

and thus $|f(x) - L| < \epsilon$. Therefore $\lim_{x \to c} f(x) = L$. ■

2.4 – Infinite Limits

An **infinite limit** is a limit that equals either $+\infty$ or $-\infty$. Thus an infinite limit is a kind of limit that does not exist in the sense that it does not equal a real number, but at least some information is conveyed about the *reason* for the nonexistence. The precise definition for what it means for a limit to equal $+\infty$ or $-\infty$ follows.

Definition 2.24. *Let f be a real-valued function, and let $c \in \mathbb{R}$ be a limit point of* $\mathrm{Dom}(f)$. *We say f has* ***limit*** $+\infty$ *at c, written*

$$\lim_{x \to c} f(x) = +\infty,$$

if for all $\alpha > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) > \alpha.$$

We say f has ***limit*** $-\infty$ *at c, written*

$$\lim_{x \to c} f(x) = -\infty,$$

if for all $\alpha > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) < -\alpha.$$

There are one-sided versions of the above definitions. For instance, $\lim_{x \to c^+} f(x) = +\infty$ means that, for any $\alpha > 0$ there is some $\delta > 0$ such that $x \in (c, c + \delta)$ implies $f(x) > \alpha$ for all $x \in \mathrm{Dom}(f)$.

Example 2.25. Prove that

$$\lim_{x \to 3} \frac{25}{(x-3)^2} = +\infty.$$

Preliminary Analysis. By Definition 2.24 we must show that, for any $\alpha > 0$, there is some $\delta > 0$ such that $0 < |x - 3| < \delta$ implies

$$\frac{25}{(x-3)^2} > \alpha. \tag{2.8}$$

However,

$$\frac{25}{(x-3)^2} > \alpha \quad \Leftrightarrow \quad (x-3)^2 < \frac{25}{\alpha} \quad \Leftrightarrow \quad |x - 3| < \frac{5}{\sqrt{\alpha}},$$

so in particular we must find some δ such that $|x - 3| < \delta$ implies $|x - 3| < 5/\sqrt{\alpha}$. Clearly we should choose $\delta = 5/\sqrt{\alpha}$. With this in mind, we proceed with the proof. ■

Proof. Let $\alpha > 0$. Choose $\delta = \frac{5}{\sqrt{\alpha}}$, and suppose $0 < |x - 3| < \delta$. Then $|x - 3| < 5/\sqrt{\alpha}$ follows, and as seen in the preliminary analysis this implies that (2.8) holds. This finishes the proof. ■

If c is a limit point of $\mathrm{Dom}(f)$, then f is said to have **vertical asymptote** $x = c$ if

$$\lim_{x \to c^-} |f(x)| = +\infty \quad \text{or} \quad \lim_{x \to c^+} |f(x)| = +\infty.$$

There is no theoretical limit to how many vertical asymptotes a function may have. Recall that the tangent function in trigonometry has an infinite number of vertical asymptotes.

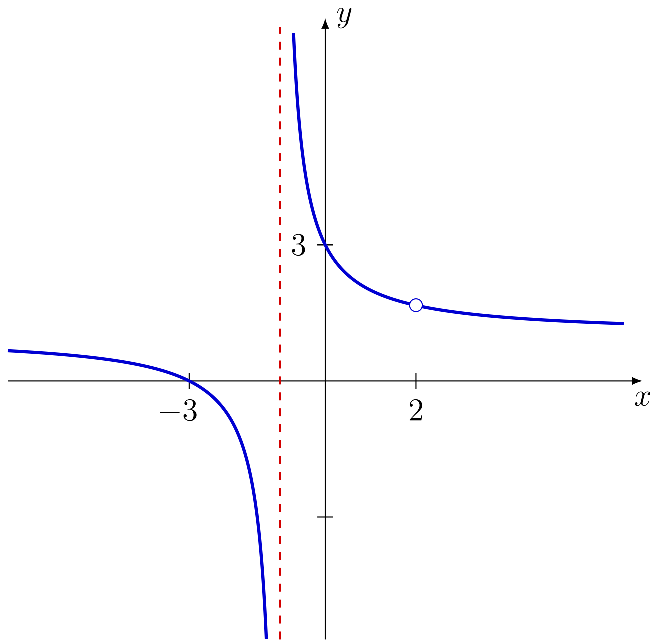


FIGURE 6.

Example 2.26. Consider the function

$$f(x) = \frac{x^2 + x - 6}{x^2 - x - 2}.$$

Factoring the numerator and denominator, we obtain

$$f(x) = \frac{(x-2)(x+3)}{(x-2)(x+1)},$$

which makes clear that the domain of f is $D = (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$. By Proposition 2.16(2) we find that

$$\lim_{x \to c} f(x) = \frac{(c-2)(c+3)}{(c-2)(c+1)} \in \mathbb{R}$$

for all $c \in D$, and so the only remaining candidates for a vertical asymptote of f are $x = 2$ and $x = -1$. However, for all $x \in B_1'(2)$ we have

$$\frac{(x-2)(x+3)}{(x-2)(x+1)} = \frac{x+3}{x+1}, \tag{2.9}$$

and so by Theorem 2.18 and Proposition 2.16(2) we obtain

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x+3}{x+1} = \frac{2+3}{2+1} = \frac{5}{3}.$$

This shows that $\lim_{x \to 2} |f(x)| \neq +\infty$, and thus $x = 2$ is not a vertical asymptote of f.

Finally we turn to $x = -1$. Equation (2.9) holds also for $x \in B_1'(-1)$, and so

$$\lim_{x \to -1} |f(x)| = \lim_{x \to -1} \frac{|x+3|}{|x+1|}$$

by Theorem 2.18. This limit does in fact equal $+\infty$, for as x approaches -1 we find that the numerator $|x+3|$ approaches 2 while the denominator $|x+1|$ approaches 0 *from the right.*

We can make this rigorous. Let $\alpha > 0$ be arbitrary. Choose $\delta = \min\{1, 1/\alpha\}$. Suppose $x \in \text{Dom}(f)$ is such that $0 < |x+1| < \delta$. In particular this implies that $|x+1| < 1$, which is to say $1 < x+3 < 3$ and hence $|x+3| > 1$. Now,

$$0 < |x+1| < \delta \quad \Rightarrow \quad |x+1| < \frac{1}{\alpha} \quad \Rightarrow \quad \frac{1}{|x+1|} > \alpha \quad \Rightarrow \quad \frac{|x+3|}{|x+1|} > \alpha,$$

and so $|f(x)| > \alpha$. In accordance with Definition 2.24 this proves that

$$\lim_{x \to -1} |f(x)| = +\infty,$$

and therefore $x = -1$ is the only vertical asymptote for f.

A graph of f is shown in Figure 6, with the vertical asymptote depicted as a dashed line. It can be seen that, in particular, $f(x) \to -\infty$ as $x \to -1^-$, and $f(x) \to +\infty$ as $x \to -1^+$. At the point $\left(2, \frac{5}{3}\right)$ there is merely a hole in the graph. ■

2.5 – Limits at Infinity

We now consider what happens to the value of a function f as x grows without bound toward either $+\infty$ or $-\infty$, which is known as a **limit at infinity**.

Definition 2.27. *Let f be a real-valued function, $L \in \mathbb{R}$, and suppose there exists some $\gamma > 0$ for which $(\gamma, \infty) \subseteq \mathrm{Dom}(f)$. We say the **limit of** $f(x)$ **as** x **approaches** $+\infty$ **is** L, written*

$$\lim_{x\to\infty} f(x) = L,$$

if for every $\epsilon > 0$ there exists some $\beta > 0$ such that $x > \beta$ implies that $|f(x) - L| < \epsilon$.

*Now suppose $(-\infty, -\gamma) \subseteq \mathrm{Dom}(f)$ for some $\gamma > 0$. We say the **limit of** $f(x)$ **as** x **approaches** $-\infty$ **is** L, written*

$$\lim_{x\to-\infty} f(x) = L,$$

if for every $\epsilon > 0$ there exists some $\beta > 0$ such that $x < -\beta$ implies that $|f(x) - L| < \epsilon$.

A function f is said to have **horizontal asymptote** $y = L$ if

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

A function can have at most two horizontal asymptotes.

Example 2.28. Determine the horizontal asymptotes, if any, of the function f given by

$$f(x) = \frac{\sqrt{x^2 + 2x + 6} - 3}{2x - 1}.$$

Solution. Recall that in general $\sqrt{x^2} = |x|$. Now, when $x \to \infty$ we have $x > 0$, so then $\sqrt{x^2} = x$ and we obtain

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{2x-1} = \lim_{x\to\infty} \frac{x\sqrt{1+2/x+6/x^2}-3}{2x-1} \\ &= \lim_{x\to\infty} \frac{\sqrt{1+2/x+6/x^2}-3/x}{2-1/x} = \frac{\sqrt{1+0+0}-0}{2-0} = \frac{1}{2}.\end{aligned}$$

On the other hand $x \to -\infty$ implies $x < 0$, so then $\sqrt{x^2} = -x$ and we obtain

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{\sqrt{x^2+2x+6}-3}{2x-1} = \lim_{x\to-\infty} \frac{-x\sqrt{1+2/x+6/x^2}-3}{2x-1} \\ &= \lim_{x\to-\infty} \frac{-\sqrt{1+2/x+6/x^2}-3/x}{2-1/x} = \frac{-\sqrt{1+0+0}-0}{2-0} = -\frac{1}{2}.\end{aligned}$$

Hence the horizontal asymptotes of f are $y = \frac{1}{2}$ and $y = -\frac{1}{2}$.

A graph of f is shown in Figure 7, with the horizontal asymptotes depicted as dashed lines. As can be seen, it is entirely possible for the graph of a function to cross one of its horizontal asymptotes. ■

The following proposition informs us that a limit at infinity, when it exists, is always equivalent to a one-sided limit at 0.

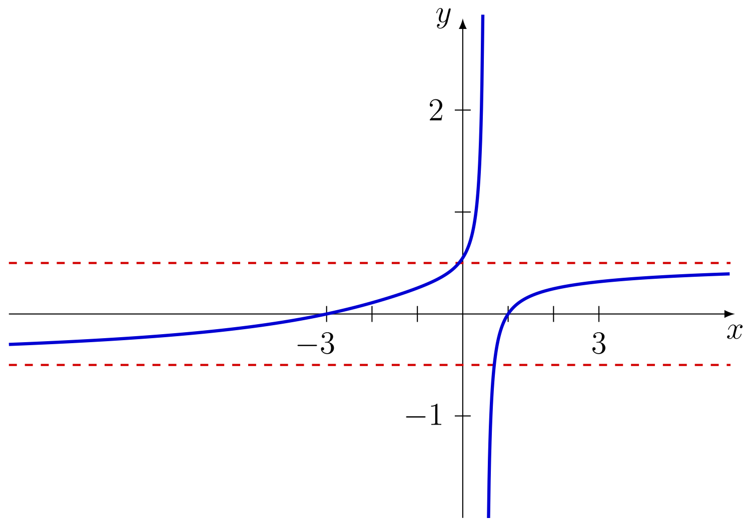


Figure 7.

Proposition 2.29. *For any function f and $L \in [-\infty, \infty]$,*

$$\lim_{x\to 0^+} f(x) = L \text{ \textit{if and only if} } \lim_{x\to\infty} f(1/x) = L,$$

and

$$\lim_{x\to 0^-} f(x) = L \text{ \textit{if and only if} } \lim_{x\to-\infty} f(1/x) = L.$$

Proof. It will suffice to prove only the first biconditional statement, since the proof of the second one is much the same.

First assume that $-\infty < L < \infty$. Suppose $\lim_{x\to 0^+} f(x) = L$, and let $\epsilon > 0$. Then there exists some $\delta > 0$ such that $0 < x < \delta$ implies that $|f(x) - L| < \epsilon$. Choose $\beta = 1/\delta$, and suppose that $x > \beta$. Then $x > 1/\delta > 0$, whence $0 < 1/x < \delta$ obtains and we get $|f(1/x) - L| < \epsilon$. Thus $\lim_{x\to\infty} f(1/x) = L$.

For the converse, suppose that $\lim_{x\to\infty} f(1/x) = L$, and again let $\epsilon > 0$. Then there exists some $\beta > 0$ such that $x > \beta$ implies that $|f(1/x) - L| < \epsilon$. Choose $\delta = 1/\beta$, and suppose that $0 < x < \delta$. Then $0 < 1/\delta = \beta < 1/x$, whence we obtain

$$|f(x) - L| = \left| f\left(\frac{1}{1/x}\right) - L \right| < \epsilon$$

and thus $\lim_{x\to 0^+} f(x) = L$.

Now we consider the case when $L = \infty$. Suppose $\lim_{x\to 0^+} f(x) = \infty$, and let $\alpha > 0$. There exists some $\delta > 0$ such that $0 < x < \delta$ implies that $f(x) > \alpha$. Choose $\beta = 1/\delta$, and suppose that $x > \beta$. Then $x > 1/\delta > 0$, whence we get $0 < 1/x < \delta$ and so $f(1/x) > \alpha$. Thus $\lim_{x\to\infty} f(1/x) = \infty$.

For the converse, suppose that $\lim_{x\to\infty} f(1/x) = \infty$, and let $\alpha > 0$. There exists some $\beta > 0$ such that $x > \beta$ implies that $f(1/x) > \alpha$. Choose $\delta = 1/\beta$, and suppose that $0 < x < \delta$. Then $0 < 1/\delta = \beta < 1/x$, whence we obtain

$$f(x) = f\left(\frac{1}{1/x}\right) > \alpha.$$

Thus $\lim_{x\to 0^+} f(x) = \infty$. ∎

Because of this proposition we can assume that all the properties of limits given in §2.3 also apply to limits at infinity. For instance, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, then with Proposition 2.29 and Theorem 2.12 we find that

$$\lim_{x\to\infty}\big[f(x)g(x)\big] = \lim_{x\to 0^+}\big[f(1/x)g(1/x)\big] = \lim_{x\to 0^+} f(1/x)\cdot \lim_{x\to 0^+} g(1/x) = \lim_{x\to\infty} f(x)\cdot \lim_{x\to\infty} g(x),$$

which is the analog of Law (4) in Theorem 2.12. For the sake of completeness we state the following proposition, which could also be proved directly using Definition 2.27 in much the same manner that Theorem 2.12 was proved using Definition 2.3.

Proposition 2.30. *Laws (1) through (6) given by Theorem 2.12 are valid in the cases when $c = \pm\infty$. If $c = \infty$, then Law (7) holds provided there's some $\gamma > 0$ such that $f(x) \geq 0$ for all $x > \gamma$. If $c = -\infty$, then Law (7) holds provided there's some $\gamma > 0$ such that $f(x) \geq 0$ for all $x < -\gamma$.*

The next proposition imposes certain conditions on a function f that ensure that $\lim_{x\to\infty} f(x)$ exists in $\mathbb{R}$. There is, however, no shortage of limits at infinity that exist for functions that do not satisfy the proposition's nondecreasing assumption. We make no use of the proposition until §8.6.

Proposition 2.31. *Suppose that f is a nondecreasing function on (a,∞); that is, $f(x_2) \geq f(x_1)$ for all $x_1, x_2 \in (a,\infty)$ such that $x_2 > x_1$, . If f is bounded above on (a,∞), then $\lim_{x\to\infty} f(x)$ exists in $\mathbb{R}$.*

Proof. Suppose f is bounded above on (a,∞), so there exists some $0 < \alpha < \infty$ such that $f(x) \leq \alpha$ for all $x \in (a,\infty)$. Then $S = \{f(x) : x > a\}$ is a subset of $\mathbb{R}$ that is bounded above, and by the Completeness Axiom there exists some $\beta \in \mathbb{R}$ such that $\sup(S) = \beta$. (That is, S has a least upper bound in $\mathbb{R}$.)

Let $\epsilon > 0$. Since β is the least upper bound for S, there exists some $\gamma > a$ such that $f(\gamma) > \beta - \epsilon$. Then, since f is nondecreasing on (a,∞) we find that $f(x) \geq f(\gamma) > \beta - \epsilon$ for all $x > \gamma$, and also $f(x) \leq \beta$ for all $x > \gamma$ since β is an upper bound for S. Combining these results gives $|f(x) - \beta| < \epsilon$ for all $x > \gamma$.

Thus for each $\epsilon > 0$ there exists some $\gamma > a$ such that $|f(x) - \beta| < \epsilon$ whenever $x > \gamma$. Noting that we can always let $\gamma > \max\{a, 0\}$ to ensure that $\gamma > 0$, it follows immediately from Definition 2.27 that $\lim_{x\to\infty} f(x) = \beta$. Because β is a real number we conclude that $\lim_{x\to\infty} f(x)$ exists in $\mathbb{R}$. ∎

Corollary 2.32. *Let f be nondecreasing on (a,∞). If $\lim_{x\to\infty} f(x)$ does not exist in $\mathbb{R}$, then $\lim_{x\to\infty} f(x) = \infty$.*

Proof. Suppose $\lim_{x\to\infty} f(x)$ does not exist in $\mathbb{R}$. Fix $\alpha > 0$. By the contrapositive of Proposition 2.31 f is not bounded above, and so there exists some $\gamma > a$ such that $f(\gamma) > \alpha$. Then, since f is nondecreasing, $f(x) > \alpha$ for all $x > \gamma$. Therefore $\lim_{x\to\infty} f(x) = \infty$. ∎

Aside from vertical and horizontal asymptotes there are also linear asymptotes that have real slope $m \neq 0$. A function f is said to have **slant asymptote** $y = mx + b$ if

$$\lim_{x\to\infty} \big|(mx+b) - f(x)\big| = 0 \quad \text{or} \quad \lim_{x\to-\infty} \big|(mx+b) - f(x)\big| = 0.$$

An alternate means of expressing the limit at left is to write $f(x) \to mx + b$ as $x \to \infty$, while the limit at right may be written as $f(x) \to mx + b$ as $x \to -\infty$. It is a fact that letting $m = 0$ in our definition of a slant asymptote would give us an alternate (but equivalent) definition of a horizontal asymptote. A function can have no more than two asymptotes that are not vertical asymptotes.

Given a polynomial function p, recall that the degree of p (denoted by $\deg(p)$) is defined to be the highest power of x in the expression $p(x)$. Thus the degree of $4x - 7x^2$ is 2, the degree of $3x^5 - 2x^3 + 9$ is 5, and so on.

The following theorem addresses all possible manners in which a rational function, in particular, may or may not have a horizontal or slant asymptote.

Theorem 2.33. *Suppose p and q are polynomial functions, so that p/q is a rational function.*

1. *If* $\deg(p) < \deg(q)$, *then* $p(x)/q(x) \to 0$ *as* $x \to \pm\infty$.
2. *If* $\deg(p) = \deg(q)$, *with* A *and* B *being the lead coefficients of* $p(x)$ *and* $q(x)$, *then* $p(x)/q(x) \to A/B$ *as* $x \to \pm\infty$.
3. *If* $\deg(p) = \deg(q) + 1$, *with* $mx + b$ *being the quotient of the division* $p(x)/q(x)$, *then* $p(x)/q(x) \to mx + b$ *as* $x \to \pm\infty$.
4. *If* $\deg(p) > \deg(q) + 1$, *then* p/q *has no horizontal or slant asymptote.*

Example 2.34. Find all asymptotes of the function

$$f(x) = \frac{3x^3 - 2x^2 + 7x + 1}{2x^2 - x + 1}.$$

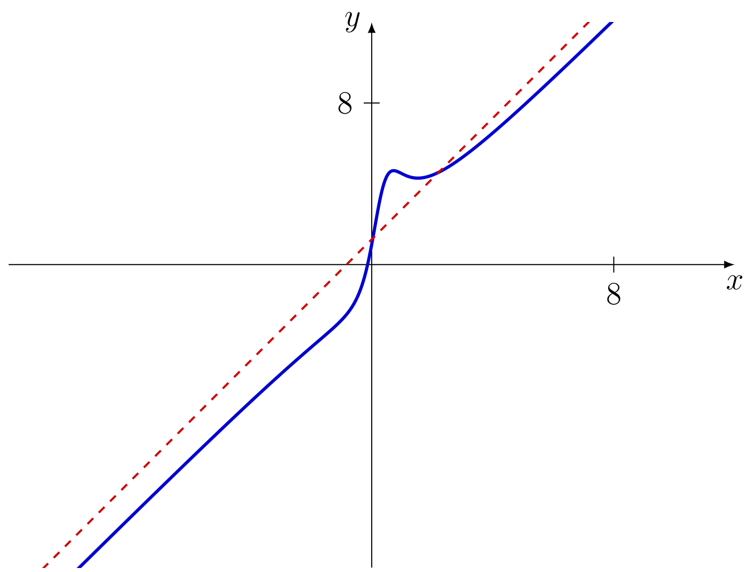


Figure 8.

Solution. Since $2x^2 - x + 1$ can be rewritten as $2(x - \frac{1}{4})^2 + \frac{7}{8}$, the denominator of $f(x)$ cannot equal zero for any real x and hence the domain of f is $(-\infty, \infty)$. In particular this means f has no vertical asymptotes.

Next we observe that the degree of the polynomial in the numerator of $f(x)$ is 3 while the degree of the polynomial in the denominator is 2. Since long division gives

$$\frac{3x^3 - 2x^2 + 7x + 1}{2x^2 - x + 1} = \left(\tfrac{3}{2}x - \tfrac{1}{4}\right) + \frac{\frac{21}{4}x + \frac{5}{4}}{2x^2 - x + 1},$$

we conclude by part (3) of Theorem 2.33 that $y = \frac{3}{2}x - \frac{1}{4}$ is a slant asymptote of the rational function f, and it is the one and only asymptote that f possesses. The graph of f and its asymptote is given in Figure 8. ■

The next example features a function that has slant asymptotes (in fact its graph is part of a hyperbola), but since the function is not rational no use can be made of Theorem 2.33.

Example 2.35. Find all asymptotes of the function

$$f(x) = 3\sqrt{\frac{x^2}{4} - 1}.$$

Solution. Polynomials have no vertical asymptotes, and since f is the square root of a polynomial, it too has no vertical asymptotes. However, upon rewriting $f(x)$ as

$$f(x) = \frac{3}{2}|x|\sqrt{1 - \frac{4}{x^2}},$$

it appears plausible that $f(x) \to 3|x|/2$ as $x \to \pm\infty$, since $\sqrt{1 - 4/x^2} \to 1$ as $x \to \pm\infty$. Specifically, it appears that the graph of f approaches $y = \frac{3}{2}x$ as $x \to \infty$, and $y = -\frac{3}{2}x$ as

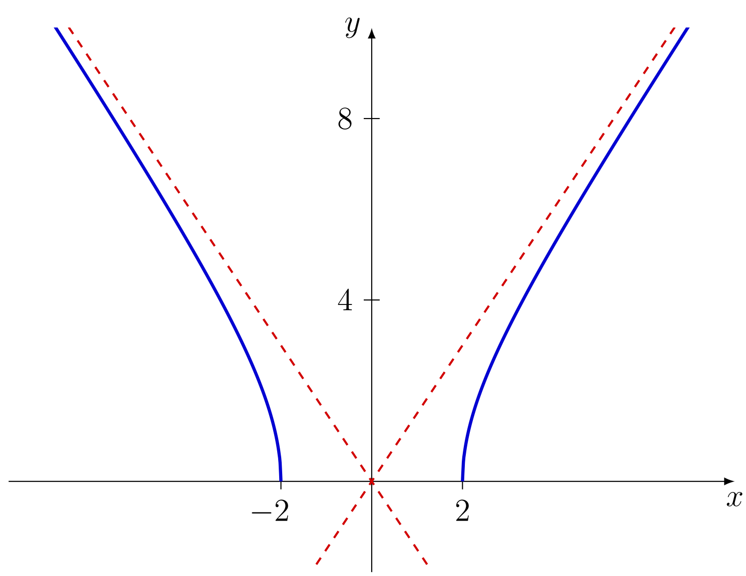


FIGURE 9.

$x \to -\infty$, and so $y = \pm\frac{3}{2}x$ are likely slant asymptotes for f. This must still be shown by working with our definition of a slant asymptote. We have

$$\begin{aligned}
\lim_{x\to\infty}\left|\frac{3}{2}x - f(x)\right| = \lim_{x\to\infty}\left|\frac{3}{2}x - \frac{3}{2}x\sqrt{1-\frac{4}{x^2}}\right| &= \lim_{x\to\infty}\frac{3}{2}x\left|1-\sqrt{1-4/x^2}\,\right| \\
&= \lim_{x\to\infty}\frac{3}{2}x\left|\frac{1-\sqrt{1-4/x^2}}{1}\cdot\frac{1+\sqrt{1-4/x^2}}{1+\sqrt{1-4/x^2}}\right| \\
&= \lim_{x\to\infty}\left(\frac{3x}{2}\cdot\frac{4/x^2}{1+\sqrt{1-4/x^2}}\right) \\
&= \lim_{x\to\infty}\frac{6/x}{1+\sqrt{1-4/x^2}} = 0,
\end{aligned}$$

and hence $y = \frac{3}{2}x$ is indeed a slant asymptote of f. Similarly, since $|x| = -x$ when $x \to -\infty$, we have

$$\lim_{x\to-\infty}\left|-\frac{3}{2}x - f(x)\right| = \lim_{x\to-\infty}\left|-\frac{3}{2}x + \frac{3}{2}x\sqrt{1-\frac{4}{x^2}}\,\right| = 0,$$

and hence $y = -\frac{3}{2}x$ is also a slant asymptote of f. There is no horizontal asymptote, since a function cannot have more than two asymptotes that are not vertical. Therefore the asymptotes of f are $y = \pm\frac{3}{2}x$. The graph of f, which has domain $(-\infty,-2]\cup[2,\infty)$, is shown along with its asymptotes in Figure 9. ∎

2.6 – Continuity

The following is the definition of continuity of a real-valued function f of a single real variable that will be used throughout these notes.

Definition 2.36. *Let $c \in \mathrm{Dom}(f)$. Then f is* ***continuous at*** *c if for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,*

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

Given a set S, we say f is ***continuous on*** *S if it is continuous at each $x \in S$. A* ***continuous function*** *is a function that is continuous on its domain.*

If a function is not continuous at some point c in its domain, then we say the function is **discontinuous at** c. A careful reading of Definition 2.36 should make clear that a function is discontinuous at any point that is not in its domain!

Theorem 2.37. *Let c be a limit point of $\mathrm{Dom}(f)$. Then f is continuous at c if and only if*

$$\lim_{x \to c} f(x) = f(c).$$

Proof. Suppose f is continuous at c. Then for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

Since

$$0 < |x - c| < \delta \quad \Rightarrow \quad |x - c| < \delta,$$

we now have the following: c is a limit point of $\mathrm{Dom}(f)$, and for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

Therefore

$$\lim_{x \to c} f(x) = f(c)$$

by Definition 2.3.

For the converse, suppose that $f(x) \to f(c)$ as $x \to c$. This immediately makes clear that $c \in \mathrm{Dom}(f)$, and since $|f(x) - f(c)| = 0$ when $x = c$, it follows that for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in \mathrm{Dom}(f)$,

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

Therefore f is continuous at c. ■

It is often said that a function is continuous if its graph can be drawn without lifting the pen, but there are many continuous functions with graphs that consist of two or more disconnected pieces! This occurs whenever a function is continuous on its domain, but the domain consists of two or more disjoint intervals of real numbers. Such functions are not hard to come by: consider $f(x) = 1/x$.

Theorem 2.38. *Suppose that functions f and g are continuous at c. Then the following functions are also continuous at c:*

1. *af for any $a \in \mathbb{R}$.*
2. *$f+g$, $f-g$, fg.*
3. *f/g provided that $g(c) \neq 0$.*
4. *f^n for any nonzero integer n, provided that $f(c) \neq 0$ if $n < 0$.*

Proof.
Proof of Part (1). Let $a \in \mathbb{R}$. Since f is continuous at c we have $\lim_{x\to c} f(x) = f(c) \in \mathbb{R}$. By Theorem 2.12(2)

$$\lim_{x\to c}(af)(x) = \lim_{x\to c} af(x) = a \lim_{x\to c} f(x) = af(c) = (af)(c),$$

which shows that af is continuous at c.

Proof of Part (2). We have $\lim_{x\to c} f(x) = f(c) \in \mathbb{R}$ and $\lim_{x\to c} g(x) = g(c) \in \mathbb{R}$. By Theorem 2.12(3)

$$\lim_{x\to c}(f \pm g)(x) = \lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = f(c) \pm g(c) = (f+g)(c),$$

which shows that $f+g$ and $f-g$ are continuous at c. The proof that fg is continuous at c is similar and uses Theorem 2.12(4).

Proof of Part (3). Suppose $g(c) \neq 0$. Since $\lim_{x\to c} f(x) = f(c) \in \mathbb{R}$ and $\lim_{x\to c} g(x) = g(c) \in \mathbb{R}$, by Theorem 2.12(5)

$$\lim_{x\to c}(f/g)(x) = \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)} = (f/g)(c),$$

which shows that f/g is continuous at c.

Proof of Part (4). Suppose that n is a positive integer. Since $\lim_{x\to c} f(x) = f(c) \in \mathbb{R}$, by Theorem 2.12(6)

$$\lim_{x\to c} f^n(x) = \lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n = [f(c)]^n = f^n(c),$$

and so f^n is continuous at c if n is positive.

Now suppose that n is a negative integer and $f(c) \neq 0$. Then $f^{-n}(c) \neq 0$, and so by Theorem 2.12(5) and 2.12(1), along with the observation that $-n$ is a positive integer, we obtain

$$\lim_{x\to c} f^n(x) = \lim_{x\to c} \frac{1}{f^{-n}(x)} = \frac{\lim_{x\to c}(1)}{\lim_{x\to c} f^{-n}(x)} = \frac{1}{f^{-n}(c)} = f^n(c).$$

Hence f^n is continuous at c if n is negative. ∎

Corollary to Theorem 2.38 are similar theorems in which the word “continuous” is replaced by “continuous from the right” or “continuous from the left”.

Theorem 2.39. *If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.*

Proof. Suppose g is continuous at c and f is continuous at $g(c)$. Fix $\epsilon > 0$. Since f is continuous at $g(c)$, there exists some $\gamma > 0$ such that, for all $y \in \mathrm{Dom}(f)$,

$$|y - g(c)| < \gamma \quad \Rightarrow \quad |f(y) - f(g(c))| < \epsilon. \tag{2.10}$$

Since g is continuous at c, there exists some $\delta > 0$ such that, for all $x \in \mathrm{Dom}(g)$,

$$|x - c| < \delta \quad \Rightarrow \quad |g(x) - g(c)| < \gamma. \tag{2.11}$$

Let $x \in \mathrm{Dom}(f \circ g)$, so that $x \in \mathrm{Dom}(g)$ and $g(x) \in \mathrm{Dom}(f)$. Suppose that $|x - c| < \delta$. Then $|f(x) - f(c)| < \gamma$ by (2.11), and since $g(x) \in \mathrm{Dom}(f)$ it follows from (2.10) that

$$|f(g(x)) - f(g(c))| < \epsilon.$$

That is,

$$|(f \circ g)(x) - (f \circ g)(c)| < \epsilon,$$

and therefore $f \circ g$ is continuous at c. ■

Lemma 2.40. *If f is continuous at b, b is an interior point of $\mathrm{Dom}(f)$, and $\lim_{x\to c} g(x) = b$, then c is a limit point of $\mathrm{Dom}(f \circ g)$.*

Proof. Suppose f is continuous at $b \in \mathrm{Int}(\mathrm{Dom}(f))$, and $g(x) \to b$ as $x \to c$. Let $\gamma > 0$ be arbitrary. Since b is an interior point of the domain of f, there exists some $\epsilon > 0$ such that $B_\epsilon(b) \subseteq \mathrm{Dom}(f)$. Now, because $\lim_{x\to c} g(x) = b$, we know that c is a limit point of $\mathrm{Dom}(g)$, and there exists some $0 < \delta < \gamma$ sufficiently small such that, for all $x \in \mathrm{Dom}(g)$,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |g(x) - b| < \epsilon.$$

That is, $x \in \mathrm{Dom}(g) \cap B'_\delta(c)$ implies $g(x) \in B_\epsilon(b) \subseteq \mathrm{Dom}(f)$, and hence $x \in \mathrm{Dom}(f \circ g)$ since $x \in \mathrm{Dom}(g)$ and $g(x) \in \mathrm{Dom}(f)$. This establishes that

$$\mathrm{Dom}(g) \cap B'_\delta(c) \subseteq \mathrm{Dom}(f \circ g).$$

Observing that $B'_\delta(c) \subseteq B'_\gamma(c)$ since $\delta < \gamma$, we then obtain

$$\mathrm{Dom}(g) \cap B'_\delta(c) \subseteq \mathrm{Dom}(f \circ g) \cap B'_\gamma(c).$$

Recalling that c is a limit point of $\mathrm{Dom}(g)$, it follows that $\mathrm{Dom}(g) \cap B'_\delta(c) \neq \varnothing$, and hence $\mathrm{Dom}(f \circ g) \cap B'_\gamma(c) \neq \varnothing$ as well. Since $\gamma > 0$ is arbitrary, we conclude that c is a limit point of $\mathrm{Dom}(f \circ g)$. ■

Proposition 2.41. *Suppose that f is continuous at b, where b is an interior point of $\mathrm{Dom}(f)$. If $\lim_{x\to c} g(x) = b$, then*

$$\lim_{x\to c} (f \circ g)(x) = f(b). \tag{2.12}$$

Proof. Suppose that $g(x) \to b$ as $x \to c$. By Lemma 2.40 it is known that c is a limit point of $\mathrm{Dom}(f \circ g)$, which is necessary (but not sufficient) in order for the limit (2.12) to exist. Let $\epsilon > 0$ be arbitrary. Since b is an interior point of $\mathrm{Dom}(f)$ and f is continuous at b, there exists some $\gamma > 0$ such that

$$|x - b| < \gamma \quad \Rightarrow \quad |f(x) - f(b)| < \epsilon.$$

Additionally, since $\lim_{x\to c} g(x) = b$ there can be found some $\delta > 0$ such that

$$0 < |x - c| < \delta \quad \Rightarrow \quad |g(x) - b| < \gamma.$$

Now, supposing that x is such that $0 < |x - c| < \delta$, it follows that $|g(x) - b| < \gamma$ and hence

$$|(f \circ g)(x) - f(b)| = |f(g(x)) - f(b)| < \epsilon.$$

This shows that $\lim_{x\to c}(f \circ g)(x) = f(b)$. ■

The conclusion of Proposition 2.41 can be written more compellingly as

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right),$$

so that in effect the proposition provides a means to bring a limit "inside" a function under certain conditions.

In what follows we define a **radical function** to be a function f given by $f(x) = \sqrt[m]{x}$ for some integer $m \geq 2$.

Proposition 2.42. *For each integer $m \geq 2$ the radical function $\sqrt[m]{x}$ is continuous on its domain.*

In the case when m is odd the proof for this proposition is given by Law (7) of limits in Theorem 2.12, and in the case when m is even the proof is given by Corollary 2.13. To prove the next proposition we cannot avoid the reality that the trigonometric functions are defined in geometrical terms in the standard mathematical curriculum. It is an unfortunate reality because calculus is analytical in essence, not geometrical. In the chapters to come calculus will be employed to give more analytical—and general—definitions for concepts such as the "length" of a curve and the "area" of a region enclosed by a curve. Such definitions may then be used to make the trigonometric functions less dependent on geometrical notions.

Proposition 2.43. *The trigonometric functions* sin, cos, tan, csc, sec, *and* cot *are continuous on their domain.*

Proof. We begin by showing the sine function is continuous at 0. Recall that for any $t \in \mathbb{R}$ the value of $\sin t$ is determined to be the y-coordinate of the point p on the unit circle

$$C_1(0,0) = \{(x,y) : x^2 + y^2 = 1\}$$

that is reached when traveling a distance of $|t|$ units on $C_1(0,0)$ (counterclockwise if $t > 0$, clockwise if $t < 0$) starting at the point $(1,0)$. Suppose that $0 < t < \pi/2$, so that the point

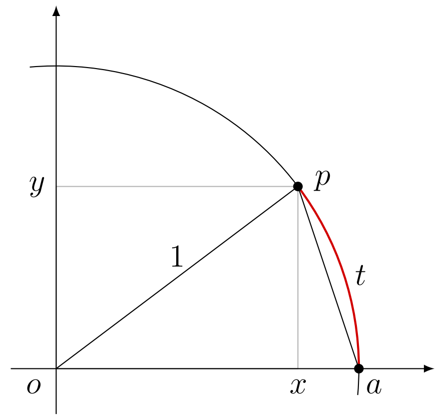


FIGURE 10.

$p = (x, y)$ on $C_1(0,0)$ is in the first quadrant as shown in Figure 10, and in particular $y > 0$. Letting $o = (0,0)$ and $a = (1,0)$, the area of the triangle $\triangle aop$ is

$$\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{\sin t}{2} > 0,$$

while the area of the circular sector bounded by the arc $\widehat{ap}$ and segments $\overline{oa}$ and $\overline{op}$ is

$$\frac{t}{2\pi}\big(\text{Area of } C_1(0,0)\big) = \frac{t}{2\pi} \cdot \pi(1)^2 = \frac{t}{2}.$$

Of course, the area of $\triangle aop$ is less than the area of the sector since the sector circumscribes the triangle, which implies that

$$0 < \frac{\sin t}{2} < \frac{t}{2}$$

and therefore $0 < \sin t < t$ for all $t \in (0, \pi/2)$. Since

$$\lim_{t\to 0^+} t = \lim_{t\to 0^+}(0) = 0,$$

by the Squeeze Theorem we conclude that

$$\lim_{t\to 0^+} \sin t = 0.$$

A nearly identical argument will show that $\sin t \to 0$ as $t \to 0^-$, whence we obtain

$$\lim_{t\to 0} \sin t = 0 = \sin 0$$

and therefore $\sin t$ is continuous at 0.

We now are in a position to show that cosine is continuous at 0. For $t \in (-\pi/2, \pi/2)$ we have

$$\cos t = \sqrt{1 - \sin^2(t)},$$

and so by Theorems 2.18 and 2.12

$$\lim_{t\to 0} \cos t = \lim_{t\to 0} \sqrt{1 - \sin^2 t} = \sqrt{1 - \Big[\lim_{t\to 0} \sin t\Big]^2} = \sqrt{1 - 0^2} = 1 = \cos 0.$$

Therefore $\cos t$ is continuous at 0.

Now let $c \in \mathbb{R}$. Using Proposition 2.23, the identity

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

and Theorems 2.18 and 2.12, we obtain

$$\begin{aligned}
\lim_{t \to c} \sin t &= \lim_{h \to 0} \sin(c + h) \\
&= \lim_{h \to 0} (\sin c \cos h + \cos c \sin h) \\
&= \sin c \lim_{h \to 0} \cos h + \cos c \lim_{h \to 0} \sin h \\
&= \sin c \cos 0 + \cos c \sin 0 \\
&= \sin c \cdot 1 + \cos c \cdot 0 = \sin c.
\end{aligned}$$

Hence $\sin t$ is continuous at c, and since $c \in \mathbb{R} = \text{Dom}(\sin)$ is arbitrary we conclude that the sine function is continuous on its domain.

Again let $c \in \mathbb{R}$. Using Proposition 2.23, the identity

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

and Theorems 2.18 and 2.12, we find that

$$\begin{aligned}
\lim_{t \to c} \cos t &= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} (\cos c \cos h - \sin c \sin h) \\
&= \cos c \lim_{h \to 0} \cos h - \sin c \lim_{h \to 0} \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \cdot 1 - \sin c \cdot 0 = \cos c,
\end{aligned}$$

and thus $\cos t$ is continuous at c. From this we conclude that the cosine function is continuous on its domain.

To show the other four trigonometric functions are continuous on their domain is straightforward. For instance let $c \in \text{Dom}(\tan)$. Since $\text{Dom}(\tan)$ is an open set there exists some $\gamma > 0$ such that $I = (c - \gamma, c + \gamma) \subseteq \text{Dom}(\tan)$. Observing that $\cos c \neq 0$ (otherwise c can't be in the domain of the tangent function) and $\tan t = \sin t / \cos t$ for all $t \in I$, we have by Theorems 2.18 and 2.12(5)

$$\lim_{t \to c} \tan t = \lim_{t \to c} \frac{\sin t}{\cos t} = \frac{\lim_{t \to c} \sin t}{\lim_{t \to c} \cos t} = \frac{\sin c}{\cos c} = \tan(c).$$

This shows that $\tan t$ is continuous at c, and therefore the tangent function is continuous on its domain.

The verification that the cosecant, secant, and cotangent functions are continuous on their domain is left to the exercises. ■

A careful examination of Proposition 2.16 should make it clear that polynomial and rational functions are continuous on their domain. Combining Propositions 2.16, 2.42, and 2.43, we have the following comprehensive theorem.

Theorem 2.44. *Polynomial, rational, radical, and trigonometric functions are continuous on their domain.*

This theorem, together with Theorems 2.38 and 2.39, can be used to show that almost all of the functions dealt with in the study and application of calculus are continuous on their domain. The foremost exceptions are piecewise-defined functions, as demonstrated in the following example.

Example 2.45. Determine where the function φ given by

$$\varphi(x) = \begin{cases} \sqrt[3]{2x-1}, & \text{if } x < \sqrt{\pi} \\ \sin(x^2), & \text{if } x \geq \sqrt{\pi} \end{cases}$$

is continuous.

Solution. First note that $\mathrm{Dom}(\varphi) = (-\infty, \infty)$. Since by Theorem 2.44 the polynomial function $g(x) = 2x - 1$ is continuous at $\sqrt{\pi}$, and the radical function $f(x) = \sqrt[3]{x}$ is continuous at $g(\sqrt{\pi}) = 2\sqrt{\pi} - 1$, by Theorem 2.39 the function

$$(f \circ g)(x) = \sqrt[3]{2x-1}$$

is continuous at $\sqrt{\pi}$, and so

$$\lim_{x \to \sqrt{\pi}} \sqrt[3]{2x-1} = \sqrt[3]{2\sqrt{\pi}-1}$$

by Theorem 2.37. Now,

$$\lim_{x \to \sqrt{\pi}^-} \varphi(x) = \lim_{x \to \sqrt{\pi}^-} \sqrt[3]{2x-1} = \sqrt[3]{2\sqrt{\pi}-1} \neq 0 = \sin(\pi) = \varphi(\sqrt{\pi})$$

shows that

$$\lim_{x \to \sqrt{\pi}} \varphi(x) \neq \varphi(\sqrt{\pi}),$$

and therefore φ is discontinuous at $\sqrt{\pi}$ by Theorem 2.37.

Next, for any $c < \sqrt{\pi}$ we have $g(x) = 2x - 1$ is continuous at c and $f(x) = \sqrt[3]{x}$ is continuous at $g(c) = 2c - 1$, and so by Theorem 2.39 $(f \circ g)(x) = \sqrt[3]{2x-1}$ is continuous at c. Since $\varphi(x) = (f \circ g)(x)$ for all $x < \sqrt{\pi}$, we conclude that φ is likewise continuous at c. That is, φ is continuous on $(-\infty, \sqrt{\pi})$.

Finally, for any $c > \sqrt{\pi}$ we have $g(x) = x^2$ is continuous at c and $f(x) = \sin x$ is continuous at $g(c) = c^2$, and so by Theorem 2.39 $(f \circ g)(x) = \sin(x^2)$ is continuous at c. Since $\varphi(x) = (f \circ g)(x)$ for all $x > \sqrt{\pi}$, we conclude that φ is likewise continuous at c. That is, φ is continuous on $(\sqrt{\pi}, \infty)$.

Therefore φ is continuous on $(-\infty, \sqrt{\pi}) \cup (\sqrt{\pi}, \infty)$, and discontinuous at $\sqrt{\pi}$. In particular φ is not continuous on its domain! ■

Theorem 2.46 (Intermediate Value Theorem). *Suppose f is continuous on $[a, b]$ and $L \in \mathbb{R}$ lies between $f(a)$ and $f(b)$. Then there exists some $c \in (a, b)$ such that $f(c) = L$.*

Proof. Without loss of generality it can be assumed that $f(a) < L < f(b)$. Define the set

$$S = \{x \in [a,b] : f(x) \leq L\}.$$

Since $a \in S$ we have $S \neq \varnothing$, and so since b is an upper bound for S the Completeness Axiom implies that there exists some $c \in \mathbb{R}$ such that $\sup(S) = c$.

Clearly $c \geq a$, since $a \in S$ and c is an upper bound for S. Also $c \leq b$, since b is an upper bound and c is a *least* upper bound for S. Therefore $c \in [a,b]$ and $f(c)$ is defined. What remains to show is that $f(c) = L$.

Suppose that $f(c) > L$. Then $c \in (a,b]$, and since f is continuous at c there exists some sufficiently small $\epsilon > 0$ such that $f(x) > L$ for all $x \in (c-\epsilon, c]$. Hence $x \notin S$ for all $c-\epsilon < x \leq c$, whereas $c = \sup(S)$ implies there should exist some $x_0 \in S$ such that $c - \epsilon < x_0 \leq c$. Given this contradiction, we conclude that $f(c) \not> L$.

Now suppose that $f(c) < L$. Then $c \in [a,b)$, and since f is continuous at c there exists some $\epsilon > 0$ such that $f(x) < L$ for all $x \in [c, c+\epsilon)$. Hence $f(c+\epsilon/2) < L$, so that $c + \epsilon/2 \in S$ and thus c cannot be an upper bound for S as indicated by $c = \sup(S)$. Given this contradiction, we conclude that $f(c) \not< L$.

Therefore $f(c) = L$, and since $L \neq f(a), f(b)$ it is clear that $c \neq a, b$ and thus $c \in (a,b)$. If we assume that $f(b) < L < f(a)$, then $-f(a) < -L < -f(b)$, and since $-f$ is continuous on $[a,b]$ we may employ the same argument as above to show that there exists some $c \in (a,b)$ such that $-f(c) = -L$, and thus $f(c) = L$ obtains once more. ■

Example 2.47. Does there exist a real number that is exactly 1 more than its cube?

Solution. The question is whether there exists some $x \in \mathbb{R}$ such that $x = x^3 + 1$. If we let $f(x) = x^3 - x + 1$, then the question becomes whether there exists some $x \in \mathbb{R}$ such that $f(x) = 0$. Since f is a polynomial function it is continuous everywhere, and so in particular is continuous on $[-2, 0]$. Now, since $f(-2) = -5$ and $f(0) = 1$, we see that 0 lies between $f(-2)$ and $f(0)$, and so by the Intermediate Value Theorem there exists some $c \in (-2, 0)$ for which $f(c) = 0$. That is, c is a real number and $c = c^3 + 1$. ■

In the example above we see that the Intermediate Value Theorem can tell us that there is indeed a real number that is 1 more than its cube, but it cannot tell us what exactly that number is, or even if it is unique. But knowing that such a number lies somewhere in the interval $(-2, 0)$ should be enough to enable a computer algebra system to determine its value to whatever degree of accuracy is desired.

Much of the remainder of this section is concerned with the development of theoretical results that will be employed much later in the text.

Definition 2.48. *Given functions $f, g : S \to \mathbb{R}$, define $f \vee g : S \to \mathbb{R}$ by*

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

and $f \wedge g : S \to \mathbb{R}$ by

$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

Given a function f, observe that if $f(x) \geq 0$ then

$$(f \vee 0)(x) = \max\{f(x), 0\} = f(x) \quad \text{and} \quad (-f \vee 0)(x) = \max\{-f(x), 0\} = 0,$$

and so
$$(f \vee 0)(x) - (-f \vee 0) = f(x) - 0 = f(x);$$
and if $f(x) < 0$ then
$$(f \vee 0)(x) = \max\{f(x), 0\} = 0 \quad \text{and} \quad (-f \vee 0)(x) = \max\{-f(x), 0\} = -f(x),$$
and so
$$(f \vee 0)(x) - (-f \vee 0) = 0 - [-f(x)] = f(x).$$
Hence
$$f = (f \vee 0) - (-f \vee 0), \tag{2.13}$$
which expresses f as a difference of two *nonnegative* functions and so will have great utility in later mathematical developments.

Proposition 2.49. *If f and g are continuous at c, then $f \vee g$ and $f \wedge g$ are continuous at c.*

Proof. Suppose that f and g are continuous at c. We will assume that $c \in \text{Int}(I)$, since if c is an endpoint of I the proof will be the same except that two-sided limits will become one-sided. Let $\epsilon > 0$. Then there exists $\delta_1, \delta_2 > 0$ such that
$$|x - c| < \delta_1 \;\;\Rightarrow\;\; |f(x) - f(c)| < \epsilon,$$
and
$$|x - c| < \delta_2 \;\;\Rightarrow\;\; |g(x) - g(c)| < \epsilon$$

Suppose that $f(c) > g(c)$, so that $(f \vee g)(c) = f(c)$. The continuity of f at c implies that there is some $\delta_3 > 0$ such that
$$|x - c| < \delta_3 \;\;\Rightarrow\;\; f(x) > g(x),$$
which is to say for all x such that $c - \delta_3 < x < c + \delta_3$ we have $(f \vee g)(x) = f(x)$. Choose $\delta = \min\{\delta_1, \delta_3\}$, and suppose that $|x - c| < \delta$. Since $|x - c| < \delta_3$ we have $(f \vee g)(x) = f(x)$, and since $|x - c| < \delta_1$ we have $|f(x) - f(c)| < \epsilon$. Now,
$$|(f \vee g)(x) - (f \vee g)(c)| = |f(x) - f(c)| < \epsilon.$$
Supposing that $f(c) < g(c)$, a similar argument to that above will show that
$$|(f \vee g)(x) - (f \vee g)(c)| = |g(x) - g(c)| < \epsilon.$$

Finally, suppose $f(c) = g(c)$, so that $(f \vee g)(c) = f(c) = g(c)$. Choose $\delta = \min\{\delta_1, \delta_2\}$, and suppose $|x - c| < \delta$. If $f(x) \geq g(x)$, then
$$|(f \vee g)(x) - (f \vee g)(c)| = |f(x) - f(c)| < \epsilon$$
since $|x - c| < \delta_1$. If $f(x) < g(x)$, then
$$|(f \vee g)(x) - (f \vee g)(c)| = |g(x) - g(c)| < \epsilon$$
since $|x - c| < \delta_2$.

We have now shown that for every $\epsilon > 0$ there exists some $\delta > 0$ such that
$$|x - c| < \delta \;\;\Rightarrow\;\; |(f \vee g)(x) - (f \vee g)(c)| < \epsilon,$$

and therefore $f \vee g$ is continuous at c. The proof that $f \wedge g$ is continuous at c is similar. ■

Corollary 2.50. *If f is continuous at c, then $|f|$ is continuous at c.*

Proof. Suppose that f is continuous at c. If $f(x) \geq 0$, then

$$(-f \vee f)(x) = \max\{-f(x), f(x)\} = f(x) = |f(x)|;$$

and if $f(x) < 0$, then

$$(-f \vee f)(x) = \max\{-f(x), f(x)\} = -f(x) = |f(x)|.$$

Thus we have $|f| = -f \vee f$, and since f and $-f$ are continuous at c, it follows by Proposition 2.49 that $|f|$ is also continuous at c. ■

2.7 – One-Sided Continuity

If a function f is not continuous at some point c in its domain, it may still be valuable to know that it is continuous in a one-sided sense. **One-sided continuity** at c refers to continuity at c from either the left or the right of c, which we now define.

Definition 2.51. *A function f is* ***continuous from the left*** *at c if $\lim_{x\to c^-} f(x) = f(c)$, and* ***continuous from the right*** *at c if $\lim_{x\to c^+} f(x) = f(c)$.*

3

Differentiation Theory

3.1 – The Derivative of a Function

Motivated historically by the slope problem that is discussed a little later, there is the following definition.

Definition 3.1. *Let c be an interior point of $\mathrm{Dom}(f)$. Then the **derivative of f at c** is*

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

*provided the limit exists. If $f'(c)$ exists, then f is said to be **differentiable at** c. If $f'(x)$ exists for all $x \in I$, then f is **differentiable on** I. Finally, if $f'(x)$ exists for all $x \in \mathrm{Dom}(f)$, then f is a **differentiable function**.*

The following proposition offers up an alternative but entirely equivalent means of finding $f'(c)$ which sometimes is more convenient.

Proposition 3.2. *Let $L \in \mathbb{R}$. Then $f'(c) = L$ if and only if*

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L, \tag{3.1}$$

and therefore

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided that the limit exists.

Proof. Suppose that $f'(c) = L$. Let $\epsilon > 0$ be arbitrary. By Definition 3.1 there exists some $\delta > 0$ such that $0 < |x - c| < \delta$ implies that

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon. \tag{3.2}$$

Suppose h is such that $0 < |h| < \delta$. Then $0 < |(c+h) - c| < \delta$ and so, substituting $c + h$ for x in (3.2), we obtain

$$\left| \frac{f(c+h) - f(c)}{(c+h) - c} - L \right| < \epsilon,$$

and thus

$$\left|\frac{f(c+h)-f(c)}{h}-L\right|<\epsilon.$$

This verifies equation (3.1) and thus completes the first part of the proof.

For the converse, suppose that (3.1) is true. Again let $\epsilon > 0$ be arbitrary. Then there exists some $\delta > 0$ such that $0 < |h| < \delta$ implies that

$$\left|\frac{f(c+h)-f(c)}{h}-L\right|<\epsilon. \tag{3.3}$$

Suppose that x is such that $0 < |x-c| < \delta$. Then, substituting $x-c$ for h in (3.3), we obtain

$$\left|\frac{f(c+(x-c))-f(c)}{x-c}-L\right|<\epsilon,$$

and thus

$$\left|\frac{f(x)-f(c)}{x-c}-L\right|<\epsilon.$$

This demonstrates that $f'(c) = L$, and so the proof is complete. ■

From a function f, then, we "derive" a new function f' whose domain consists of all $x \in \mathrm{Dom}(f)$ for which $f'(x)$ exists in $\mathbb{R}$. This is why the term "derivative" is used. The process of finding the derivative of a function is known as **differentiation**.

Definition 3.3. *For a function f, the **derivative of f** is the function f' given by*

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

for all x for which $f'(x) \in \mathbb{R}$.

It is customary to let $f'_-(x)$ and $f'_+(x)$ denote the left-hand and right-hand derivative limit, which is to say

$$f'_-(x) = \lim_{t\to x^-}\frac{f(t)-f(x)}{t-x} \quad \text{and} \quad f'_+(x) = \lim_{t\to x^+}\frac{f(t)-f(x)}{t-x}.$$

This notation will be used now and again, and more will be said about so-called one-sided derivatives at the end of the section.

In addition to the "prime notation" that denotes the derivative of f by f', there is "Leibniz notation" df/dx and "operator notation" $\partial_x f$. Moreover, if we set $y = f(x)$ then we can also represent f' by either y' in the prime notation or dy/dx in the Leibniz notation. All of the symbols

$$f', \quad y', \quad \frac{df}{dx}, \quad \frac{dy}{dx}, \quad \partial_x f$$

are used to represent the derivative of f, which is a *function*. To represent the *value* of the function f' at x, where x is considered to be a variable, there are prime notation symbols such as

$$f'(x), \quad [f(x)]', \quad y'(x), \quad [y(x)]',$$

as well as Leibniz and operator notation symbols such as

$$\frac{df}{dx}(x), \quad \frac{d}{dx}[f(x)], \quad \frac{dy}{dx}(x), \quad \frac{d}{dx}[y(x)], \quad \partial_x f(x), \quad \partial_x[f(x)].$$

If we wish to indicate specifically the value of f' when $x = c$, in addition to the symbols $f'(c)$ and $y'(c)$ there are

$$\frac{df}{dx}(c), \quad \left.\frac{df}{dx}\right|_{x=c}, \quad \frac{dy}{dx}(c), \quad \left.\frac{dy}{dx}\right|_{x=c}, \quad \partial_x f(c).$$

One other matter to bear in mind is that in practice (and especially in the study of differential equations) the symbols y' and dy/dx are often used to denote $f'(x)$ in the interests of brevity. Thus the symbols y' and dy/dx have two possible interpretations: they can represent f' or $f'(x)$, and only context makes clear which is intended.

Example 3.4. Given $f(x) = x^3$, find f' and its domain.

Solution. Using the limit in Proposition 3.2,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3hx^2 + 3h^2x + h^3) - x^3}{h} = \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3}{h} \\ &= \lim_{h\to 0}(3x^2 + 3hx + h^2) = 3x^2 + 3(0)x + (0)^2 = 3x^2. \end{aligned}$$

The steps taken to evaluate the limit are valid for any $x \in \mathbb{R}$, and the result is the real number $3x^2$. That is, $f'(x)$ is defined to be a real number for each $x \in \mathbb{R}$, and therefore $\text{Dom}(f') = (-\infty, \infty)$. ■

In Example 3.4, to indicate that the derivative of the function f given by $f(x) = x^3$ is the function f' given by $f'(x) = 3x^2$, we may simply write

$$(x^3)' = 3x^2 \quad \text{or} \quad \frac{d}{dx}(x^3) = 3x^2.$$

That is, to write $(x^3)' = 3x^2$ is to state that the derivative of the function $x \mapsto x^3$ is the function $x \mapsto 3x^2$. This practice is especially convenient when we are finding the derivative of a function to which we have not given a name such as f.

Example 3.5. Given $f(x) = \sqrt{3x+1}$, find f' and its domain.

Solution. Using the limit in Proposition 3.2,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\ &= \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} = \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \end{aligned}$$

$$= \frac{3}{\sqrt{3(x+0)+1} + \sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}.$$

So f' is seen to be a function with domain given by

$$\mathrm{Dom}(f') = \left\{x : \frac{3}{2\sqrt{3x+1}} \in \mathbb{R}\right\} = \{x : 3x+1 > 0\} = \left(-\tfrac{1}{3}, \infty\right).$$

Contrast this with $\mathrm{Dom}(f) = [-1/3, \infty)$ to see that, though $-1/3$ is in the domain of f, the function is not differentiable there. ■

One of the questions that motivated the discovery (some would say invention) of calculus was the so-called "slope problem." The question is, given a curve in $\mathbb{R}^2$ defined by the function $y = f(x)$, what is the "slope" of the curve at any one of its points $(x, f(x))$? Equivalently one might ask what angle the curve makes with the positive x-axis at $(x, f(x))$, but in any case the problem is that unless the curve is a line (to say "straight line" is redundant) we should expect the answer to vary from one point to the next. The issue is not entirely academic. If $s = s(t)$ gives the position s of an object at time t, we shall see in the exercises that the question of the velocity of the particle at some particular instant in time t_0 amounts to asking for the slope of the curve defined by $s = s(t)$ at the point $(t_0, s(t_0))$. The following definition settles the matter with the use of derivatives.

Definition 3.6. *Let C be a curve given by $y = f(x)$. The **slope** of C at a point $(c, f(c))$ is $f'(c)$ provided that f is differentiable at c, in which case the **tangent line** to C at $(c, f(c))$ is given by*

$$y - f(c) = f'(c)(x - c).$$

If $|f'_-(c)| = \infty$ and $|f'_+(c)| = \infty$, then C has no slope at $(c, f(c))$, and the vertical line $x = c$ is designated to be the tangent line.

As we might have guessed, a curve $y = f(x)$ has a horizontal tangent line at any point where $f'(c) = 0$. If $f'(c)$ does not exist in $\mathbb{R}$ and either $|f'_-(c)| \neq \infty$ or $|f'_+(c)| \neq \infty$, then there is no tangent line at $(c, f(c))$ whatsoever.

Example 3.7. Find all points, if any, where the curve C given by $y = \sqrt{3x+1}$ has a slope of 1. At each such point find the equation of the tangent line.

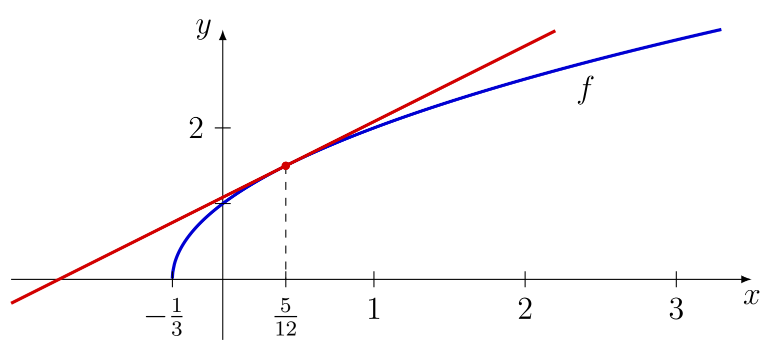


FIGURE 11. The tangent line to $y = \sqrt{3x+1}$ at $\left(\frac{5}{12}, \frac{3}{2}\right)$.

Solution. Setting $y = f(x)$, in Example 3.5 it was found that

$$f'(x) = \frac{3}{2\sqrt{3x+1}}.$$

What must be done is to find all $x \in \mathrm{Dom}(f') = (-1/3, \infty)$ for which $f'(x) = 1$. This is a simple matter of algebra:

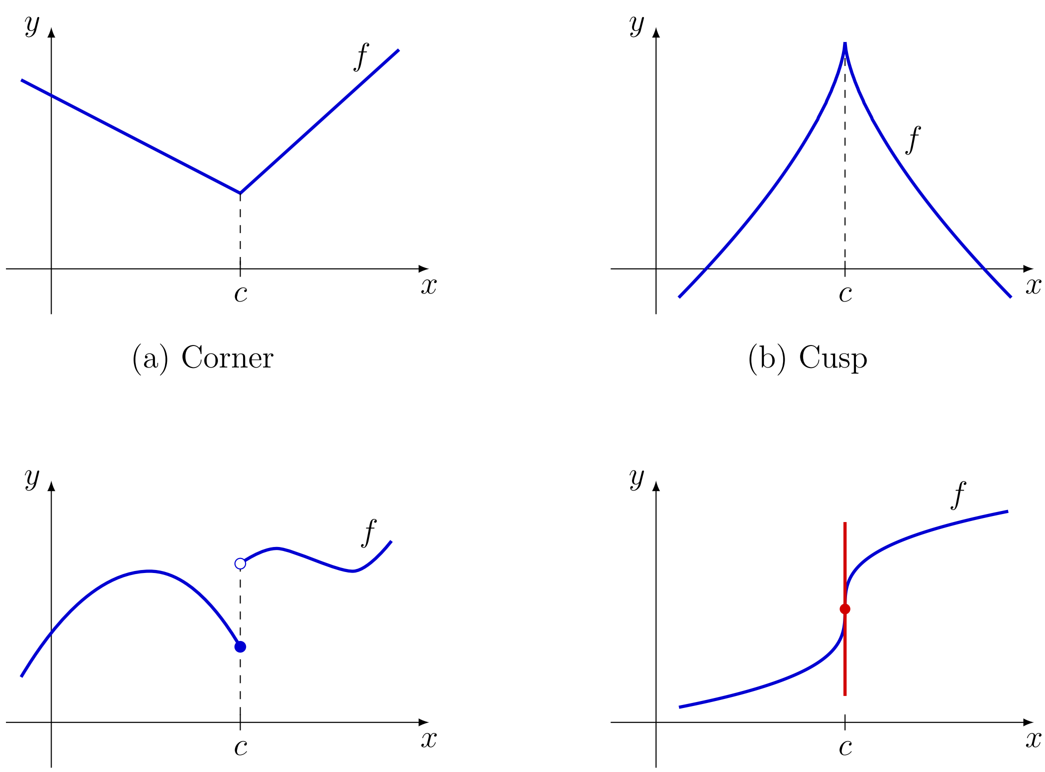
$$\frac{3}{2\sqrt{3x+1}} = 1 \;\Rightarrow\; 2\sqrt{3x+1} = 3 \;\Rightarrow\; 4(3x+1) = 9 \;\Rightarrow\; x = \frac{5}{12}.$$

Hence C has a slope of 1 only at the point $(5/12, f(5/12)) = (5/12, 3/2)$, and nowhere else. The tangent line to C at $(5/12, 3/2)$ is given by the equation

$$y - \frac{3}{2} = 1 \cdot \left(x - \frac{5}{12}\right),$$

which simplifies to become $y = x + 13/12$. See Figure 11. ■

There are many ways that a function f can fail to be differentiable at some point c in its domain, a few of which will now be discussed. Not uncommon is for a function to have a **corner**, which is a point $c \in \mathrm{Dom}(f)$ where f is continuous and $f'_-(c)$ and $f'_+(c)$ are real numbers, but $f'_-(c) \neq f'_+(c)$. Clearly $f'_-(c) \neq f'_+(c)$ implies that $f'(c)$ cannot exist. Figure 12(a) depicts a corner, and Example 3.8 shows that $f(x) = |x - 2|$ has a corner at $x = 2$.



(a) Corner

(b) Cusp

(c) Jump Discontinuity

(d) Vertical Tangent Line

Figure 12. Some common scenarios in which differentiability fails.

Another kind of point where differentiability can fail is a **cusp**, which is defined to be a point $c \in \mathrm{Dom}(f)$ where f is continuous, and yet the one-sided derivative limits go to opposite infinities: either $f_-(c) = +\infty$ and $f_+(c) = -\infty$ as in Figure 12(b), or $f_-(c) = -\infty$ and $f_+(c) = +\infty$ as in Example 3.9. Notice that this means f has a vertical tangent line at $(c, f(c))$.

Next there is the classic jump discontinuity, as in Figures 12(c) and 14, which defeats differentiability as surely as any other kind of discontinuity.

A more subtle scenario where the derivative of a function f fails to exist is at a point $(c, f(c))$ that is *not* an extreme point and yet the function has a vertical tangent line there. Of course, in light of the old Vertical Line Test no relation that is a function can have a graph that intersects a vertical line at more than one point, so the only way a function can have a vertical tangent line and still remain a function is for it to have vertical tangent lines at isolated points. See Figure 12(d). Are such functions as these very unusual? Not really. In Example 3.11 the familiar cube root function $f(x) = \sqrt[3]{x}$ is discovered to have a vertical tangent line at the origin.

Example 3.8. Show that $f(x) = |x-2|$ is not differentiable at $x = 2$.

Solution. By Proposition 3.2

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h},$$

so to show f is not differentiable at 2 means to show that the above limit does not exist. We can do this by showing that the corresponding one-sided limits do not agree. On one hand we have

$$\begin{aligned}\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0^-} \frac{|(2+h)-2| - |2-2|}{h} \\ &= \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1,\end{aligned}$$

where $|h| = -h$ since h is approaching 0 from the left and therefore $h < 0$; on the other hand we have

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{|(2+h)-2| - |2-2|}{h}$$

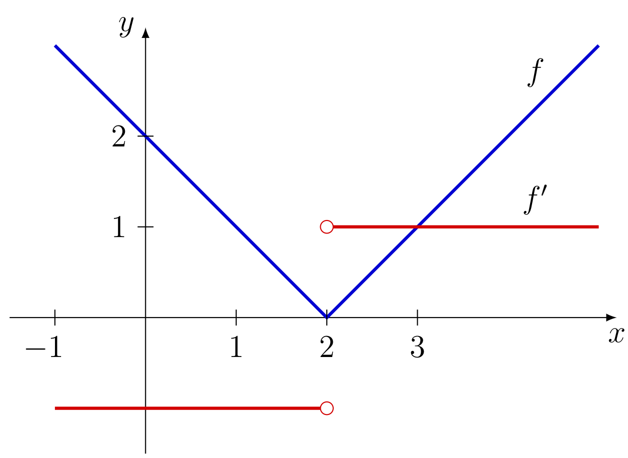


Figure 13. The graph of $|x-2|$ and its derivative.

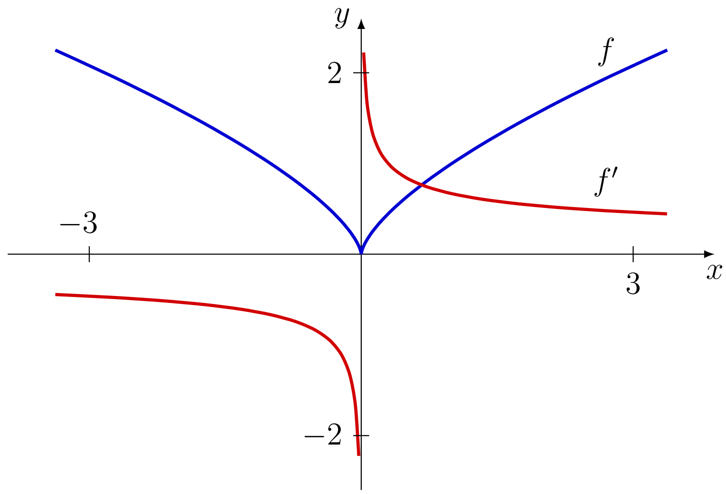


FIGURE 14. The graph of $x^{2/3}$ and its derivative.

$$= \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} (1) = 1,$$

where $|h| = h$ since h is approaching 0 from the right and therefore $h > 0$. Since $\lim_{h\to 0^-} = -1 \neq 1 = \lim_{h\to 0^+}$, it follows that $\lim_{h\to 0}$ does not exist. Therefore f is not differentiable at 2.

Figure 13 shows the graph of both f and f'. Note the corner in the graph of f at $x = 2$, and the corresponding jump discontinuity in the graph of f'. It can be seen that

$$f'(x) = \begin{cases} -1, & \text{if } x < 2 \\ 1, & \text{if } x > 2 \end{cases}$$

so while $\mathrm{Dom}(f) = (-\infty, \infty)$, we have $\mathrm{Dom}(f') = (-\infty, 0) \cup (0, \infty)$. That is, f is differentiable everywhere except at 2. ■

Example 3.9. Show that $f(x) = x^{2/3}$ is not differentiable at $x = 0$.

Solution. By definition

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t\to 0} \frac{t^{2/3}}{t} = \lim_{t\to 0} \frac{1}{\sqrt[3]{t}},$$

which clearly does not exist. In fact $f'_-(0) = -\infty$ and $f'_+(0) = +\infty$, so f has a cusp (and hence a vertical tangent line) at the point $(0, 0)$. See Figure 14. ■

Example 3.10. Show that

$$f(x) = \begin{cases} x, & \text{if } x \leq 2 \\ x + 1, & \text{if } x > 2 \end{cases}$$

is not differentiable at $x = 2$.

Solution. The function was engineered to be devious. A glance at Figure 15 would seem to suggest that the slope of the curve given by f must equal 1 at every point on the curve, including

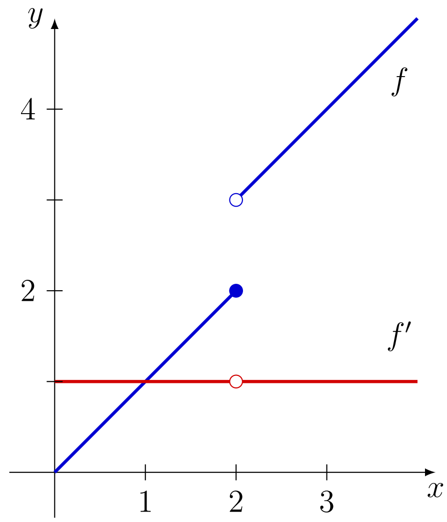


FIGURE 15.

the point $(2, f(2)) = (2, 2)$. So shouldn't $f'(x) = 1$ for all $x \in (-\infty, \infty)$? Alas, not quite. We do have

$$\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{(2+h) - 2}{h} = \lim_{h\to 0^-} \frac{h}{h} = \lim_{h\to 0^-} (1) = 1,$$

however

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{[(2+h)+1] - 2}{h} = \lim_{h\to 0^+} \frac{h+1}{h} = \lim_{h\to 0^+} \left(1 + \frac{1}{h}\right) = \infty$$

demonstrates that $f'(2)$ not only fails to equal 1, it fails to exist at all. ∎

Example 3.11. Show that $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$, and find f',

Solution. By definition

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t\to 0} \frac{\sqrt[3]{t}}{t} = \lim_{t\to 0} \frac{1}{\sqrt[3]{t^2}} = \infty,$$

so $f'(0) \notin \mathbb{R}$ and it's concluded that f' is not differentiable at 0. (However, the curve $y = f(x)$ is seen to have $x = 0$ as a vertical tangent line at $(0, 0)$.)

Next, for any $x \neq 0$ we have

$$\begin{aligned}
f'(x) &= \lim_{t\to x} \frac{f(x) - f(t)}{x - t} = \lim_{t\to x} \frac{\sqrt[3]{x} - \sqrt[3]{t}}{x - t} = \lim_{t\to x} \frac{\sqrt[3]{x} - \sqrt[3]{t}}{\left(\sqrt[3]{x}\right)^3 - \left(\sqrt[3]{t}\right)^3} \\
&= \lim_{t\to x} \frac{\sqrt[3]{x} - \sqrt[3]{t}}{\left(\sqrt[3]{x} - \sqrt[3]{t}\right)\left[\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}\sqrt[3]{t} + \left(\sqrt[3]{t}\right)^2\right]} \\
&= \lim_{t\to x} \frac{1}{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}\sqrt[3]{t} + \left(\sqrt[3]{t}\right)^2} \\
&= \frac{1}{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3},
\end{aligned}$$

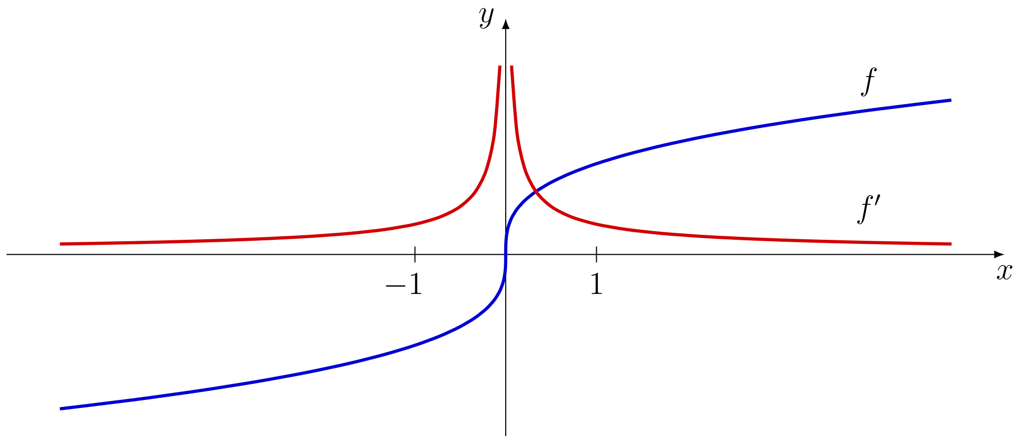


FIGURE 16. The graph of $\sqrt[3]{x}$ and its derivative.

recalling the factoring formula $u^3 - v^3 = (u-v)(u^2 + uv + v^2)$. Thus it's seen that f is differentiable for all $x \neq 0$. See Figure 16. ■

Any discontinuity at a point in the domain of a function f will preclude differentiability of f at that point, as the contrapositive of the next proposition makes clear.

Proposition 3.12. *If f is differentiable at x, then f is continuous at x.*

Proof. Suppose that f is differentiable at x, so x is an interior point of $\mathrm{Dom}(f)$, and the limit

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

exists in $\mathbb{R}$. Now, using limit laws established in §2.3, we obtain

$$\begin{aligned}
\lim_{t\to x} f(t) = \lim_{t\to x}[f(t) - f(x) + f(x)] &= \lim_{t\to x}\left[\frac{f(t) - f(x)}{t - x}\cdot (t-x) + f(x)\right] \\
&= \lim_{t\to x}\left[\frac{f(t) - f(x)}{t - x}\cdot (t-x)\right] + \lim_{t\to x} f(x) \\
&= \lim_{t\to x}\left[\frac{f(t) - f(x)}{t - x}\right]\cdot \lim_{t\to x}(t-x) + f(x) \\
&= f'(x)\cdot (x-x) + f(x) = f(x).
\end{aligned}$$

Therefore f is continuous at x. ■

The converse of Proposition 3.12 is not true in general. That is, if f is continuous at x, it does not necessarily follow that f is differentiable at x. For proof of this one need look no further than Example 3.8.

3.2 – Rules of Differentiation

Finding derivatives of functions using Definition 3.1 or the limit in Proposition 3.2 can be tedious at best and nearly impossible at worst. Fortunately there are properties that may be employed under most circumstances that reduce the process to a routine calculation.

Theorem 3.13 (**Rules of Differentiation**). *Suppose f and g are differentiable functions at x, and let $c \in \mathbb{R}$. Then the following hold.*

1. *Constant Multiple Rule: cf is differentiable at x, with*
$$(cf)'(x) = cf'(x).$$
2. *Sum/Difference Rule: $f \pm g$ is differentiable at x, with*
$$(f \pm g)'(x) = f'(x) \pm g'(x).$$
3. *Product Rule: fg is differentiable at x, with*
$$(fg)'(x) = f(x)g'(x) + f'(x)g(x).$$
4. *Quotient Rule: If $g(x) \neq 0$, then f/g is differentiable with*
$$(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Proof. The differentiability of f and g at x implies that the limits
$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \quad \text{and} \quad \lim_{t \to x} \frac{g(t) - g(x)}{t - x} \tag{3.4}$$
exist. It also implies x is an interior point of both $\text{Dom}(f)$ and $\text{Dom}(g)$, so there exists some $\gamma > 0$ such that $t \in \text{Dom}(f) \cap \text{Dom}(g)$ for all $t \in (x - \gamma, x + \gamma)$, and hence $f(t)$ and $g(t)$ are defined for all t sufficiently close to x. We use these facts in what follows.

Proof of Part 3. First observe that since g is differentiable at x, by Proposition 3.12 it is also continuous at x and so
$$\lim_{t \to x} g(t) = g(x).$$
Now, recalling the existence of the limits in (3.4) and employing usual limit laws, we have
$$\begin{aligned}
(fg)'(x) &= \lim_{t \to x} \frac{(fg)(t) - (fg)(x)}{t - x} = \lim_{t \to x} \frac{f(t)g(t) - f(x)g(x)}{t - x} \\
&= \lim_{t \to x} \frac{f(t)g(t) - f(x)g(t) + f(x)g(t) - f(x)g(x)}{t - x} \\
&= \lim_{t \to x} \left[\frac{f(t) - f(x)}{t - x} g(t) + \frac{g(t) - g(x)}{t - x} f(x) \right] \\
&= \lim_{t \to x} \left[\frac{f(t) - f(x)}{t - x} g(t) \right] + \lim_{t \to x} \left[\frac{g(t) - g(x)}{t - x} f(x) \right] \\
&= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot \lim_{t \to x} g(t) + f(x) \lim_{t \to x} \frac{g(t) - g(x)}{t - x} \\
&= f'(x)g(x) + f(x)g'(x),
\end{aligned}$$

as was to be shown.

Proof of Part 4. Suppose that $g(x) \neq 0$. In Part (3) we established that g is continuous at x, and so

$$\lim_{t\to x} \frac{1}{g(t)} = \frac{1}{\lim_{t\to x} g(t)} = \frac{1}{g(x)}. \tag{3.5}$$

Now,

$$\begin{aligned}
(f/g)'(x) &= \lim_{t\to x} \frac{(f/g)(t) - (f/g)(x)}{t-x} = \lim_{t\to x} \frac{f(t)/g(t) - f(x)/g(x)}{t-x} \\
&= \lim_{t\to x} \frac{f(t)g(x) - f(x)g(t)}{(t-x)g(t)g(x)} = \lim_{t\to x}\left[\frac{f(t)g(x) - f(x)g(t)}{t-x} \cdot \frac{1}{g(t)g(x)}\right] \\
&= \lim_{t\to x}\left[\frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{t-x} \cdot \frac{1}{g(t)g(x)}\right] \\
&= \lim_{t\to x}\left[\left(\frac{f(t)-f(x)}{t-x}g(x) - \frac{g(t)-g(x)}{t-x}f(x)\right) \cdot \frac{1}{g(t)g(x)}\right],
\end{aligned}$$

and since the limits in (3.4) and (3.5) exist we obtain, via usual limit laws,

$$\begin{aligned}
(f/g)'(x) &= \left[\lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}g(x)\right) - \lim_{t\to x}\left(\frac{g(t)-g(x)}{t-x}f(x)\right)\right] \cdot \lim_{t\to x} \frac{1}{g(t)g(x)} \\
&= \left[g(x)\lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right) - f(x)\lim_{t\to x}\left(\frac{g(t)-g(x)}{t-x}f(x)\right)\right] \cdot \frac{1}{g(x)} \lim_{t\to x} \frac{1}{g(t)} \\
&= \left[g(x) \cdot f'(x) - f(x) \cdot g'(x)\right] \cdot \frac{1}{g(x)} \cdot \frac{1}{g(x)} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.
\end{aligned}$$

This completes the proof. ∎

Theorem 3.13 gives derivatives of new functions in terms of the derivatives of old functions, whereas the next theorems give explicit formulas for the derivatives of constant and monomial functions.

Theorem 3.14 (**Constant Rule**). *For any constant $c \in \mathbb{R}$, if $f \equiv c$ then $f' \equiv 0$.*

Proof. Suppose $f \equiv c$, which is to say $f(x) = c$ for all x. For any x we use the definition of derivative to obtain

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{t\to x} \frac{c-c}{t-x} = \lim_{t\to x} \frac{0}{t-x} = \lim_{t\to x}(0) = 0;$$

that is, $f'(x) = 0$ for any x, and therefore $f' \equiv 0$. ∎

Letting the symbol c itself represent the constant function $f \equiv c$, as is typical, the Constant Rule can be written as $(c)' = 0$ or

$$\frac{d}{dx}(c) = 0.$$

Theorem 3.15 (Power Rule). *For any nonzero constant $r \in \mathbb{R}$,*

$$(x^r)' = rx^{r-1}.$$

Here we shall prove the Power Rule when r is any nonzero integer, and then in §3.5 the proof will be extended to include any nonzero rational number r. The treatment in the case when r is irrational must wait until Chapter 7. Whenever $r = 0$ we will regard x^r to be the constant function 1, in which case $(x^0)' = (1)' = 0$ according to Theorem 3.14.

Proof for Integer Powers. First we consider the case when $r = 1$, so that $x^r = x^1 = x$. Let $f(x) = x$. Then

$$(x^r)' = f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{t - x}{t - x} = \lim_{t \to x}(1) = 1 = 1 \cdot x^0 = rx^{r-1},$$

or more concisely $(x)' = 1$. This establishes the base case of an inductive argument.

Now let r be any positive integer, and suppose that $(x^r)' = rx^{r-1}$. Using the Product Rule of Theorem 3.13, and the fact that $(x)' = 1$, we obtain

$$(x^{r+1})' = (x \cdot x^r)' = x(x^r)' + (x)'x^r = x \cdot rx^{r-1} + 1 \cdot x^r = rx^r + x^r = (r+1)x^r.$$

By the Principle of Induction it follows that $(x^r)' = rx^{r-1}$ holds for any positive integer r. ■

We now have enough established to be able to find the derivative of any polynomial or rational function with relative ease. Moreover, though the proof of Theorem 3.15 is not yet complete, we shall nonetheless make full use of the result whilst cleaving fast to our faith that it shall someday be proven for any $r \in \mathbb{R}$.

Example 3.16. Find the derivative of each function.
(a) $f(x) = \sqrt{x}$.
(b) $g(x) = 3x^8 - 4x^3 + 7$.
(c) $h(x) = \dfrac{\sqrt{x}}{3x^8 - 4x^3 + 7}$

Solution.
(a) Observing that $\sqrt{x} = x^{1/2}$, we use the Power Rule to obtain

$$f'(x) = \frac{d}{dx}(x^{1/2}) = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

(b) Using the Sum/Difference Rule, then the Constant Multiple Rule and Constant Rule, and finally the Power Rule, we obtain

$$\begin{aligned} g'(x) &= \frac{d}{dx}(3x^8) - \frac{d}{dx}(4x^3) + \frac{d}{dx}(7) = 3\frac{d}{dx}(x^8) - 4\frac{d}{dx}(x^3) \\ &= 3 \cdot 8x^7 - 4 \cdot 3x^2 = 24x^7 - 12x^2. \end{aligned}$$

(c) Since $h = f/g$, we employ the Quotient Rule to find that

$$h'(x) = (f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$= \frac{\dfrac{1}{2\sqrt{x}} \cdot (3x^8 - 4x^3 + 7) - \sqrt{x} \cdot (24x^7 - 12x^2)}{(24x^7 - 12x^2)^2}$$
$$= \frac{3x^8 - 48x^7 - 4x^3 + 24x^2 + 7}{2\sqrt{x}(24x^7 - 12x^2)^2}.$$

■

Example 3.17. Find the derivative of

$$h(x) = \begin{cases} \dfrac{x-1}{x+1}, & \text{if } x \le 1 \\ \frac{1}{2}x - \frac{1}{2}, & \text{if } x > 1 \end{cases}$$

Solution. For any $x \in (-\infty, 1)$ with $x \neq -1$ there is an open neighborhood N of x such that

$$h(t) = \frac{t-1}{t+1}$$

for all $t \in N$, and so the Quotient Rule can be applied to obtain

$$h'(x) = \frac{2}{(x+1)^2}$$

for all $x \in (-\infty, -1) \cup (-1, 1)$. Note that $h'(-1)$ does not exist since $h(-1)$ is undefined.

For any $x \in (1, \infty)$ there is an open neighborhood N of x such that

$$h(t) = \frac{1}{2}t - \frac{1}{2}$$

for all $t \in N$, and so the Sum/Difference, Constant Multiple, Power, and Constant Rules can be used to find that $h'(x) = \frac{1}{2}$ for all $x \in (1, \infty)$.

At $x = 1$ there is an obstacle to using any rules of differentiation to find the derivative: any open neighborhood N of 1 necessarily contains values of t for which $h(t)$ is $(t-1)/(t+1)$, and other values of t for which $h(t)$ is $t/2 - 1/2$. This leaves us with no option other than to evaluate $h'(1)$ by using the definition of derivative. In particular we will need to evaluate the one-sided limits $h'_-(1)$ and $h'_+(1)$ and see whether they have the same value. We have

$$h'_-(1) = \lim_{t \to 1^-} \frac{h(t) - h(1)}{t-1} = \lim_{t \to 1^-} \frac{\dfrac{t-1}{t+1} - 0}{t-1} = \lim_{t \to 1^-} \frac{1}{t+1} = \frac{1}{2},$$

and

$$h'_+(1) = \lim_{t \to 1^+} \frac{h(t) - h(1)}{t-1} = \lim_{t \to 1^+} \frac{\frac{1}{2}t - \frac{1}{2} - 0}{t-1} = \lim_{t \to 1^+} \frac{1}{2} = \frac{1}{2}.$$

Therefore $h'(1) = \frac{1}{2}$, and we can write

$$h'(x) = \begin{cases} \dfrac{2}{(x+1)^2}, & \text{if } x < 1 \\ \frac{1}{2}, & \text{if } x \ge 1 \end{cases}$$

Note that writing $x \le 1$ and $x > 1$ instead of $x < 1$ and $x \ge 1$ would also be correct. ■

3.3 – Derivatives of Trigonometric Functions

To determine the derivatives of the trigonometric functions, we first need to evaluate two special limits.

Lemma 3.18.

$$\lim_{t\to 0}\frac{\sin t}{t}=1 \quad \textit{and} \quad \lim_{t\to 0}\frac{1-\cos t}{t}=0$$

Proof. For the first limit, suppose that $t\in(0,\pi/2)$. Proceeding counterclockwise a distance of t units from the point $a=(1,0)$ on the unit circle $C_1(0,0)$, we arrive at the point $p=(x,y)$ in the first quadrant as shown at left in Figure 17, and thereby obtain $\sin t=y$ by the definition of the sine function. As shown in the proof of Proposition 2.43 the area of $\triangle aop$ is $A_1=\sin t/2$, and the area of the circular sector with vertices a, o, and p is $A_2=t/2$.

Let q be the point where the vertical line $x=1$ (i.e. the vertical line containing the point a) intersects the ray $\overrightarrow{op}$. If we let $q=(1,z)$, then the triangle $\triangle aoq$ shown at right in Figure 17 has a base of length 1 and a height of length z. We know that $\tan t=y/x$, and since $\triangle xop$ and $\triangle aoq$ are similar triangles we obtain

$$\frac{y}{x}=\frac{z}{1},$$

and thus $z=\tan t$. The area of $\triangle aoq$ is therefore $A_3=\tan t/2$.

Now, since $0<A_1<A_2<A_3$ we obtain

$$0<\frac{\sin t}{2}<\frac{t}{2}<\frac{\tan t}{2}.$$

Multiplying this by $2/\sin t$ gives

$$1<\frac{t}{\sin t}<\frac{1}{\cos t},$$

and finally

$$\cos t<\frac{\sin t}{t}<1$$

for all $t\in(0,\pi/2)$. From $\lim_{t\to 0^+}\cos t=\cos 0=1$ and $\lim_{t\to 0^+}(1)=1$, it follows by the Squeeze Theorem that

$$\lim_{t\to 0^+}\frac{\sin t}{t}=1 \tag{3.6}$$

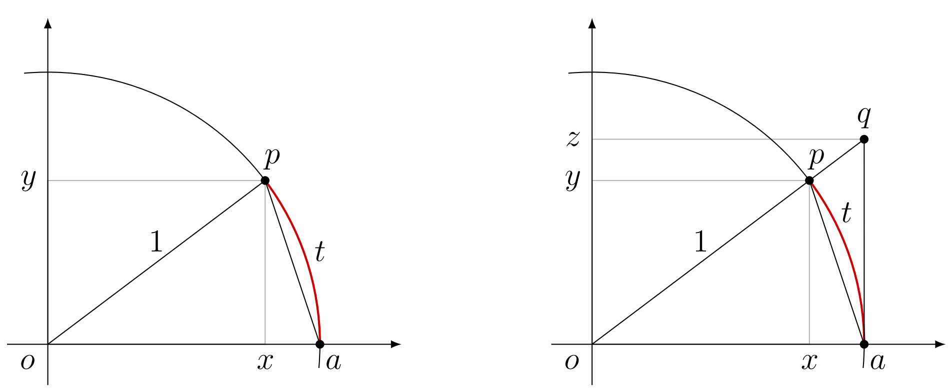


Figure 17.

as well.

In the case when $t \in (-\pi/2, 0)$ we proceed a distance of $|t|$ units *clockwise* on $C_1(0,0)$ from $a = (1,0)$ to a point $p = (x,y)$ in the fourth quadrant, so that $x > 0$ and $y < 0$. Once again $\sin t = y$, and the situation is as shown at left in Figure 18. Let $q = (1,z)$ be the point of intersection of the ray $\overrightarrow{op}$ and the line $x = 1$, so that $z < 0$, shown at right in Figure 18. Note that $z < 0$. We have

$$\begin{aligned} A_1 &= \text{Area of } \triangle aop = \frac{1}{2}|y| = \frac{|\sin t|}{2} = -\frac{\sin t}{2}, \\ A_2 &= \text{Area of circular sector} = \frac{|t|}{2} = -\frac{t}{2}, \\ A_3 &= \text{Area of } \triangle aoq = \frac{1}{2}|z| = \frac{|\tan t|}{2} = -\frac{\tan t}{2}, \end{aligned}$$

where $|z| = |y|/x = |\tan t|$ by similar triangles. From $0 < A_1 < A_2 < A_3$ comes

$$0 < -\frac{\sin t}{2} < -\frac{t}{2} < -\frac{\tan t}{2},$$

which when multiplied by $-2/\sin t$ (a positive quantity) yields

$$0 < 1 < \frac{t}{\sin t} < \frac{1}{\cos t},$$

and finally

$$\cos t < \frac{\sin t}{t} < 1$$

as before. This inequality holds for all $-\pi/2 < t < 0$ and since $\cos t \to 1$ as $t \to 0^-$, the Squeeze Theorem implies that

$$\lim_{t \to 0^-} \frac{\sin t}{t} = 1. \tag{3.7}$$

Combining (3.6) and (3.7), we conclude that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1,$$

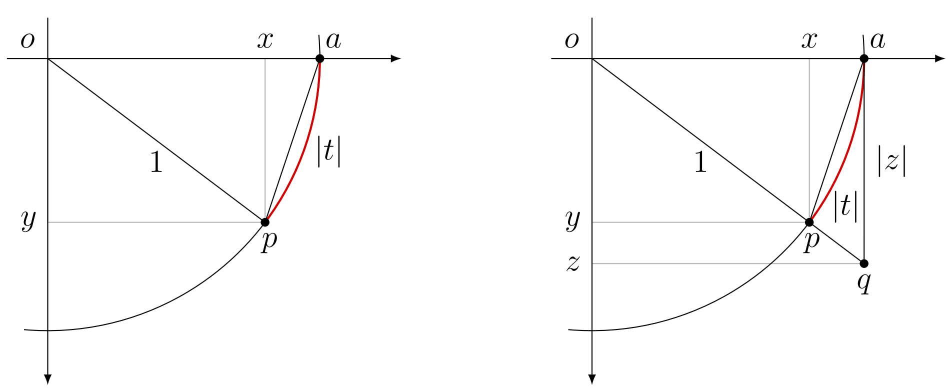


FIGURE 18.

which verifies the first limit in the lemma. As for the second limit, a straightforward calculation is all that is required:

$$\begin{aligned}
\lim_{t\to 0}\frac{1-\cos t}{t} &= \lim_{t\to 0}\left[\frac{1-\cos t}{t}\cdot\frac{1+\cos t}{1+\cos t}\right] = \lim_{t\to 0}\frac{1-\cos^2 t}{t(1+\cos t)} \\
&= \lim_{t\to 0}\frac{\sin^2 t}{t(1+\cos t)} = \lim_{t\to 0}\frac{\sin t}{t}\cdot\lim_{t\to 0}\frac{\sin t}{1+\cos t} \\
&= 1\cdot\frac{\sin 0}{1+\cos 0} = 1\cdot 0 = 0.
\end{aligned}$$

This completes the proof. ∎

Theorem 3.19. *For each t in the domain of each function,*

1. $\sin' t = \cos t$
2. $\cos' t = -\sin t$
3. $\tan' t = \sec^2 t$
4. $\csc' t = -\csc t\cot t$
5. $\sec' t = \sec t\tan t$
6. $\cot' t = -\csc^2 t$

Proof.
Proof of Part (1). For any $t\in\mathbb{R}$, using the identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

we have

$$\begin{aligned}
\sin' t &= \lim_{h\to 0}\frac{\sin(t+h)-\sin t}{h} = \lim_{h\to 0}\frac{\sin t\cos h + \cos t\sin h - \sin t}{h} \\
&= \lim_{h\to 0}\left[\cos t\left(\frac{\sin h}{h}\right) - \sin t\left(\frac{1-\cos h}{h}\right)\right] \\
&= \cos t\cdot\lim_{h\to 0}\frac{\sin h}{h} - \sin t\cdot\lim_{h\to 0}\frac{1-\cos h}{h} \\
&= \cos t\cdot 1 - \sin t\cdot 0 = \cos t
\end{aligned}$$

by Lemma 3.18.

Proof of Part (2). For any $t\in\mathbb{R}$, using the identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

we have

$$\begin{aligned}
\cos' t &= \lim_{h\to 0}\frac{\cos(t+h)-\cos t}{h} = \lim_{h\to 0}\frac{\cos t\cos h - \cos t\cos h - \cos t}{h} \\
&= \lim_{h\to 0}\left[\cos t\left(\frac{\cos h - 1}{h}\right) - \sin t\left(\frac{\sin h}{h}\right)\right] \\
&= \cos t\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin t\cdot\lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos t\cdot 0 - \sin t\cdot 1 = -\sin t
\end{aligned}$$

by Lemma 3.18.

Proof of Part (3). For any $t \in \text{Dom}(\tan)$, by the Quotient Rule together with parts (1) and (2),

$$\begin{aligned}\tan' t &= \left(\frac{\sin t}{\cos t}\right)' = \frac{\cos t \sin' t - \cos' t \sin t}{\cos^2 t} = \frac{\cos t \cos t + \sin t \sin t}{\cos^2 t} \\ &= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t.\end{aligned}$$

■

As with the tangent function, since $\csc = 1/\sin$, $\sec = 1/\cos$, and $\cot = 1/\tan$, the proofs of the last three parts of Theorem 3.19 may also be done using the Quotient Rule.

Example 3.20. Find y' for

$$y = \frac{2\cos x}{1 + \sin x}.$$

Solution. By the Quotient Rule,

$$y' = \frac{(1+\sin x)(2\cos x)' - (2\cos x)(1+\sin x)'}{(1+\sin x)^2} = \frac{(1+\sin x)(-2\sin x) - (2\cos x)(\cos x)}{(1+\sin x)^2},$$

which becomes

$$y' = -\frac{2}{\sin x + 1}$$

after some algebraic simplification. ■

3.4 – The Chain Rule

We have seen rules for finding the derivatives of sums, differences, products, and quotients of functions. Now we establish a rule for finding the derivative of a composition of two functions.

Theorem 3.21 (**Chain Rule**). *If g is differentiable at c and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c, and*

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

Proof. Suppose that g is differentiable at c and f is differentiable at $g(c)$. Then c is in the interior of $\mathrm{Dom}(g)$, $g(c)$ is in the interior of $\mathrm{Dom}(f)$, and since g is continuous at c there exists some $\gamma > 0$ such that $g(c-\gamma, c+\gamma) \subseteq \mathrm{Dom}(f)$. Thus c is in the interior of $\mathrm{Dom}(f \circ g)$ and it is legitimate to investigate the differentiability of $f \circ g$ at c. Define the function ρ by

$$\rho(y) = \begin{cases} \dfrac{f(y) - f(g(c))}{y - g(c)} - f'(g(c)), & \text{if } y \neq g(c) \\ 0, & \text{if } y = g(c) \end{cases}$$

and observe that $g(c)$ is in the interior of $\mathrm{Dom}(\rho)$. Now, by the differentiability of f at $g(c)$,

$$\begin{aligned} \lim_{y \to g(c)} \rho(y) &= \lim_{y \to g(c)} \left[\frac{f(y) - f(g(c))}{y - g(c)} - f'(g(c)) \right] \\ &= \lim_{y \to g(c)} \frac{f(y) - f(g(c))}{y - g(c)} - \lim_{y \to g(c)} f'(g(c)) \\ &= f'(g(c)) - f'(g(c)) = 0 = \rho(g(c)), \end{aligned}$$

which shows that ρ is continuous at $g(c)$.

Since $g(x) \to g(c)$ as $x \to c$, $g(c) \in \mathrm{Int}(\mathrm{Dom}(\rho))$, and ρ is continuous at $g(c)$, by Proposition 2.41 we obtain

$$\lim_{x \to c} \rho(g(x)) = \rho\left(\lim_{x \to c} g(x)\right) = \rho(g(c)) = 0.$$

Now, for any $g(x) \in \mathrm{Dom}(\rho)$ such that $g(x) \neq g(c)$, we find that

$$\rho(g(x)) = \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} - f'(g(c))$$

and hence

$$f(g(x)) - f(g(c)) = [f'(g(c)) + \rho(g(x))][g(x) - g(c)]. \tag{3.8}$$

Since (3.8) also holds whenever $g(x) = g(c)$, we conclude that it holds for all $x \in (c-\gamma, c+\gamma)$ and so

$$\begin{aligned} (f \circ g)'(c) &= \lim_{x \to c} \frac{(f \circ g)(x) - (f \circ g)(c)}{x - c} = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\ &= \lim_{x \to c} \frac{[f'(g(c)) + \rho(g(x))][g(x) - g(c)]}{x - c} \\ &= \lim_{x \to c} [f'(g(c)) + \rho(g(x))] \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \end{aligned}$$

$$= [f'(g(c)) + 0] \cdot g'(c) = f'(g(c))g'(c),$$

which completes the proof. ■

The reason for the Chain Rule's name becomes evident when we consider the derivative of a composition of three or more functions. If h is differentiable at c, g is differentiable at $h(c)$, and f is differentiable at $g(h(c))$, then

$$(f \circ g \circ h)'(c) = f'((g \circ h)(c))g'(h(c))h'(c) = f'(g(h(c))) \cdot g'(h(c)) \cdot h'(c). \tag{3.9}$$

We show this as follows. Let $\varphi = g \circ h$, and note that φ is differentiable at c by Theorem 3.21, and by the same theorem we obtain

$$\varphi'(c) = (g \circ h)'(c) = g'(h(c))h'(c).$$

Now, since φ is differentiable at c and f is differentiable at $g(h(c)) = \varphi(c)$, by Theorem 3.21 again we obtain

$$(f \circ \varphi)'(c) = f'(\varphi(c))\varphi'(c),$$

and hence

$$(f \circ \varphi)'(c) = f'(\varphi(c))g'(h(c))h'(c).$$

Substituting $g \circ h$ for φ in this equation then yields (3.9). More generally a proof by induction gives

$$(f_1 \circ \cdots \circ f_n)'(x) = \big[f_1'((f_2 \circ \cdots \circ f_n)(x))\big]\big[f_2'((f_3 \circ \cdots \circ f_n)(x))\big] \cdots \big[f_{n-1}'(f_n(x))\big]\big[f_n'(x)\big].$$

Example 3.22. Find the derivative of

$$H(x) = \sin\left(\tan\left(\sqrt{\cos x}\,\right)\right).$$

Solution. Here $H = f \circ g \circ h \circ p$ with

$$f(x) = \sin x, \quad g(x) = \tan x, \quad h(x) = \sqrt{x} = x^{1/2}, \quad p(x) = \cos x.$$

We obtain

$$\begin{aligned} H'(x) &= f'(g(h(p(x)))) \cdot g'(h(p(x))) \cdot h'(p(x)) \cdot p'(x) \\ &= \cos\left(\tan\left(\sqrt{\cos x}\,\right)\right) \cdot \sec^2\left(\sqrt{\cos x}\,\right) \cdot \tfrac{1}{2}(\cos x)^{-1/2} \cdot (-\sin x) \\ &= -\frac{\cos\left(\tan\left(\sqrt{\cos x}\,\right)\right)\sec^2\left(\sqrt{\cos x}\,\right)\sin x}{2\sqrt{\cos x}} \end{aligned}$$

for all $x \in \mathrm{Int}(\mathrm{Dom}(H))$ where the differentiability conditions are satisfied. ■

With the Chain Rule we now have the tools needed to consider the example that is often given to demonstrate that the derivative of a function is not necessarily continuous on its domain.

Example 3.23. The function

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable on $(-\infty, \infty)$, and yet f' has a discontinuity at 0! To see this we must find $f'(x)$ for all x.

When $x \neq 0$ the function f is simply given by $f(x) = x^2 \sin(1/x)$. Since $1/x$, $\sin(x)$, and x^2 are each clearly differentiable on $I = (-\infty, 0) \cup (0, \infty)$, we can employ the usual differentiation laws to f on I to obtain

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right). \tag{3.10}$$

Because $1/x$ is not differentiable at $x = 0$ we cannot use differentiation laws to find $f'(0)$, and anyway (3.10) fails when $x = 0$. This does not necessarily mean that f itself is not differentiable at 0, however. To find $f'(0)$ the only recourse is to use the definition of derivative:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0,$$

where the last equality follows from an example of the Squeeze Theorem in §2.3. So $f'(0) = 0$, and to show that f' is not continuous at 0 we need only that $\lim_{x \to 0} f'(x) \neq f'(0)$. Employing (3.10), we obtain

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right].$$

Suppose this limit exists and equals $L \in \mathbb{R}$. Since

$$\lim_{x \to 0} 2x \sin\left(\frac{1}{x}\right) = 2 \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 2 \cdot 0 = 0,$$

a law of limits gives

$$\begin{aligned} \lim_{x \to 0} -\cos\left(\frac{1}{x}\right) &= \lim_{x \to 0} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) - 2x \sin\left(\frac{1}{x}\right)\right] \\ &= \lim_{x \to 0} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right] - \lim_{x \to 0} 2x \sin\left(\frac{1}{x}\right) \\ &= L - 0 = L, \end{aligned}$$

and thus

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right) = -L \in \mathbb{R}.$$

But this is impossible, since it was shown in §2.2 that this limit does not exist! Hence $\lim_{x \to 0} f'(x)$ does not exist, and in particular $\lim_{x \to 0} f'(x) \neq 0 = f'(0)$. Therefore f' is not continuous at $x = 0$.

What we have found is that

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

So $\mathrm{Dom}(f) = \mathrm{Dom}(f') = (-\infty, \infty)$, with f continuous on $(-\infty, \infty)$[6] and f' continuous on $(-\infty, 0) \cup (0, \infty)$. ■

[6]Note that $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin(1/x) = \lim_{x\to 0} x \cdot \lim_{x\to 0} x \sin(1/x) = 0 \cdot 0 = 0 = f(0)$.

3.5 – Implicit Differentiation

An equation containing two variables x and y, with the y not isolated on one side, may be looked upon as (at least potentially) **implicitly** defining y as a function of x in one or more ways. For example, $x^2 + y^2 = 9$ results in two functions that cast x as the independent variable and y as the dependent variable:

$$y = \sqrt{9 - x^2} \quad \text{and} \quad y = -\sqrt{9 - x^2}.$$

Thus the equation $x^2 + y^2 = 9$ implicitly defines $y = f(x)$ for two choices of f: namely, $f(x) = \pm\sqrt{9 - x^2}$. The domain of both possible functions f is $[-3, 3]$, and from $x^2 + y^2 = 9$ we find that $x^2 + [f(x)]^2 = 9$ for all $x \in [-3, 3]$. Noting that f is differentiable on $(-3, 3)$, it follows that

$$\frac{d}{dx}\left(x^2 + [f(x)]^2\right) = \frac{d}{dx}(9)$$

for all $x \in (-3, 3)$, and then by various differentiation rules (including the Chain Rule) we obtain

$$2x + 2f(x)f'(x) = 0,$$

or equivalently $x + yy' = 0$ if we let $y' = f'(x)$. Now we can solve for y' in term of x and y to get

$$y' = -\frac{x}{y},$$

at least whenever $y \neq 0$.

Implicit differentiation is the operation of identifying an equation with two variables (such as $x^2 + y^2 = 9$) as implicitly defining y as a differentiable function of x, and then differentiating both sides of the equation with respect to x so as to obtain a new equation featuring x, y, and y' (such as $x + yy' = 0$). One of the great advantages of implicit differentiation is that it allows for finding dy/dx in situations when it would be quite difficult or outright impossible to isolate y in an equation involving both x and y; situations, that is, in which putting the equation into the form $y = f(x)$ for some suitable function f is simply not practical.

Example 3.24. Assume that

$$x^4 + 2x^2y^2 + y^4 = \tfrac{25}{4}xy^2. \tag{3.11}$$

implicitly defines y as a function of x.

(a) Use implicit differentiation to find dy/dx.

(b) Determine an equation of the tangent line to the curve at the point $(1, 2)$.

Solution.

(a) Let the symbol $'$ signify differentiation with respect to the variable x. Since $y = f(x)$ for some (unknown) function f, we have in particular

$$(y^2)' = \frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2yy'$$

by the Chain Rule. Similarly $(y^4)' = 4y^3y'$, and so on. Differentiating both sides of (3.11) with respect to x, then, we find that

$$4x^3 + 4xy^2 + 4x^2yy' + 4y^3y' = \frac{25}{4}y^2 + \frac{25}{2}xyy'.$$

Solving for y' gives

$$y' = \frac{25y^2 - 16xy^2 - 16x^3}{16x^2y + 16y^3 - 50xy}. \tag{3.12}$$

Of course $dy/dx = y'$, so we are done.

(b) At $(x, y) = (1, 2)$ we obtain $y' = \frac{1}{3}$ from (3.12), and so the tangent line has point $(1, 2)$ and slope $\frac{1}{3}$. Using the point-slope formula, the equation of the tangent line is $y - 2 = \frac{1}{3}(x - 1)$, or

$$y = \tfrac{1}{3}x + \tfrac{5}{3}$$

in slope-intercept form. ■

As something of an aside, it is possible to isolate y in the equation (3.11). Multiplying by 4 and collecting terms, we may write (3.11) as

$$4y^4 + (8x^2 - 25x)y^2 + 4x^4 = 0,$$

from which the quadratic formula gives

$$y^2 = \frac{(25x - 8x^2) \pm \sqrt{(8x^2 - 25x)^2 - 64x^4}}{8},$$

and hence

$$y = \pm\sqrt{\frac{(25x - 8x^2) + 5\sqrt{25x^2 - 8x^3}}{8}} \quad \text{or} \quad y = \pm\sqrt{\frac{(25x - 8x^2) - 5\sqrt{25x^2 - 8x^3}}{8}}.$$

That is, y can equal any one of four different functions, none of which would have a particularly nice derivative. Thus, even when it's possible to isolate y as a function of x, applying implicit differentiation may still be the better strategy.

In the next example there is no reasonable way to isolate y in the given equation, and so implicit differentiation is the only game in town.

Example 3.25. Find the value of y'' at the point on the curve

$$xy + y^3 = 1$$

where $x = 0$.

Solution. The equation $xy + y^3 = 1$ implicitly defines y as a function of x, and so we proceed to first find $y' = dy/dx$ using implicit differentiation:

$$xy + y^3 = 1 \;\Rightarrow\; \frac{d}{dx}(xy + y^3) = \frac{d}{dx}(1) \;\Rightarrow\; xy' + y + 3y^2y' = 0 \;\Rightarrow\; y' = -\frac{y}{x + 3y^2}.$$

When $x = 0$ the original equation becomes $y^3 = 1$, for which $y = 1$ is the only real-valued solution, and thus $(0, 1)$ is the only point on the curve where $x = 0$. At this point we find that

$$y' = -\frac{1}{0 + 3(1)^2} = -\frac{1}{3}.$$

Next, using the Quotient Rule we have

$$y'' = \frac{d}{dx}\left(-\frac{y}{x + 3y^2}\right) = -\frac{(x + 3y^2)y' - (1 + 6yy')y}{(x + 3y^2)^2}.$$

Since $y' = -\frac{1}{3}$ when $(x,y) = (0,1)$, we find that

$$y'' = -\frac{\big[0+3(1)^2\big]\big(-\frac{1}{3}\big) - \big[1+6(1)\big(-\frac{1}{3}\big)\big](1)}{[0+3(1)^2]^2} = 0$$

at the point where $x = 0$. ■

In §3.2 the Power Rule of differentiation was proven for integer exponents. Now, with implicit differentiation, we are in a position to expand the proof to include rational exponents. For convenience we restate the Power Rule here.

Theorem 3.26 (**Power Rule**). *For any nonzero constant $r \in \mathbb{R}$,*

$$(x^r)' = rx^{r-1}.$$

Proof for Rational Powers. Let r be any nonzero rational number, so $r = m/n$ for some integers $m \neq 0$ and $n \geq 1$. Set $y = x^r = x^{m/n}$, so y is a function of x. From this we find that

$$y^n = (x^{m/n})^n = x^m,$$

which is an equation suitable for implicit differentiation. Since the Power Rule has already been proven to work for integer exponents, it follows that

$$y^n = x^m \quad \Rightarrow \quad \frac{d}{dx}(y^n) = \frac{d}{dx}(x^m) \quad \Rightarrow \quad ny^{n-1}y' = mx^{m-1},$$

and therefore

$$y' = \frac{mx^{m-1}}{ny^{n-1}} = \frac{m}{n}x^{m-1}(x^{m/n})^{1-n} = \frac{m}{n}x^{(m-1)+(m/n-m)} = \frac{m}{n}x^{m/n-1} = rx^{r-1}.$$

That is, $(x^r)' = rx^{r-1}$, and the Power Rule is now proven for any rational exponent r. ■

3.6 – Rates of Change

Example 3.27. A rope passing through a capstan on a dock is tied at water level to a boat offshore. If the capstan is 5 ft above the water and Barnacle Bill the Sailor turns the capstan so as to pull the rope in at a constant rate of 3 ft/s, how fast is the boat traveling when it is 10 ft from the dock?

Solution. Let x be the distance between the boat and the dock, and r the length of the rope. As seen in Figure 19 there is a right triangle involved so that $x^2 + 5^2 = r^2$ by the Pythagorean Theorem. Solving this for x gives

$$x = \sqrt{r^2 - 25}. \tag{3.13}$$

Here x and r are implicitly functions of time, which is simply to say they change as time changes, and so we differentiate (3.13) with respect to time to obtain

$$x'(t) = \frac{r(t)r'(t)}{\sqrt{r^2(t) - 25}}. \tag{3.14}$$

Now, we're given that $r'(t) = -3$ ft/s. Moreover, at the time t when $x(t) = 10$ feet we have $r(t) = 5\sqrt{5}$ feet by (3.13). Putting these facts into (3.14) gives

$$x'(t) = \frac{(5\sqrt{5}\,)(-3)}{\sqrt{\left(5\sqrt{5}\,\right)^2 - 25}} = -\frac{3\sqrt{5}}{2}\ \text{ft/s}.$$

This tells us that the distance between the boat and the dock is *decreasing* at a rate of $3\sqrt{5}/2$ ft/s at the instant when the boat is 10 ft from the dock. Thus, when the boat is 10 ft from the dock it is traveling at a speed of $3\sqrt{5}/2$ ft/s. ■

Example 3.28. A 2-meter-tall man walks at 1 m/s toward a street light that is 7 meters above the ground. What is the rate of change of the length of his shadow when he is 5 meters from the street light? At what rate is the tip of his shadow moving?

Solution. Let x be the distance between the man and the base of the street light, and let ℓ be the length of the man's shadow. The triangles $\triangle ABC$ and $\triangle OBD$ in Figure 20 are similar, and so we have

$$\frac{\ell + x}{7} = \frac{\ell}{2}.$$

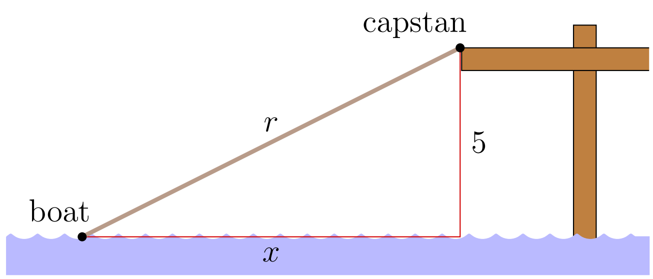


Figure 19.

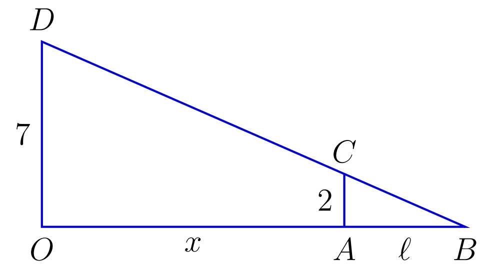


FIGURE 20.

Solving this for ℓ and observing that ℓ and x are both functions of time t, we obtain

$$\ell(t) = \frac{2}{5}x(t).$$

Differentiating both sides with respect to t gives

$$\ell'(t) = \frac{2}{5}x'(t) = \frac{2}{5} \cdot (-1) = -\frac{2}{5} \text{ m/s}.$$

Thus at any time t the length of the shadow is growing shorter at a rate of $\frac{2}{5}$ m/s, which includes the time when the man is 5 meters from the street light!

Regarding the rate at which the tip of the shadow is moving, since point B in Figure 20—which is the tip of the shadow—is moving toward A at $-\frac{2}{5}$ m/s, and A is moving toward O at -1 m/s, is follows that B is moving toward O at $-1\frac{2}{5}$ m/s. ■

4

Applications of Differentiation

4.1 – Extrema of Functions

Definition 4.1. *Let f be a real-valued function, $S \subseteq \mathrm{Dom}(f) \subseteq \mathbb{R}$, and $c \in S$.*

*If $f(c) \geq f(x)$ for all $x \in S$, then we say f has a **maximum on S at c** and call $f(c)$ the **maximum value of f on S**. In the event that $S = \mathrm{Dom}(f)$ we say f has a **global maximum at c** and call $f(c)$ the **global maximum value of f** (or simply the **maximum value of f**).*

*If $f(c) \leq f(x)$ for all $x \in S$, then we say f has a **minimum on S at c** and call $f(c)$ the **minimum value of f on S**. In the event that $S = \mathrm{Dom}(f)$ we say f has a **global minimum at c** and call $f(c)$ the **global minimum value of f** (or simply the **minimum value of f**).*

The maxima and minima (or maximums and minimums) of a function are collectively referred to as the function's **extrema** (or **extremums**), with the maximum and minimum values being called the **extreme values**. The next theorem makes clear that any function that is continuous on a closed, bounded interval I will attain both a maximum value and a minimum value on I.

Theorem 4.2 (Extreme Value Theorem). *If f is continuous on a closed interval $I = [a, b]$, then there exist $x_1, x_2 \in I$ such that $f(x_1)$ is a maximum value of f on I and $f(x_2)$ is a minimum value of f on I.*

The proof of this theorem is beyond the scope of this text, but can be found in advanced calculus or mathematical analysis texts.

Definition 4.3. *Suppose c is a point in the interior of $\mathrm{Dom}(f)$. Then f has a **local maximum at c** if there exists some $\gamma > 0$ such that $f(c) \geq f(x)$ for all $x \in (c-\gamma, c+\gamma)$, in which case $f(c)$ is called a **local maximum value of f**. Similarly, f has a **local minimum at c** if there exists some $\gamma > 0$ such that $f(c) \leq f(x)$ for all $x \in (c-\gamma, c+\gamma)$, in which case $f(c)$ is called a **local minimum value of f**.*

The local maxima and minima of a function are collectively called the function's **local extrema**. Comparing Definitions 4.1 and 4.3, we see that a global extremum for f at c is necessarily also a local extremum provided that c lies in the interior of $\mathrm{Dom}(f)$.

The following proposition, like the Squeeze Theorem, gives some inequality results for limits of functions; and like the Squeeze Theorem it will be used occasionally in certain proofs.

Proposition 4.4.

1. *Let $c \in \mathbb{R}$, and suppose there exists some $\gamma > 0$ such that $f(x) \le g(x)$ for all $x \in B'_\gamma(c)$. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then*
$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$
2. *Let $c \in (-\infty, \infty]$. If $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^-} g(x)$ exist, and there is some $\gamma > 0$ such that $f(x) \le g(x)$ for all $x \in (c-\gamma, c)$, then*
$$\lim_{x\to c^-} f(x) \le \lim_{x\to c^-} g(x).$$
3. *Let $c \in [-\infty, \infty)$. If $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^+} g(x)$ exist, and there is some $\gamma > 0$ such that $f(x) \le g(x)$ for all $x \in (c, c+\gamma)$, then*
$$\lim_{x\to c^+} f(x) \le \lim_{x\to c^+} g(x).$$

Proof. We prove only the first part here, since the proofs of the other two parts would run along similar lines.

Suppose that $\lim_{x\to c} f(x) = L \in \mathbb{R}$ and $\lim_{x\to c} g(x) = M \in \mathbb{R}$. Assume $L > M$. Let $\epsilon = (L-M)/2$, so that $\epsilon > 0$. Now, there exists some $\delta_1 > 0$ such that
$$0 < |x-c| < \delta_1 \;\;\Rightarrow\;\; |f(x) - L| < \epsilon,$$
and there exists some $\delta_2 > 0$ such that
$$0 < |x-c| < \delta_2 \;\;\Rightarrow\;\; |g(x) - M| < \epsilon.$$
Choosing $\delta = \min\{\delta_1, \delta_2, \gamma\}$, suppose that x is such that $0 < |x-c| < \delta$. Then we obtain
$$|f(x) - L| < \frac{L-M}{2} \quad \text{and} \quad |g(x) - M| < \frac{L-M}{2},$$
with the first inequality yielding
$$f(x) > \frac{L+M}{2}$$
and the second inequality yielding
$$g(x) < \frac{L+M}{2},$$
and hence
$$f(x) > \frac{L+M}{2} > g(x). \tag{4.1}$$
But $0 < |x-c| < \delta$ implies that $0 < |x-c| < \gamma$, which is to say $x \in B'_\gamma(c)$ and so we must have $f(x) \le g(x)$ by hypothesis. Since (4.1) contradicts this hypothesis, we conclude that $L \le M$. ■

Theorem 4.5 (**Fermat's Theorem**). *If f has a local extremum at c and $f'(c)$ exists, then $f'(c) = 0$.*

Proof. Suppose f has a local maximum at c, so there is some $\gamma > 0$ such that $f(c) \geq f(x)$ for all $x \in (c-\gamma, c+\gamma)$. Suppose $f'(c)$ exists, so that

$$f'(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \in \mathbb{R}.$$

Now, for all $x \in (c-\gamma, c)$ we have $f(x) - f(c) \leq 0$ and $x - c < 0$, which implies

$$\frac{f(x)-f(c)}{x-c} \geq 0$$

and thus by Proposition 4.4(2)

$$f'(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq \lim_{x \to c^-} (0) = 0.$$

Also for all $x \in (c, c+\gamma)$ we have $f(x) - f(c) \leq 0$ and $x - c > 0$, which implies

$$\frac{f(x)-f(c)}{x-c} \leq 0$$

and thus by Proposition 4.4(3)

$$f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq \lim_{x \to c^-} (0) = 0.$$

Since $f'(c) \geq 0$ and $f'(c) \leq 0$, we conclude that $f'(c) = 0$.

The proof is similar when supposing that f has a local minimum at c. ■

Corollary 4.6. *If f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.*

The corollary makes it clear where we should focus our attention when searching for local extrema of a function f: only those points $c \in \mathrm{Dom}(f)$ where $f'(c)$ is zero or does not exist. Finding the global extrema of f, if any, is often more challenging, but with the help of the Extreme Value Theorem we will devise a method for locating global extrema at least in situations when the domain of f is restricted to a closed, bounded interval.

Definition 4.7. *A **critical point** of a function f is any point c in the interior of* $\mathrm{Dom}(f)$ *for which either $f'(c) = 0$ or $f'(c)$ does not exist.*

Theorem 4.8 (**Closed Interval Method**)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $K \subseteq [a, b]$ is the set of critical points of f. Let $c \in K$.*

1. *If f has a maximum on K at c, then f has a maximum on $[a, b]$ at c.*
2. *If f has a minimum on K at c, then f has a minimum on $[a, b]$ at c.*

Proof.
Proof of Part (1). Suppose that f has a maximum on K at c. Now, by the Extreme Value Theorem there is some $\hat{c} \in [a, b]$ such that $f(\hat{c})$ is a maximum value of f on $[a, b]$, and so f has a local maximum at $\hat{c}$. By Corollary 4.6 either $f'(\hat{c}) = 0$ or $f'(\hat{c})$ does not exist, which is to say that $\hat{c}$ is a critical point of f and thus $\hat{c} \in K$. Since f has a maximum on K at c we have $f(c) \geq f(\hat{c})$. Since $f(\hat{c})$ is a maximum of f on $[a, b]$ we have $f(c) \leq f(\hat{c})$. Therefore $f(c) = f(\hat{c})$, which shows that f has a maximum on $[a, b]$ at c.

Proof of Part (2). Left as an exercise. ■

In practice the following procedure is what is often referred to as the Closed Interval Method, and it follows directly from the preceding theorem.

Procedure. To find the extreme values of a continuous function $f : [a, b] \to \mathbb{R}$:

- Step 1. Find the critical points of f on (a, b);
- Step 2. Evaluate f at all critical points on (a, b);
- Step 3. Evaluate f at the endpoints a and b.

The greatest and least values obtained in Steps (2) and (3) are the maximum and minimum values of f on $[a, b]$, respectively. ∎

Example 4.9. Find the global extrema of the function $f(x) = x\sqrt{x - x^2}$.

Solution. First we observe that the domain of f is the set

$$\operatorname{Dom}(f) = \{x : x - x^2 \geq 0\} = \{x : x(1 - x) \geq 0\} = [0, 1],$$

a closed and bounded interval. Since f is continuous on $[0, 1]$, we employ the Closed Interval Method. To start, we find any critical points of f that lie on $(0, 1)$; that is, we find all $0 < x < 1$ for which $f'(x)$ is zero or does not exist.

Setting $f'(x) = 0$ we obtain

$$\begin{aligned} f'(x) = 0 \quad &\Rightarrow \quad \frac{x(1-2x)}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = 0 \quad \Rightarrow \quad \sqrt{x - x^2} = -\frac{x(1-2x)}{2\sqrt{x - x^2}} \\ &\Rightarrow \quad 2(x - x^2) = -x(1 - 2x) \quad \Rightarrow \quad 4x^2 - 3x = 0 \\ &\Rightarrow \quad x(4x - 3) = 0 \quad \Rightarrow \quad x = 3/4. \end{aligned}$$

(Of course $x(4x - 3) = 0$ by itself implies $x = 0$ is also a solution, but not only does this not lie in $(0, 1)$, it is also an extraneous solution since it results in division by zero in the original equation $f'(x) = 0$.) There is no $x \in (0, 1)$ for which $f'(x)$ does not exist.

We now evaluate f at the critical point $3/4 \in (0, 1)$, and also at the endpoints 0 and 1:

$$f(0) = 0 \cdot \sqrt{0 - 0^2} = 0$$

$$f(3/4) = \frac{3}{4}\sqrt{\frac{3}{4} - \frac{9}{16}} = \frac{3\sqrt{3}}{16}$$

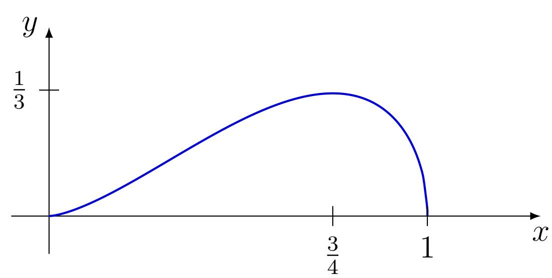


Figure 21.

$$f(1) = 1 \cdot \sqrt{1 - 1^2} = 0$$

Thus $f(3/4) = 3\sqrt{3}/16$ is the maximum value of f on $[0, 1]$, and $f(0) = f(1) = 0$ is the minimum value of f on $[0, 1]$. Because $\mathrm{Dom}(f) = [0, 1]$, it follows that we have found the global extrema of f. See Figure 21. ■

Example 4.10. Find the global extrema of the function $f(x) = |x + x^2|$, $x \in [-1.5, 0.8]$.

Solution. Because the functions $g(x) = x + x^2$ and $h(x) = |x|$ are each continuous everywhere, by Theorem 2.39 it follows that $(h \circ g)(x) = |x + x^2|$ is also continuous everywhere, and hence f is continuous on its given domain $[-1.5, 0.8]$ in particular.

Now, the equation $g(x) = 0$, or equivalently $x(1 + x) = 0$, has solutions $x = -1, 0$. Since $g(-2) > 0$ we conclude that $g > 0$ on $(-\infty, -1)$. Indeed, if there were some $c \in (-\infty, -1)$ such that $g(c) < 0$, then the Intermediate Value Theorem would imply that there is some number a between -1.25 and c, and hence in $(-\infty, -1)$, such that $g(a) = 0$, which is a contradiction. By the same reasoning, since $g(-0.5) < 0$ we have $g < 0$ on $(-1, 0)$, and since $g(1) > 0$ we have $g > 0$ on $(0, \infty)$. Noting that $f(x) = |g(x)|$ on $[-1.5, 0.8]$, we obtain

$$f(x) = \begin{cases} x + x^2, & \text{if } -1.5 \le x \le -1 \\ -x - x^2, & \text{if } -1 < x < 0 \\ x + x^2, & \text{if } 0 \le x \le 0.8 \end{cases} \tag{4.2}$$

We now begin the Closed Interval Method by finding any critical points of f that lie on $(-1.5, 0.8)$. From (4.2) we obtain

$$f'(x) = \begin{cases} 1 + 2x, & \text{if } -1.5 < x < -1 \\ -1 - 2x, & \text{if } -1 < x < 0 \\ 1 + 2x, & \text{if } 0 < x < 0.8, \end{cases}$$

from which $f'(-1)$ and $f'(0)$ are readily found to not exist by using the definition of derivative, and so -1 and 0 are critical points.

Setting $f'(x) = 0$ gives rise to the equation $1 + 2x = 0$ on $I_1 = (-1.5, -1) \cup (0, 0.8)$, which has no solution belonging to I_1. However $f'(x) = 0$ is the equation $-1 - 2x = 0$ on $I_2 = (-1, 0)$, which has a solution $x = -0.5 \in I_2$. That is, $f'(-0.5) = 0$ and so -0.5 is a critical point.

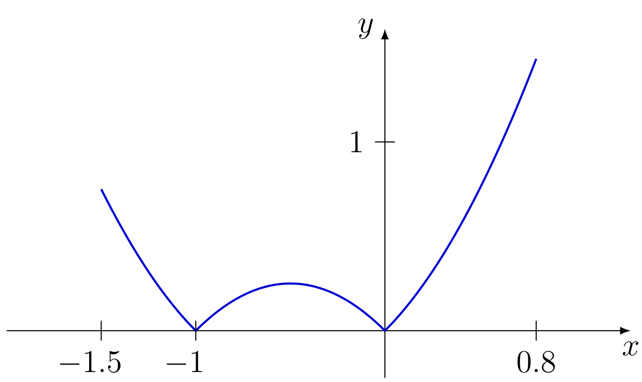


FIGURE 22.

We now evaluate f at the critical points -1.5, -1, -0.5, 0, and 0.8:

$$\begin{aligned}
f(-1.5) &= |-1.5+(-1.5)^2| = 0.75\\
f(-1) &= |-1+(-1)^2| = 0\\
f(-0.5) &= |-0.5+(-0.5)^2| = 0.25\\
f(0) &= |0+0^2| = 0\\
f(0.8) &= |0.8+0.8^2| = 1.44
\end{aligned}$$

Therefore $f(0.8) = 1.44$ is the maximum value of f on $[-1.5, 0.8]$, and $f(-1) = f(0) = 0$ is the minimum value of f on $[-1.5, 0.8]$. Since $\mathrm{Dom}(f) = [-1.5, 0.8]$ we have found the global extrema of f. See Figure 22. ■

4.2 – The Mean Value Theorem

We start with Rolle's Theorem, which is a special instance of the Mean Value Theorem and in fact will be used to prove it.

Theorem 4.11 (**Rolle's Theorem**). *Let f be differentiable on (a, b) and continuous on $[a, b]$. If $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$*

Proof. Suppose $f(a) = f(b)$. Since f is continuous on $[a, b]$, the Extreme Value Theorem implies that f has both a maximum and a minimum on $[a, b]$. There are two mutually exclusive possibilities: (i) At least one extremum lies in (a, b); and (ii) No extremum lies in (a, b).

Assume (i) is the case. Then there exists some $c \in (a, b)$ at which f has an extremum on $[a, b]$, which implies that f has a local extremum at c. Now, $f'(c)$ exists since f is differentiable on (a, b), and therefore $f'(c) = 0$ by Theorem 4.5.

Assume (ii) is the case. Then the extrema of f lie at the endpoints a and b. From $f(a) = f(b)$ it follows that the minimum of f on $[a, b]$ equals the maximum, and thus $f(a)$ is simultaneously the maximum value and minimum value attained by f on $[a, b]$. So, for any $x \in [a, b]$ we find that $f(x) \leq f(a)$ and $f(x) \geq f(a)$ are both true, which implies that $f(x) = f(a)$ and therefore f is in fact a constant function; that is, $f(x) = f(a)$ for all $x \in [a, b]$, which immediately leads to the conclusion that $f'(c) = 0$ for *any* $c \in (a, b)$. ■

Rolle's Theorem in conjunction with the Intermediate Value Theorem can be used to uncover a surprising assortment of things about the solution sets of many equations that do not readily lend themselves to being solved by algebraic means.

Example 4.12. Show that

$$x^3 - 2x^2 + 4x - 2 = 0 \tag{4.3}$$

has exactly one real root.

Solution. We first set about showing that the equation has at least one real root. Let

$$f(x) = x^3 - 2x^2 + 4x - 2.$$

Now, $f(0) = -2 < 0$ and $f(1) = 1 > 0$, and since f is a polynomial function it is continuous on $[0, 1]$. So, since 0 lies between $f(0)$ and $f(1)$, by the Intermediate Value Theorem there exists some $r \in (0, 1)$ such that $f(r) = 0$. This demonstrates that r is a real root of (4.3).

Now that it's been shown that the equation has one real root, it must next be shown that it cannot have *more than* one real root. Toward that end, suppose the equation has two real roots a and b with $a < b$, so that $f(a) = f(b) = 0$. Since f is differentiable, it follows from Rolle's Theorem that there is some real number $c \in (a, b)$ such that $f'(c) = 0$. Thus, because $f'(x) = 3x^2 - 4x + 4$, we conclude that c is a real number for which $3c^2 - 4c + 4 = 0$. However, when we apply the quadratic formula to solve

$$3x^2 - 4x + 4 = 0, \tag{4.4}$$

we find only the complex-valued solutions

$$\frac{2}{3} \pm \frac{2\sqrt{2}}{3}i.$$

We have arrived at a contradiction: c is a real-valued root for (4.4), and yet (4.4) has no real-valued roots! It follows that (4.3) cannot have more than one real root. Therefore (4.3) has exactly one real root. ∎

Example 4.13. Show that

$$x^4 + 4x + k = 0 \tag{4.5}$$

has at most two real roots, where $k \in \mathbb{R}$ is any constant.

Solution. When $k = 0$ the equation (4.5) becomes $x^4 + 4x = 0$, or equivalently $x(x^3 + 4) = 0$, and thus there are two real roots: $x = 0$ and $x = \sqrt[3]{-4}$. So there exists a real number k for which (4.5) has two real roots. It remains to show that no matter what k equals, (4.5) cannot have *more than* two real roots.

Suppose there exists some real number k such that (4.5) has more than two real roots. Denote three of these roots by r_1, r_2 and r_3, where $r_1 < r_2 < r_3$. Let

$$f(x) = x^4 + 4x + k,$$

so we have $f(r_1) = f(r_2) = f(r_3) = 0$. Since f is a polynomial function it is continuous and differentiable everywhere. By Rolle's Theorem, then, there exists some $c_1 \in (r_1, r_2)$ such that $f'(c_1) = 0$, and there exists some $c_2 \in (r_2, r_3)$ such that $f'(c_2) = 0$. Note that we have $r_1 < c_1 < r_2 < c_2 < r_3$.

Now, $f'(x) = 4x^3 + 4$, so $f'(c_1) = 0$ implies $4c_1^3 = -4$ and thus $c_1 = -1$; and $f'(c_2) = 0$ implies $c_2 = -1$ as well. We now have $r_1 < -1 < r_2 < -1 < r_3$, which leads to the conclusion that $-1 < -1$. Since this is a contradiction, it follows that (4.5) cannot have more than two real roots no matter what k is. Therefore (4.5) has at most two real roots. ∎

We now employ Rolle's Theorem for the purpose of proving a more general result called the Mean Value Theorem, which in turn will later be used to prove the Fundamental Theorem of Calculus.

Theorem 4.14 (Mean Value Theorem). *Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define the function g by

$$g(x) = \frac{f(b) - f(a)}{b - a}(a - x) + f(x),$$

so g is continuous on $[a, b]$ and differentiable on (a, b) since f has these properties, and $g(a) = f(a) = g(b)$. By Rolle's Theorem there exists some $c \in (a, b)$ such that $g'(c) = 0$, and since

$$g'(x) = -\frac{f(b) - f(a)}{b - a} + f'(x),$$

it follows that

$$-\frac{f(b)-f(a)}{b-a}+f'(c)=0.$$

Therefore

$$f'(c)=\frac{f(b)-f(a)}{b-a},$$

as was to be shown. ■

Example 4.15. Let

$$f(x)=\frac{x+1}{x-1}.$$

Show there's no $c\in[0,2]$ such that $f'(c)=(f(2)-f(0))/2$. Why does this not contradict the Mean Value Theorem?

Solution. Differentiating f, we obtain

$$f'(x)=-\frac{2}{(x-1)^2}.$$

We attempt to find some $0\leq c\leq 2$ such that

$$-\frac{2}{(c-1)^2}=f'(c)=\frac{f(2)-f(0)}{2}=\frac{3-(-1)}{2}=2,$$

However, a little algebra leads to $(c-1)^2=-1$, which clearly has no real solution whatsoever, much less a solution in the interval $[0,2]$. The reason that this outcome does not dethrone the kingly Mean Value Theorem is that f is not differentiable (or even continuous) on the interval $(0,2)$, since $1\notin \mathrm{Dom}(f)$. ■

Example 4.16. A function f is a **Lipschitz function** if there exists some constant $M\in\mathbb{R}$ such that

$$|f(x_2)-f(x_1)|\leq M|x_2-x_1| \tag{4.6}$$

for all $x_1,x_2\in\mathrm{Dom}(f)$. Show that if $f:(a,b)\to\mathbb{R}$ satisfies $|f'(x)|\leq M$ for all $x\in(a,b)$, then f is a Lipschitz function.

Solution. Suppose $|f'(x)|\leq M$ for all $x\in(a,b)$. Let $x_1,x_2\in(a,b)$, and let $\alpha=\min\{x_1,x_2\}$ and $\beta=\max\{x_1,x_2\}$. Since f is differentiable on (a,b) and $[\alpha,\beta]\subseteq(a,b)$, we know that f is differentiable on (α,β) and continuous on $[\alpha,\beta]$. By the Mean Value Theorem there exists some $c\in(\alpha,\beta)$ such that

$$f'(c)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha},$$

and hence

$$f(\beta)-f(\alpha)=f'(c)(\beta-\alpha).$$

Since $|f'(x)|\leq M$ for all $x\in(\alpha,\beta)$, we next obtain

$$|f(\beta)-f(\alpha)|=M|\beta-\alpha|. \tag{4.7}$$

If $x_1<x_2$, then $\alpha=x_1$ and $\beta=x_2$, and (4.7) implies (4.6); and if $x_2<x_1$, then $\alpha=x_2$ and $\beta=x_1$, and again (4.7) implies (4.6). Therefore f is a Lipschitz function. ■

We conclude this section by using the Mean Value Theorem to prove the following proposition, which will in turn be used to prove the proposition that follows.

Proposition 4.17. *Let I be an open interval.*

1. *If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*
2. *If $f'(x) = g'(x)$ for all $x \in I$, then f and g differ by a constant on I.*

Proof.
Proof of Part (1). Suppose that $f'(x) = 0$ for all $x \in I$. Let $x_1, x_2 \in I$, where $x_1 < x_2$. The differentiability of f on I implies the continuity of f on I, and since $[x_1, x_2] \subseteq I$ it follows that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Now the Mean Value Theorem concludes that there is some $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

and since $c \in I$ implies $f'(c) = 0$ we obtain

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0.$$

So $f(x_2) - f(x_1) = 0$ and therefore $f(x_1) = f(x_2)$. But x_1 and x_2 are arbitrary points in I, so f must be constant on I.

Proof of Part (2). Suppose that $f'(x) = g'(x)$ for all $x \in I$. Then

$$(f - g)'(x) = (f' - g')(x) = f'(x) - g'(x) = 0$$

for all $x \in I$, and by part (1) we conclude that $f - g$ must be constant on I. That is, there exists some $c \in \mathbb{R}$ such that

$$f(x) - g(x) = (f - g)(x) = c$$

for all $x \in I$, which shows that f and g differ by c on I. ■

Proposition 4.18. *Let I be an arbitrary interval, and suppose that f and g are continuous on I. If $f'(x) = g'(x)$ for all x in the interior of I, then f and g differ by a constant on I.*

Proof. Suppose that $f'(x) = g'(x)$ for all x in the interior of I. Proposition 4.17 implies there exists some $c \in \mathbb{R}$ such that $f - g = c$ on the interior of I, since it is an open interval. Suppose that I includes its left endpoint, which we'll denote by a. Since f and g are continuous on I, so too is $f - g$; then, since there is some $\delta > 0$ such that $[a, a + \delta) \subseteq I$, we have

$$\lim_{x \to a^+} (f - g)(x) = (f - g)(a). \tag{4.8}$$

However, $(f - g)(x) = c$ for all $a < x < a + \delta$ so that

$$\lim_{x \to a^+} (f - g)(x) = \lim_{x \to a^+} c = c. \tag{4.9}$$

Combining (4.8) and (4.9), we find that $(f - g)(a) = c$.

A similar argument will show that $(f - g)(b) = c$ if I contains its right endpoint b. Therefore $f - g = c$ on all I. ■

Both of the propositions above will be put to good use when antiderivatives are discussed in section 4.8.

Exercises

1. Show $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

2. Show the equation $x^7 + x^3 = -x^5 - 1$ has exactly one real solution.

3. Show that, for any constant k, the equation $x^3 - 15x + k = 0$ has at most one root in the interval $[-2, 2]$.

4. Show $x^{101} + x^{51} + x - 1 = 0$ has exactly one real root.

5. Show $2x - 1 - \sin x = 0$ has exactly one real root.

6. Show $2x^4 + 5x^2 + x - 3 = 0$ has exactly two real roots.

7. Let $f(x) = x^4 - x^3 + 7x^2 + 3x - 11$. Prove that the graph of f has at least one horizontal tangent line.

8. Show that a 3rd-degree polynomial equation has at most three real roots. (Hint: start with a general cubic equation $ax^3 + bx^2 + cx + d = 0$.)

9. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval, then find the numbers c that satisfy the theorem's conclusion.
   (a) $f(x) = 3x^2 + 2x + 5, \quad [-1, 1]$
   (b) $g(x) = \sqrt[3]{x}, \quad [0, 1]$
   (c) $h(x) = \dfrac{x}{x+2}, \quad [1, 4]$

10. Let $\varphi(x) = |x - 1|$. Show that there's no $c \in [0, 3]$ such that
$$\varphi'(c) = \frac{\varphi(3) - \varphi(0)}{3}.$$
Why does this not contradict the Mean Value Theorem?

11. For what values of a, b, and m does the function
$$f(x) = \begin{cases} 3, & x = 0 \\ a + 3x - x^2, & x \in (0, 1) \\ mx + b, & x \in [1, 2] \end{cases}$$
satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

12. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x?

13. Suppose f is continuous on $[6, 15]$ and differentiable on $(6, 15)$. Also suppose that $f(6) = -8$ and $f'(x) \leq 12$ for all $x \in (6, 15)$. What is the largest possible value for $f(15)$?

4.3 – Strict Monotonicity and Concavity

Definition 4.19. *Let f be a function and $I \subseteq \mathrm{Dom}(f)$ an interval.*

If $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$ such that $x_1 < x_2$, then f is ***increasing*** *on I. In the event that $I = \mathrm{Dom}(f)$ we say that f is an* ***increasing function****.*

If $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$ such that $x_1 < x_2$, then f is ***decreasing*** *on I. In the event that $I = \mathrm{Dom}(f)$ we say that f is a* ***decreasing function****.*

A function is said to be **strictly monotonic** on an interval I if it is either increasing or decreasing on I. A useful fact: if f is increasing or decreasing I, then $-f$ is decreasing or increasing on I, respectively. The following theorem will prove invaluable in determining whether a function f has the property of strict monotonicity on a given interval, so long as f fulfills certain continuity and differentiability requirements.

Theorem 4.20 (**Monotonicity Test**). *Let I be an interval. Suppose f is continuous on I and differentiable on $\mathrm{Int}(I)$.*

1. *If $f' > 0$ on $\mathrm{Int}(I)$, then f is increasing on I.*
2. *If $f' < 0$ on $\mathrm{Int}(I)$, then f is decreasing on I.*

Proof.
Proof of Part (1). Suppose $f' > 0$ on $\mathrm{Int}(I)$. Let $x_1, x_2 \in I$ such that $x_1 < x_2$. Since $[x_1, x_2] \subseteq I$ and $(x_1, x_2) \subseteq \mathrm{Int}(I)$, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), and so by the Mean Value Theorem there exists some $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Observing that $c \in \mathrm{Int}(I)$, we obtain $f'(c) > 0$ and thus

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

From this we conclude that $f(x_2) - f(x_1) > 0$, which is to say $f(x_1) < f(x_2)$ and therefore f is increasing on I.

Proof of Part (2). Suppose $f' < 0$ on $\mathrm{Int}(I)$. Then $-f' > 0$ on $\mathrm{Int}(I)$, so that $-f$ is increasing on I by Part (1), and therefore f itself is decreasing on I. ∎

Like the Mean Value Theorem itself, the Monotonicity Test is often used to establish inequalities that might otherwise be quite challenging to establish by purely algebraic means. The next example illustrates a typical treatment.

Example 4.21. Show that

$$2\sqrt{x} > 3 - \frac{1}{x}$$

for all $x > 1$.

Solution. Let

$$f(x) = 2\sqrt{x} + \frac{1}{x} - 3,$$

so the problem becomes one of showing that $f(x) > 0$ for all $x > 1$.

For any $x > 1$ we have

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2}.$$

But we also have $\sqrt{x} > 1$, so that

$$x^2 = x \cdot x > x \cdot 1 = x = \sqrt{x} \cdot \sqrt{x} > \sqrt{x} \cdot 1 = \sqrt{x},$$

hence

$$\frac{1}{x^2} < \frac{1}{\sqrt{x}},$$

and therefore $f'(x) > 0$. So $f' > 0$ on $(1, \infty)$, and since f is continuous on $[1, \infty)$, the Monotonicity Test implies that f is increasing on $[1, \infty)$. In particular this means that $f(x) > f(1) = 0$ for all $x > 1$. ∎

Theorem 4.22 (First Derivative Test). *Let c be a critical point of f, and suppose there exist some $\gamma > 0$ such that f is continuous on $B_\gamma(c)$.*

1. *If $f' > 0$ on $(c - \gamma, c)$ and $f' < 0$ on $(c, c + \gamma)$, then f has a local maximum at c.*
2. *If $f' < 0$ on $(c - \gamma, c)$ and $f' > 0$ on $(c, c + \gamma)$, then f has a local minimum at c.*
3. *If $f' > 0$ on $B'_\gamma(c)$, or $f' < 0$ on $B'_\gamma(c)$, then f has no local extremum at c.*

Proof.
Proof of Part (1). Suppose $f' > 0$ on $(c-\gamma, c)$ and $f' < 0$ on $(c, c+\gamma)$. Since f is continuous on $B_\gamma(c)$ it is also continuous on $[c-\gamma/2, c]$. Since f is differentiable on $(c-\gamma, c)$ it is differentiable on $(c - \gamma/2, c)$. Now, $f' > 0$ on $(c - \gamma/2, c)$, and so f is increasing on $[c - \gamma/2, c]$ by Theorem 4.20(1).

Next, since f is continuous on $B_\gamma(c)$ it is continuous on $[c, c + \gamma/2]$. Since f is differentiable on $(c, c+\gamma)$ it is differentiable on $(c, c+\gamma/2)$. Now, $f' < 0$ on $(c, c+\gamma/2)$, and so f is decreasing on $[c, c + \gamma/2]$ by Theorem 4.20(2).

Since f is increasing on $[c - \gamma/2, c]$ we have $f(c) \geq f(x)$ for all $c - \gamma/2 \leq x \leq c$, and since f is decreasing on $[c, c + \gamma/2]$ we have $f(c) \geq f(x)$ for all $c \leq x \leq c + \gamma/2$. Thus $f(c) \geq f(x)$ for all $x \in B_{\gamma/2}(c)$, and therefore f has a local maximum at c.

Proof of Part (2). Suppose $f' < 0$ on $(c - \gamma, c)$ and $f' > 0$ on $(c, c + \gamma)$. Then $-f' > 0$ on $(c - \gamma, c)$ and $f' < 0$ on $(c, c + \gamma)$, so $-f$ has a local maximum at c by Part (1), and therefore f has a local minimum at c.

Proof of Part (3). Left as an exercise. ∎

Example 4.23. Find the intervals of increase or decrease for the function

$$f(x) = 2\sqrt{x} - 4x^2,$$

and find any local extrema.

Solution. First we get the derivative of f,

$$f'(x) = \frac{1}{\sqrt{x}} - 8x,$$

which is continuous on its domain $(0, \infty)$. Clearly $f'(x)$ exists for all $x > 0$; however,

$$f'(x) = 0 \quad \Leftrightarrow \quad \frac{1}{\sqrt{x}} - 8x = 0 \quad \Leftrightarrow \quad 1 - 8x^{3/2} = 0 \quad \Leftrightarrow \quad x^{3/2} = \frac{1}{8} \quad \Leftrightarrow \quad x = \frac{1}{4},$$

and so f has $x = \frac{1}{4}$ as its sole critical point.

Since $f'\left(\frac{1}{16}\right) = \frac{7}{2} > 0$, the Intermediate Value Theorem implies that $f' > 0$ on the interval $\left(0, \frac{1}{4}\right)$; and since $f'(1) = -7 < 0$, it follows that $f' < 0$ on $\left(\frac{1}{4}, \infty\right)$. By the Monotonicity Test we conclude that f is increasing on $\left(0, \frac{1}{4}\right)$ and decreasing on $\left(\frac{1}{4}, \infty\right)$. The First Derivative Test tells us that f has a local maximum at $x = \frac{1}{4}$, with local maximum value $f\left(\frac{1}{4}\right) = \frac{3}{4}$. ∎

Definition 4.24. *Let f be differentiable on an open interval I. If f' is increasing on I, then f is* ***concave up*** *on I. If f' is decreasing on I, then f is* ***concave down*** *on I.*

If f is continuous at c and there exists some $\gamma > 0$ such that f is concave in one sense on $(c - \gamma, c)$ and concave in the opposite sense on $(c, c + \gamma)$, then $(c, f(c))$ is an ***inflection point*** *of the graph of f.*

Notice that here the property of concavity is only defined on open intervals, unlike the property of strict monotonicity.

Example 4.25. Consider any linear function $f(x) = mx + b$, where m and b are constants. Where is f concave up, and where is it concave down? We have $f'(x) = m$, a constant function. By Definition 4.19, f' is neither increasing nor decreasing on any interval $I \subseteq \mathbb{R}$, and thus Definition 4.24 makes clear that f is neither concave up nor concave down on any open interval in $\mathbb{R}$. As a consequence, no line possesses the property of concavity anywhere along its length, and there are no inflection points. ∎

Theorem 4.26 (Concavity Test). *Suppose that f'' exists on an open interval I.*

1. *If $f'' > 0$ on I, then f is concave up on I.*
2. *If $f'' < 0$ on I, then f is concave down on I.*

Proof. We start by observing that the existence of f'' on the open interval I implies that f' is differentiable on I, which in turn implies that f' is continuous on I and f is differentiable on I.

Proof of Part (1). Suppose that $f'' > 0$ on I. Since $f'' = (f')'$, by Theorem 4.20(1) it follows that f' is increasing on I, and thus by definition f is concave up on I.

Proof of Part (2). Suppose that $f'' < 0$ on I. Since $f'' = (f')'$, by Theorem 4.20(2) it follows that f' is decreasing on I, and thus by definition f is concave down on I. ∎

Example 4.27. Find the intervals of concavity for the function

$$g(x) = 200 + 8x^3 + x^4,$$

and find any inflection points.

Solution. We start by obtaining the second derivative of g,

$$g''(x) = 48x + 12x^2.$$

Clearly g'' is defined everywhere, but $g''(x) = 0$ yields the solutions $x = -4$ and $x = 0$. We next evaluate $g(x)$ at convenient values in the intervals $(-\infty, -4)$, $(-4, 0)$, and $(0, \infty)$; we have $g''(-5) = 60 > 0$, $g''(-1) = -36 < 0$, and $g''(1) = 60 > 0$. Since g'' is continuous everywhere, by the Intermediate Value Theorem we conclude that $g'' > 0$ on intervals $(-\infty, -4)$ and $(0, \infty)$, while $g'' < 0$ on $(-4, 0)$. The Concavity Test then implies that g is concave up on $(-\infty, -4)$, concave down on $(-4, 0)$, and concave up on $(0, \infty)$. Because g is continuous at -4 and 0 in particular, we conclude that

$$(-4, g(-4)) = (-4, -56) \quad \text{and} \quad (0, g(0)) = (0, 200)$$

are inflection points of the graph of g. ■

4.4 – The Graphs of Functions

The various tests introduced in the previous section, taken together with the study of asymptotes made in §2.4 and §2.5, furnish a means of determining the general shape of the graph of many kinds of functions without plotting very many points. Indeed, the tools of differential calculus developed thus far can enable us to spot features in a graph that might otherwise be all but indiscernible on the pixelated displays of computer algebra systems. In particular, inflection points and local extrema that are not apparent on a calculator screen are often easily apprehended in the course of studying a function's first or second derivative.

Example 4.28. Let

$$f(x) = \frac{x}{\sqrt{x^2+1}}.$$

(a) Find the domain of f.
(b) Find the intercepts of f.
(c) Find the asymptotes of f, and any points where $y = f(x)$ intersects an asymptote.
(d) Use f' to find intervals of increase and decrease, then obtain critical points and find local extrema.
(e) Use f'' to find intervals where f is concave up and concave down, and identify inflection points.
(f) Sketch the graph of f.

Solution.

(a) The domain is $(-\infty, \infty)$.

(b) Since $f(x) = 0$ has only $x = 0$ as a solution, the point $(0, 0)$ is the sole x-intercept, which happens also to be the y-intercept.

(c) There is a no vertical asymptote. However, since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{|x|\sqrt{1+x^{-2}}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1+x^{-2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+x^{-2}}} = 1,$$

$y = 1$ is a horizontal asymptote; and since

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{|x|\sqrt{1+x^{-2}}} = \lim_{x\to\infty} \frac{x}{-x\sqrt{1+x^{-2}}} = \lim_{x\to\infty} \frac{-1}{\sqrt{1+x^{-2}}} = -1,$$

another horizontal asymptote is $y = -1$.

Any point where the graph of $y = f(x)$ intersects a horizontal asymptote will be a point (x, y) for which $f(x) = \pm 1$. But $f(x) = \pm 1$ implies $x = \pm\sqrt{x^2+1}$ and thus $x^2 = x^2 + 1$, which is impossible. Therefore no intersections occur between the graph of f and its horizontal asymptotes.

(d) With the quotient rule of differentiation,

$$f'(x) = \frac{\sqrt{x^2+1} - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1} = \frac{1}{(x^2+1)^{3/2}},$$

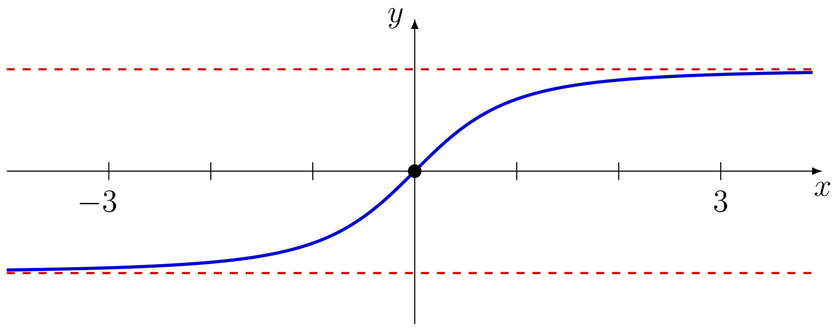


FIGURE 23.

which shows that $f' > 0$ on $(-\infty, \infty)$, and thus f is increasing everywhere by the Monotonicity Test. There are no critical points, and hence no local extrema.

(e) Differentiating f' gives

$$f''(x) = -\frac{3x}{(x^2+1)^{5/2}}.$$

Clearly $f'' > 0$ on $(-\infty, 0)$ and $f'' < 0$ on $(0, \infty)$. By the Concavity Test we conclude that f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. Since concavity changes at 0 and $0 \in \text{Dom}(f)$, it follows that $(0, 0)$ is the only inflection point on the graph of f.

(f) The sketch of the graph of f is in Figure 23, with the inflection point marked in black. ■

Example 4.29. Let

$$f(x) = \frac{x^3 - 1}{x^3 + 1}.$$

(a) Find the domain of f.
(b) Find the intercepts of f.
(c) Find the asymptotes of f.
(d) Use f' to find intervals of increase and decrease, then obtain critical points and find local extrema.
(e) Use f'' to find intervals where f is concave up and concave down, and identify inflection points.
(f) Sketch the graph of f.

Solution.

(a) The domain is $\text{Dom}(f) = \{x \in \mathbb{R} : x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$.

(b) Since $f(0) = -1$, the y-intercept of f is $(0, -1)$. As for any x-intercepts, set $f(x) = 0$ and solve for x:

$$f(x) = 0 \quad \Rightarrow \quad \frac{x^3 - 1}{x^3 + 1} = 0 \quad \Rightarrow \quad x^3 - 1 = 0 \quad \Rightarrow \quad x = 1;$$

Thus f has x-intercept $(1, 0)$.

(c) We have

$$\lim_{x\to -1^-} \frac{x^3-1}{x^3+1} = +\infty \quad \text{and} \quad \lim_{x\to -1^+} \frac{x^3-1}{x^3+1} = -\infty,$$

so there is a vertical asymptote $x = -1$. Also

$$\lim_{x\to \pm\infty} \frac{x^3-1}{x^3+1} = 1,$$

so there is a horizontal asymptote $y = 1$.

(d) Differentiating f gives

$$f'(x) = \frac{(x^3+1)(3x^2) - (x^3-1)(3x^2)}{(x^3+1)^2} = \frac{6x^2}{(x^3+1)^2}.$$

It can be seen that $f' > 0$ on $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$, which shows that f is increasing everywhere on its domain except at 0.

The only critical point for f that lies in $\mathrm{Dom}(f)$ is $x = 0$, since $f'(0) = 0$. However, because $f' > 0$ for all x on $(-1, 0) \cup (0, \infty)$, by the First Derivative Test no local extremum lies at $x = 0$.

(e) Differentiating f' gives

$$f''(x) = \frac{(x^3+1)^2(12x) - (6x^2)\cdot 2(x^3+1)(3x^2)}{(x^3+1)^4} = \frac{12x(1-2x^3)}{(x^3+1)^3}.$$

Setting $f''(x) = 0$ implies that either $12x = 0$ or $1 - 2x^3 = 0$, from which we obtain solutions $x = 0, 2^{-1/3}$.

Now, $f''(-2) \approx 1.19 > 0$, and since $f''(x) \neq 0$ for all $x \in (-\infty, -1)$ and f'' is continuous on $(-\infty, -1)$, the Intermediate Value Theorem implies that $f'' > 0$ on $(-\infty, -1)$. Therefore f is concave up on $(-\infty, -1)$.

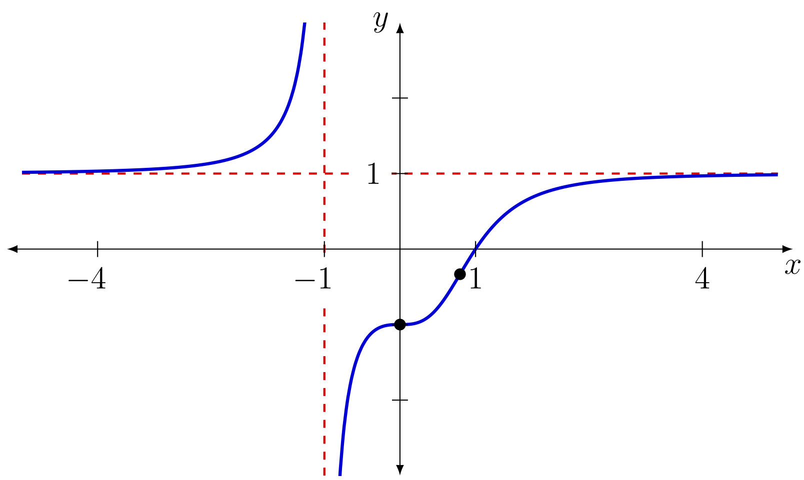


Figure 24.

Next, $f''(-1/2) \approx -11.20 < 0$, and since $f''(x) \neq 0$ for all $x \in (-1, 0)$ and f'' is continuous on $(-1, 0)$, the IVT implies $f'' < 0$ on $(-1, 0)$. Therefore f is concave down on $(-1, 0)$.

A convenient value lying between 0 and $2^{-1/3}$ is $1/2$. We have $f''(1/2) \approx 3.16 > 0$, and since $f''(x) \neq 0$ for all $x \in (0, 2^{-1/3})$ and f'' is continuous on $(0, 2^{-1/3})$, the IVT implies $f'' > 0$ on $(0, 2^{-1/3})$. Therefore f is concave up on $(0, 2^{-1/3})$.

Finally, $f''(1) = -3/2 < 0$, and since $f''(x) \neq 0$ for all $x \in (2^{-1/3}, \infty)$ and f'' is continuous on $(2^{-1/3}, \infty)$, the IVT implies $f'' < 0$ on $(2^{-1/3}, \infty)$. Therefore f is concave down on $(2^{-1/3}, \infty)$.

The inflection points of f are $(0, -1)$ and $(2^{-1/3}, -1/3)$. Note that there is not an inflection point at $x = -1$ since f is undefined there!

(f) The sketch of the graph of f is in Figure 24, with inflection points marked in black. ■

Example 4.30. Let $f(x) = x^{1/3}(x+3)^{2/3}$.

(a) Find the domain of f.
(b) Find the intercepts of f.
(c) Find the asymptotes of f.
(d) Use f' to find intervals of increase and decrease, then obtain critical points and find local extrema.
(e) Use f'' to find intervals where f is concave up and concave down, and identify inflection points.
(f) Sketch the graph of f.

Solution.

(a) The domain is $\mathrm{Dom}(f) = (-\infty, \infty)$.

(b) Since $f(0) = 0$, the y-intercept of f, which doubles as an x-intercept, is $(0, 0)$. As for any x-intercepts besides the origin, we set $f(x) = 0$ and solve for x:

$$f(x) = 0 \quad \Rightarrow \quad x^{1/3}(x+3)^{2/3} = 0 \quad \Rightarrow \quad x = -3, 0.$$

Thus f has x-intercepts $(0, 0)$ and $(-3, 0)$.

(c) Since the domain of f is $(-\infty, \infty)$ there can be no vertical asymptotes. And since $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$, there are no horizontal asymptotes either.

(d) Differentiating f gives

$$\begin{aligned} f'(x) &= x^{1/3} \cdot \frac{2}{3}(x+3)^{-1/3} + \frac{1}{3}x^{-2/3} \cdot (x+3)^{2/3} = \frac{2x^{1/3}}{3(x+3)^{1/3}} + \frac{(x+3)^{2/3}}{3x^{2/3}} \\ &= \frac{2x}{3x^{2/3}(x+3)^{1/3}} + \frac{x+3}{3x^{2/3}(x+3)^{1/3}} = \frac{3x+3}{3x^{2/3}(x+3)^{1/3}} = \frac{x+1}{x^{2/3}(x+3)^{1/3}} \end{aligned}$$

for any $x \neq -3, 0$. We see that $f' > 0$ if $x + 1 > 0$ and $x + 3 > 0$, which implies that $x > -1$; also $f' > 0$ if $x + 1 < 0$ and $x + 3 < 0$, which implies that $x < -3$. Thus f is increasing on

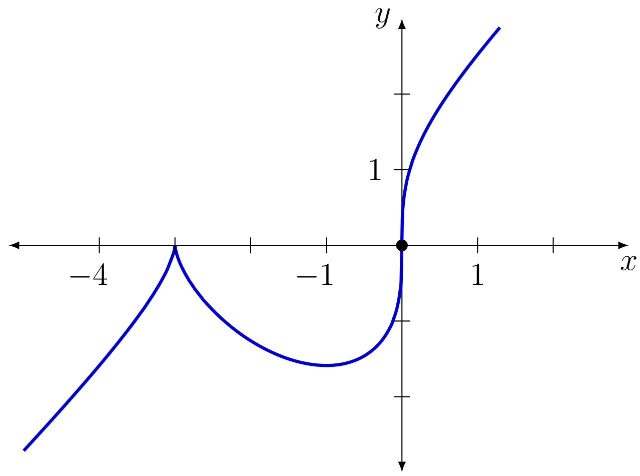


FIGURE 25.

$(-\infty, -3)$ and $(-1, \infty)$ by the Monotonicity Test. Since $f' < 0$ on $(-3, -1)$ we conclude that f is decreasing on this interval.

Now we find the critical points for f. Setting $f'(x) = 0$ gives $x = -1$, which is one critical point. As for x values for which $f'(x)$ does not exist, we have $x = -3, 0$, which are two more critical points. Since $f' > 0$ to the left of -3 and $f' < 0$ to the right of -3, by the First Derivative Test it follows that f has a local maximum at -3, with local maximum value of $f(-3) = 0$. Since $f' < 0$ to the left of -1 and $f' > 0$ to the right of -1, f has a local minimum at -1, with local minimum value of $f(-1) = -\sqrt[3]{4}$. Finally, since $f' > 0$ to the left and right of 0, there is no local extremum for f at 0.

(e) Next, we have

$$\begin{aligned} f''(x) &= \frac{(x^3 + 3x^2)^{1/3} - (x+1) \cdot \frac{1}{3}(x^3 + 3x^2)^{-2/3}(3x^2 + 6x)}{(x^3 + 3x^2)^{2/3}} \\ &= \frac{(x^3 + 3x^2) - (x+1)(x^2 + 2x)}{(x^3 + 3x^2)^{4/3}} = -\frac{2x}{(x^3 + 3x^2)^{4/3}} \end{aligned}$$

for all $x \neq -3, 0$. Since

$$(x^3 + 3x^2)^{4/3} = \left(\sqrt[3]{x^2(x+3)}\right)^4 > 0$$

for all $x \neq -3, 0$, we see that $f'' > 0$ on $(-\infty, -3) \cup (-3, 0)$ and $f'' < 0$ on $(0, \infty)$, and so by the Concavity Test f is concave up on $(-\infty, -3) \cup (-3, 0)$ and concave down on $(0, \infty)$. The point $(0, 0)$ is therefore an inflection point. At the point $(-3, 0)$ concavity does not change so there is no inflection point there.

(f) The sketch of the graph of f is in Figure 25, with inflection point marked in black. ∎

4.5 – Optimization Problems

Example 4.31. A square-based, box-shaped shipping crate is designed to have a volume of 16 cubic meters. The material used to make the base costs twice as much (per m^2) as the material in the sides, and the material used to make the top costs half as much (per m^2) as the material in the sides. What are the dimensions of crate that minimize the cost of materials?

Solution. The base of the box is square, so let x be the length of the base edges (the length and width of the box), and let y be the height of box. The volume must be 16 m^3, so $x^2y = 16$ and hence $y = 16/x^2$. Let k be the cost per square meter (in dollars) for the material in the sides of the box. The cost C of the box is a function of x as follows:

$$C(x) = \underbrace{\left(\frac{2k \text{ dollars}}{\text{m}^2}\right)(x^2 \text{ m}^2)}_{\text{cost of base}} + \underbrace{\left(\frac{0.5k \text{ dollars}}{\text{m}^2}\right)(x^2 \text{ m}^2)}_{\text{cost of top}} + \underbrace{4\left(\frac{k \text{ dollars}}{\text{m}^2}\right)\left(x \cdot \frac{16}{x^2} \text{ m}^2\right)}_{\text{cost of sides}},$$

and therefore

$$C(x) = \frac{5}{2}kx^2 + \frac{64k}{x}.$$

We wish to find x such that the cost is minimized. We have

$$C'(x) = 5kx - \frac{64k}{x^2},$$

and so if we set $C'(x) = 0$ we obtain

$$5kx - \frac{64k}{x^2} = 0 \;\Rightarrow\; 5kx^3 - 64k = 0 \;\Rightarrow\; 5x^3 = 64 \;\Rightarrow\; x = \frac{4}{\sqrt[3]{5}}.$$

Thus $C(x)$ is minimized if we set $x = 4/\sqrt[3]{5}$. That is, the box should have dimensions

$$\frac{4}{\sqrt[3]{5}} \text{ m} \times \frac{4}{\sqrt[3]{5}} \text{ m} \times \sqrt[3]{25} \text{ m}$$

in order to minimize the cost of its construction. ■

Example 4.32. The intensity of a light source at a given location is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources, one twice as strong as the other, are 12 meters apart. At what point on the line segment joining the sources is the intensity the weakest?

Solution. Let L_1 and L_2 be the weaker and stronger light sources, respectively, and let I_1 and I_2 be their intensities. If p is the point on the line segment joining L_1 and L_2 that is a distance of x from L_1, then I_1 and I_2 may be characterized as functions of x:

$$I_1(x) = \frac{ks_1}{x^2} \quad \text{and} \quad I_2(x) = \frac{ks_2}{(12-x)^2},$$

where $k > 0$ is a constant of proportionality (dependent on what kind of unit is being used to quantify "intensity"), and $s_1, s_2 > 0$ are the "strengths" of L_1 and L_2. The total light intensity I at p is thus

$$I(x) = I_1(x) + I_2(x) = \frac{ks_1}{x^2} + \frac{2ks_1}{(12-x)^2}, \quad x \in (0, 12),$$

where we use the fact that $s_2 = 2s_1$.

We must find the global minimum for I, which is the minimum value $I(x)$ attains for $0 < x < 12$. We have

$$I'(x) = -\frac{2ks_1}{x^3} + \frac{4ks_1}{(12-x)^3} = \frac{2ks_1[2x^3 - (12-x)^3]}{x^3(12-x)^3},$$

and so from $I'(x) = 0$ we obtain the equation $2x^3 - (12-x)^3 = 0$, where

$$2x^3 - (12-x)^3 = 0 \;\Rightarrow\; 2x^3 = (12-x)^3 \;\Rightarrow\; x\sqrt[3]{x} = 12 - x \;\Rightarrow\; x = \frac{12}{1+\sqrt[3]{2}} := x^*,$$

which is approximately 5.31 and so is a critical point for I that lies in $(0,12)$. There is no $x \in (0,12)$ for which $I'(x)$ does not exist, so there are no other critical points in $(0,12)$. Since I is continuous on $(0,12)$ and

$$I'(4) = -\frac{3}{128}ks_1 < 0 \quad \text{and} \quad I'(8) = \frac{15}{256}ks_1 > 0,$$

we conclude by the Intermediate Value Theorem that $I' < 0$ on $(0, x^*)$, and $I' > 0$ on $(x^*, 12)$. By the First Derivative Test I has a local minimum at x^*, and since I is decreasing on $(0, x^*)$ and increasing on $(x^*, 12)$, we conclude that the local minimum at x^* is in fact a global minimum.

Therefore the intensity of light between L_1 and L_2 is weakest a distance of

$$\frac{12}{1+\sqrt[3]{2}} \approx 5.31 \text{ m}$$

from the weaker light source L_1. ■

Example 4.33. An island under the sovereignty of the Empire is 3.5 km from the nearest point on a straight shoreline, and that point is in turn 8 km away from a nuclear power plant on the shore (see at left in Figure 26). There is a prison filled with dissidents on the island, and to power it the Empire plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the nuclear plant. Assume that it costs \$2400 per kilometer to lay underwater cable and \$1200 per kilometer to lay underground cable. At what point should the underwater cable meet the shore in order to minimize the cost of the Empire's project?

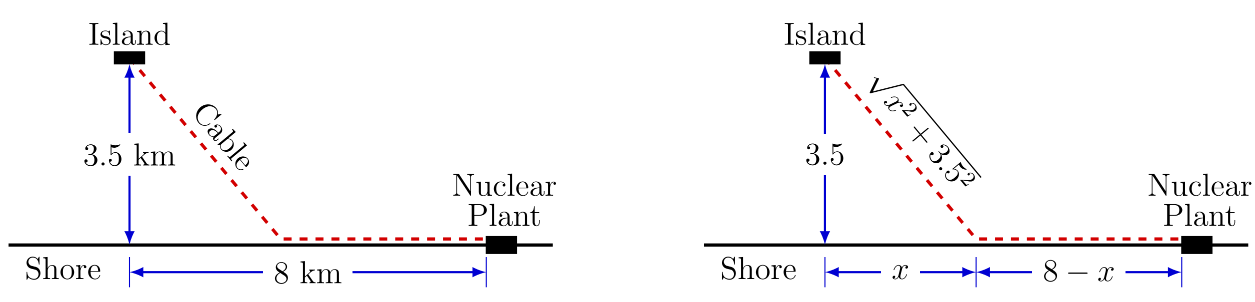


Figure 26.

Solution. Let x be as show at right in Figure 26; that is, x is the distance between the nearest point on the shore to the island and the point where the cable will meet the shore. The cost function is:

$$C(x) = \left(\frac{\$2400}{\text{km}}\right)\left(\sqrt{x^2+3.5^2}\text{ km}\right) + \left(\frac{\$1200}{\text{km}}\right)(8-x\text{ km}) = 2400\sqrt{x^2+\frac{49}{4}} + 1200(8-x).$$

We take the derivative:

$$C'(x) = \frac{2400x}{\sqrt{x^2+49/4}} - 1200.$$

Note that there is no x value for which $C'(x)$ does not exist. On the other hand,

$$C'(x) = 0 \quad\Rightarrow\quad \frac{2x}{\sqrt{x^2+49/4}} - 1 = 0 \quad\Rightarrow\quad 2x = \sqrt{x^2+\frac{49}{4}} \quad\Rightarrow\quad 4x^2 = x^2 + \frac{49}{4}$$

$$\Rightarrow\quad x^2 = \frac{49}{12} \quad\Rightarrow\quad x = \sqrt{\frac{49}{12}} = \frac{7}{2\sqrt{3}} = \frac{7\sqrt{3}}{6} \approx 5.98.$$

Thus, if the cable meets the shore at a point about $8 - 5.98 = 2.02$ km away from the nuclear plant, the cost will be minimized. ■

4.6 – Linear Approximation

Example 4.34. We use linear approximation to approximate the value of $\sqrt[5]{33}$. Let $f(x) = \sqrt[5]{x}$. Since we know that $\sqrt[5]{32} = 2$, and 32 is reasonably close to 33, we find the linear function L that provides a linear approximation of f at $x = 32$. From $f'(x) = \frac{1}{5}x^{-4/5}$ we obtain

$$f'(32) = \frac{1}{5}(32)^{-4/5} = \frac{1}{80},$$

which is the slope of the line generated by L. A point on the line is $(32, f(32)) = (32, 2)$, and so by the point-slope formula the line has equation

$$y - 2 = \frac{1}{80}(x - 32),$$

and hence

$$L(x) = \frac{1}{80}x + \frac{8}{5}$$

is the linear approximation of f at 32. That is, $f(x) \approx L(x)$ for all x near 32. In particular

$$\sqrt[5]{33} = f(33) \approx L(33) = \frac{1}{80}(33) + \frac{8}{5} = 2.0125,$$

which is quite close to the actual value 2.01234661.... The percent error is

$$\left|\frac{\text{Actual Value} - \text{Approximate Value}}{\text{Actual Value}}\right| \times 100\% \approx 0.0076\%.$$

The linear function L also provides a good approximate of, say, $\sqrt[5]{37}$:

$$\sqrt[5]{37} = f(37) \approx L(37) = \frac{1}{80}(37) + \frac{8}{5} = 2.0625,$$

which is still close to the actual value 2.05892413.... The percent error is about 0.174%. ■

4.7 – L'Hôpital's Rule

Recall the following arithmetic conventions: for any $a \in \mathbb{R}$, we define

$$a \pm (+\infty) = \pm\infty, \quad a/\pm\infty = 0, \quad \text{and} \quad a \cdot (\pm\infty) = \begin{cases} \pm\infty, & a > 0 \\ \mp\infty, & a < 0. \end{cases}$$

Also, letting ∞ stand for $+\infty$ as we frequently do, we define $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$. All addition and multiplication operations are taken to be commutative.

In contrast to the expressions above, expressions such as

$$\frac{0}{0}, \quad \frac{+\infty}{+\infty}, \quad \frac{-\infty}{-\infty}, \quad \frac{+\infty}{-\infty}, \quad \frac{-\infty}{+\infty} \tag{4.10}$$

(the last four we henceforth lump together as ∞/∞) have no definitive real-number or infinite value, and thus are called **indeterminant forms**. This is not quite the same as saying the forms are "undefined" in the sense that, say, $3/0$ is undefined. The form ∞/∞, for instance, arises in attempting to evaluate the limit

$$\lim_{x\to\infty} \frac{x^2}{x^3}.$$

Since $x^2 \to \infty$ and $x^3 \to \infty$ as $x \to \infty$, we see that $x^2/x^3 \to \infty/\infty$ as $x \to \infty$. However, we know this does not necessarily mean the limit does not exist: with a little algebra the limit is easily resolved into a real-number value:

$$\lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

So in this case we find that ∞/∞ resolves into 0. With another limit we may find ∞/∞ resolves into 1 or $-\frac{1}{2}$.

The following three theorems taken together are called **L'Hôpital's Rule**. The rule is a technique that helps to evaluate many limits that present an indeterminant form. It is possible to express the rule as a single theorem, but it would come at the cost of either clarity or precision.

Theorem 4.35 (**L'Hôpital's Rule: Left-Hand Version**). *Let f and g be differentiable on (a, b) with $-\infty \le a < b \le \infty$, and let $g'(x) \ne 0$ for all $x \in (a, b)$. Suppose that $f'(x)/g'(x) \to L$ as $x \to b^-$ for some $-\infty \le L \le \infty$. If*

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^-} g(x) = 0 \quad \text{or} \quad \lim_{x\to b^-} |g(x)| = +\infty,$$

then $f(x)/g(x) \to L$ as $x \to b^-$.

Theorem 4.36 (**L'Hôpital's Rule: Right-Hand Version**). *Let f and g be differentiable on (a, b), where $-\infty \le a < b \le \infty$, and let $g'(x) \ne 0$ for all $x \in (a, b)$. Suppose that $f'(x)/g'(x) \to L$ as $x \to a^+$ for some $-\infty \le L \le \infty$. If*

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0 \quad \text{or} \quad \lim_{x\to a^+} |g(x)| = +\infty,$$

then $f(x)/g(x) \to L$ as $x \to a^+$.

The two one-sided versions of L'Hôpital's Rule taken together readily imply a special two-sided version of the rule, stated as follows.

Theorem 4.37 (L'Hôpital's Rule: Two-Sided Version). *Let f and g be differentiable on $B'_\gamma(c)$ for some $c \in \mathbb{R}$ and $\gamma > 0$, with $g'(x) \neq 0$ for all $x \in B'_\gamma(c)$. Suppose that $\lim_{x\to c} f'(x)/g'(x) = L$ for some $-\infty \leq L \leq \infty$. If*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \quad \textit{or} \quad \lim_{x\to c} |g(x)| = +\infty,$$

then $\lim_{x\to c} f(x)/g(x) = L$.

Note that in Theorem 4.35 the left-hand limit can even take $x \to +\infty$, and in Theorem 4.36 the right-hand limit can take $x \to -\infty$. However, in Theorem 4.37 we must have $x \to c$ for some $c \in \mathbb{R}$ (so $c \neq \pm\infty$).

Example 4.38. Evaluate the limit

$$\lim_{x\to 0} \frac{\sin x - x}{7x^3}.$$

Solution. Since $\sin x - x \to 0$ and $7x^3 \to 0$ as $x \to 0$, we see the limit presents us with a $0/0$ indeterminant form. Let $f(x) = \sin x - x$ and $g(x) = 7x^3$, so the limit under consideration may be written as $\lim_{x\to 0} f(x)/g(x)$. Note that both f and g are differentiable everywhere, so in particular they are differentiable on $B'_\gamma(0)$ for, say, $\gamma = 1$. Also $g'(x) = 21x^2 \neq 0$ for all $x \in B'_1(0)$. Now, since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(\sin x - x) = 0 \quad \text{and} \quad \lim_{x\to 0} g(x) = \lim_{x\to 0} 7x^3 = 0,$$

Theorem 4.37 implies that

$$\lim_{x\to 0} \frac{\sin x - x}{7x^3} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos x - 1}{21x^2}. \tag{4.11}$$

However, the limit at right in (4.11) is also a $0/0$ indeterminant form. Can we use L'Hôpital's Rule rule again? Since f' and g' are differentiable on $B'_1(0)$, $g''(x) = 42x \neq 0$ for all $x \in B'_1(0)$, and

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}(\cos x - 1) = 0 \quad \text{and} \quad \lim_{x\to 0} g'(x) = \lim_{x\to 0} 21x^2 = 0,$$

Theorem 4.37 implies that

$$\lim_{x\to 0} \frac{\cos x - 1}{21x^2} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{-\sin x}{42x}. \tag{4.12}$$

But the limit at right in (4.12) is once again a $0/0$ form! Once again we find we can use Theorem 4.37 to obtain

$$\lim_{x\to 0} \frac{-\sin x}{42x} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{f'''(x)}{g'''(x)} = \lim_{x\to 0} \frac{-\cos x}{42} = -\frac{1}{42}.$$

Therefore

$$\lim_{x\to 0} \frac{\sin x - x}{7x^3} = -\frac{1}{42}.$$

■

There are other indeterminant forms other than the ones given in (4.10). The ones we consider in this section are

$$+\infty-\infty, \quad -\infty+\infty, \quad 0\cdot(\pm\infty), \quad (\pm\infty)\cdot 0.$$

We let $\infty-\infty$ denote the first two, and $0\cdot\infty$ denote the last two. These forms are not directly addressed by L'Hôpital's Rule, but usually some simple algebraic prestidigitation within a limit will turn these forms into either $0/0$ or ∞/∞.

Example 4.39. Evaluate the limit

$$\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right).$$

Solution. Let $h(x)=1/x-\cot x$, and note that $\lim_{x\to 0^+}h(x)$ and $\lim_{x\to 0^-}h(x)$ present the indeterminant forms $+\infty-\infty$ and $-\infty+\infty$, respectively. Neither of these forms is addressed in our L'Hôpital's Rule theorems above. However, for all $x\in B_1'(0)$ we have

$$\frac{1}{x}-\cot x=\frac{1}{x}-\frac{\cos x}{\sin x}=\frac{\sin x-x\cos x}{x\sin x},$$

and so

$$\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)=\lim_{x\to 0}\frac{\sin x-x\cos x}{x\sin x} \tag{4.13}$$

by Theorem 2.18. Now, the limit at right in (4.13) presents a $0/0$ indeterminant form. By Theorems 4.37 and 2.12, and Lemma 3.18, we finally obtain

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x-x\cos x}{x\sin x}&=\lim_{x\to 0}\frac{(\sin x-x\cos x)'}{(x\sin x)'}=\lim_{x\to 0}\frac{x\sin x}{\sin x+x\cos x}\\&=\lim_{x\to 0}\left(\frac{x\sin x}{\sin x+x\cos x}\cdot\frac{1/x}{1/x}\right)=\lim_{x\to 0}\frac{\sin x}{\dfrac{\sin x}{x}+\cos x}\\&=\frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}\dfrac{\sin x}{x}+\lim\limits_{x\to 0}\cos x}=\frac{0}{1+1}=0.\end{aligned}$$

Therefore

$$\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)=0.$$

■

4.8 – Antiderivatives and Indefinite Integrals

We begin with the notion of an antiderivative of a function f, which is quite simply any function whose derivative is equal to f.

Definition 4.40. *Let $S \subseteq \mathbb{R}$ be an interval or a disjoint union of intervals, and let f be a function with $S \subseteq \mathrm{Dom}(f)$. An* ***antiderivative*** *for f on S is a function $F : S \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in S$.*

Example 4.41. If $f(x) = 0$ for all $x \in (-\infty, \infty)$, then an antiderivative for f on $(-\infty, \infty)$ is the function F given by $F(x) = 1$ for all $x \in (-\infty, \infty)$. More concisely we say that an antiderivative for 0 is 1, with the understanding that here 0 and 1 represent constant functions and not numbers. ■

Example 4.42. An antiderivative for $\sin x$ on $(-\infty, \infty)$ is $-\cos x$, and an antiderivative for $\cos x$ on $(-\infty, \infty)$ is $\sin x$.

An antiderivative for $\sec^2 x$ on $(-\pi/2, \pi/2)$ is $\tan x$. In fact, $\tan x$ is an antiderivative for $\sec^2 x$ on $(-\pi/2 + n\pi, \pi/2 + n\pi)$ for any integer n, and as a result $\tan x$ is an antiderivative for $\sec^2 x$ on the set $S \subseteq \mathbb{R}$ given by

$$S = \bigcup_{n\in\mathbb{Z}} \left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right),$$

which is the union of all intervals of the form $(-\pi/2 + n\pi, \pi/2 + n\pi)$, $n \in \mathbb{Z}$. ■

If the set S in Definition 4.40 contains any left endpoints a or right endpoints b, then $F'(a)$ and $F'(b)$ are defined by appropriate one-sided limits,

$$F'_+(a) = \lim_{x\to a^+} \frac{F(x) - F(a)}{x - a} \quad \text{and} \quad F'_-(b) = \lim_{x\to b^-} \frac{F(x) - F(b)}{x - b},$$

which is to say only one-sided differentiability of F is required at endpoints. For instance if F is an antiderivative for f on $S = (-\infty, -1] \cup [1, \infty)$, then at 1 there is only required to be a right-hand derivative, and at -1 there is only a left-hand derivative.

If F is an antiderivative for f on some interval I, so that $F'(x) = f(x)$ for all $x \in I$, then for any constant c we find that the function $F + c$ is also an antiderivative on I:

$$(F + c)'(x) = (F' + c')(x) = F'(x) + c'(x) = f(x) + 0 = f(x) \tag{4.14}$$

for all $x \in I$, with the derivative operator $'$ indicating the appropriate one-sided derivative at any endpoint of I. Thus if a function has one antiderivative on I, then in fact it has an infinite number of antiderivatives on I which differ from one another by a constant term. A question naturally arises: are there any antiderivatives for f on I that do *not* differ from the antiderivative F by a mere constant? The next theorem makes clear the answer is no.

Theorem 4.43. *Let f be a function, and let $I \subseteq \mathrm{Dom}(f)$ be an interval on which there is an antiderivative Φ for f. Then*

$$\{F : F \textit{ is an antiderivative for } f \textit{ on } I\} = \{\Phi + c : c \in \mathbb{R}\}$$

Proof. The fact that

$$\{\Phi + c : c \in \mathbb{R}\} \subseteq \{F : F \text{ is an antiderivative for } f \text{ on } I\}$$

has already been demonstrated by equation (4.14) above.

For the reverse containment, suppose F is any antiderivative for f on I. Then F and Φ are differentiable on I, which implies that F and Φ are continuous on I. Moreover we have $F'(x) = \Phi'(x) = f(x)$ for $x \in \mathrm{Int}(I)$, and so by Proposition 4.18 there exists some constant c_0 such that $F(x) - \Phi(x) = c_0$ for $x \in I$. Thus $F = \Phi + c_0$, and therefore $F \in \{\Phi + c : c \in \mathbb{R}\}$. ∎

This is a wonderful result, because it means that once *one* antiderivative for f on I has been found, then *all* the antiderivatives for f on I are known.

Definition 4.44. *Let $I \subseteq \mathbb{R}$ be an interval, and let f be a function with $I \subseteq \mathrm{Dom}(f)$. The **indefinite integral** of f on I, denoted by $\int f$ or $\int f(x)\,dx$, is the family of all antiderivatives for f on I. That is,*

$$\int f = \{F : F \text{ is an antiderivative for } f \text{ on } I\}. \tag{4.15}$$

*In $\int f$ the function f is called the **integrand**.*

The interval I mentioned in Definition 4.44 is left purposely vague and is not built into the symbol $\int f$. This is why $\int f$ is referred to as an "indefinite" integral. Commonly I it taken to be the largest interval in $\mathbb{R}$ on which f possesses an antiderivative, but there are times when it is convenient to consider some smaller interval, and other times when two or more disjoint intervals must be considered. In fact, as we'll see in Example 4.49 below, if a function has an antiderivative that is only valid on two or more disjoint intervals, then a different arbitrary constant term may be chosen for each interval of validity.

Proposition 4.45. *If F is an antiderivative of f on an interval I, then*

$$\int f = \{F + c : c \in \mathbb{R}\}. \tag{4.16}$$

Proof. Suppose F is an antiderivative of f on I. Then by Theorem 4.43 the family of all antiderivatives of f on I is the set $\{F + c : c \in \mathbb{R}\}$, and so we see that (4.16) follows directly from (4.15). ∎

In practice (4.16) is written as either

$$\int f = F + c \quad \text{or} \quad \int f(x)\,dx = F(x) + c,$$

where the x in the latter symbol may be replaced with some other letter.

For any $\alpha \in \mathbb{R}$ and function f that has an antiderivative on some interval I we define

$$\alpha \int f = \{\alpha F : F \text{ is an antiderivative for } f \text{ on } I\} = \left\{\alpha F : F \in \int f\right\}. \tag{4.17}$$

If f and g are functions that each have an antiderivative on I, then we define

$$\int f + \int g = \{F + G : F, G \text{ are antiderivatives for } f, g \text{ on } I\}$$

$$= \left\{ F + G : F \in \int f \text{ and } G \in \int g \right\}. \tag{4.18}$$

As the next theorem shows, the indefinite integral $\int$ largely satisfies linearity properties with respect to the operations defined by (4.17) and (4.18), except in the case of (4.17) when $\alpha = 0$. Before stating the theorem, we observe that if f, g have antiderivatives F, G on I, then αf and $f + g$ likewise have antiderivatives on I due to the linearity properties of the differentiation operation. In explicit terms, since

$$(\alpha F)' = \alpha F' = \alpha f \quad \text{and} \quad (F + G)' = F' + G' = f + g,$$

we see that αF is an antiderivative for αf on I, and $F + G$ is an antiderivative for $f + g$ on I.

Theorem 4.46. *If f and g each possess an antiderivative on an interval I, then the following hold.*

1. $\int(\alpha f) = \alpha \int f$ *whenever* $\alpha \neq 0$.
2. $\int(f + g) = \int f + \int g$.

Proof.
Proof of Part (1). Fix $\alpha \neq 0$. Let Φ be an antiderivative for f on I, so that $\alpha\Phi$ is an antiderivative for αf on I. Two applications of Proposition 4.45 yields

$$\begin{aligned} \int \alpha f &= \{\alpha\Phi + c : c \in \mathbb{R}\} = \{\alpha\Phi + \alpha c : c \in \mathbb{R}\} = \{\alpha(\Phi + c) : c \in \mathbb{R}\} \\ &= \{\alpha F : F \text{ is an antiderivative for } f \text{ on } I\} = \alpha \int f, \end{aligned}$$

where the fact that $c \in \mathbb{R}$ is arbitrary implies that the terms c and αc can each be any real number.

Proof of Part (2). Let Φ, Ψ be antiderivatives for f, g on I, respectively. Since $\Phi + \Psi$ is an antiderivative for $f + g$ on I, two applications of Proposition 4.45 yields

$$\begin{aligned} \int (f + g) &= \{(\Phi + \Psi) + c : c \in \mathbb{R}\} = \{(\Phi + \Psi) + (c_1 + c_2) : c_1, c_2 \in \mathbb{R}\} \\ &= \{(\Phi + c_1) + (\Psi + c_2) : c_1, c_2 \in \mathbb{R}\} \\ &= \{F + G : F, G \text{ are antiderivatives for } f, g \text{ on } I\} = \int f + \int g, \end{aligned}$$

where the fact that $c, c_1, c_2 \in \mathbb{R}$ are arbitrary implies that the terms c and $c_1 + c_2$ can each attain any real value. ∎

As already indicated in the proof of the theorem, if F, G are particular antiderivatives for f, g on I, then by Proposition 4.45 we can rewrite (4.17) as

$$\alpha \int f = \{\alpha F + c : c \in \mathbb{R}\}, \tag{4.19}$$

and equation (4.18) as

$$\int f + \int g = \{(F + G) + c : c \in \mathbb{R}\}. \tag{4.20}$$

In practice (4.19) and (4.20) may be written simply as

$$\alpha \int f = \alpha F + c \quad \text{and} \quad \int f + \int g = (F+G) + c.$$

Now from Theorem 4.46(1) we see that to find $\int(\alpha f)$, it suffices to find a *particular* antiderivative F for f on I, and then add an arbitrary constant c to αF. Similarly, Theorem 4.46(2) shows that to find $\int(f+g)$ it's enough to find *particular* antiderivatives F, G for functions f, g on I, and then add an arbitrary constant c to $F+G$.

Example 4.47. Find the indefinite integral

$$\int 9x^2\,dx.$$

Solution. By "find the indefinite integral" is meant to determine explicitly (up to an arbitrary constant) an antiderivative for the function $9x^2$, and also every interval I on which the antiderivative is valid. In the case of $9x^2$ an antiderivative is $3x^3$, which is valid on $(-\infty, \infty)$. Thus for arbitary constant c we have

$$\int 9x^2\,dx = 3x^3 + c$$

for all $x \in (-\infty, \infty)$. ■

Example 4.48. Find the indefinite integral

$$\int |x|\,dx.$$

Solution. Let $f(x) = |x|$. Since $f(x) = x$ for all $x > 0$, an antiderivative for f on $(0, \infty)$ is $F_1(x) = \frac{1}{2}x^2$. On the other hand for $x < 0$ we find that $f(x) = -x$, and so $F_2(x) = -\frac{1}{2}x^2$ is an antiderivative for f on $(-\infty, 0)$. Noting that $F_1(0) = F_2(0) = 0$, we can combine F_1 and F_2 to obtain a single continuous function F defined on $\mathbb{R}$:

$$F(x) = \frac{1}{2}x|x|. \tag{4.21}$$

We have $F'(x) = f(x)$ for all $x \neq 0$, and so F is an antiderivative for f on $(-\infty, 0) \cup (0, \infty)$. What about $x = 0$? We must check to see if $F'(0) = f(0)$. We have

$$F'_+(0) = \lim_{h\to 0^+} \frac{F(h) - F(0)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{2}|h|h - 0}{h} = \frac{1}{2}\lim_{h\to 0^+} h = 0,$$

and

$$F'_-(0) = \lim_{h\to 0^-} \frac{F(h) - F(0)}{h} = \lim_{h\to 0^-} \frac{\frac{1}{2}|h|h - 0}{h} = -\frac{1}{2}\lim_{h\to 0^-} h = 0,$$

and so since $F'_+(0) = F'_-(0) = 0$, it follows that $F'(0) = 0 = f(0)$. Hence F given by (4.21) is an antiderivative for f on $(-\infty, \infty)$, and we conclude that

$$\int |x|\,dx = \frac{1}{2}x|x| + c, \quad x \in (-\infty, \infty).$$

We have our answer: an antiderivative $\frac{1}{2}x|x|$, an arbitrary constant c added to the antiderivative, and an interval of validity $(-\infty, \infty)$ that is as inclusive as it can be. We have "found" the indefinite integral. ■

Example 4.49. Find the indefinite integral

$$\int \frac{1}{3\sqrt[3]{x^2}}\,dx.$$

Solution. Let

$$f(x) = \frac{1}{3\sqrt[3]{x^2}}.$$

Since

$$\left(\sqrt[3]{x}\,\right)' = \left(x^{1/3}\right)' = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} = f(x)$$

for $x \neq 0$, it follows that $\sqrt[3]{x}$ is an antiderivative for f on the intervals $(-\infty, 0)$ and $(0, \infty)$. There is no hope of there being an antiderivative for f at $x = 0$ since 0 is not in the domain of f. As there is nothing that compels the constant term in an antiderivative of f on $(-\infty, 0)$ to be the same as the constant term in an antiderivative on $(0, \infty)$, we have, most generally,

$$\int \frac{1}{3\sqrt[3]{x^2}}\,dx = \begin{cases} \sqrt[3]{x} + c_1, & \text{if } x < 0 \\ \sqrt[3]{x} + c_2, & \text{if } x > 0. \end{cases}$$

Here c_1 and c_2 are arbitrary constants, not necessarily equal. ■

Example 4.50. Find the indefinite integral

$$\int \left(2|x| + \frac{1}{\sqrt[3]{x^2}}\right) dx.$$

Solution. By Theorem 4.46,

$$\int \left(2|x| + \frac{1}{\sqrt[3]{x^2}}\right) dx = \int 2|x|\,dx + \int \frac{1}{\sqrt[3]{x^2}}\,dx = 2\int |x|\,dx + 3\int \frac{1}{3\sqrt[3]{x^2}}\,dx. \tag{4.22}$$

The two indefinite integrals at right in (4.22) were already found in Examples 4.48 and 4.49: the first has expression $\frac{1}{2}x|x| + c$ for $x \in (-\infty, \infty)$, while the second has expression $\sqrt[3]{x} + c_1$ for $x < 0$ and $\sqrt[3]{x} + c_2$ for $x > 0$. Therefore from (4.22) we conclude that

$$\begin{aligned} \int \left(2|x| + \frac{1}{\sqrt[3]{x^2}}\right) dx &= \begin{cases} 2\left(\frac{1}{2}x|x|\right) + 3\sqrt[3]{x} + c_1, & \text{if } x < 0 \\ 2\left(\frac{1}{2}x|x|\right) + 3\sqrt[3]{x} + c_2, & \text{if } x > 0 \end{cases} \\ &= \begin{cases} x|x| + 3\sqrt[3]{x} + c_1, & \text{if } x < 0 \\ x|x| + 3\sqrt[3]{x} + c_2, & \text{if } x > 0. \end{cases} \end{aligned}$$

■

Though technical exactitude is desirable when first making a study of a new mathematical concept, we shall henceforth be a bit less formal and simply write things like

$$\int \left(2|x| + \frac{1}{\sqrt[3]{x^2}}\right) dx = x|x| + 3\sqrt[3]{x} + c,$$

with the understanding that the arbitrary constant c is free to assume different values in different intervals of validity.

Example 4.51. Find the indefinite integral
$$\int \tan^2 x\,dx.$$

Solution. Since $\tan^2 x = \sec^2 x - 1$ and $(\tan x)' = \sec^2 x$, we find that
$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + c$$
As discussed in Example 4.42, each interval of validity has the form $I_n = (-\pi/2 + n\pi, \pi/2 + n\pi)$, where n is an integer; and in each interval I_n the arbitrary constant c may assume a different value c_n. ■

5

Integration Theory

5.1 – The Riemann Integral

Let f be a continuous function such that $f(x) \geq 0$ for all $a \leq x \leq b$, so that $y = f(x)$ graphs as a curve that lies above the x-axis. The question arises: what is the area of the region in the xy-plane that is contained within the graphs of the equations $x = a$, $x = b$, $y = 0$, and $y = f(x)$? Figure 27(a) provides an illustration of this region, which is customarily referred to as the "region under f between a and b". A better question might be: what should be the mathematical *definition* for the area $\mathcal{A}$ under f between a and b be?

What we have on hand from geometry is a precise definition for the area of a rectangle, length times width, and so it seems reasonable to set out by first using, say, 10 rectangles to get a rough approximation of $\mathcal{A}$, as shown in Figure 27(b). To get a better approximation we increase the number of rectangles to 20 as in Figure 27(c), and then to 50 as in Figure 27(d), and so on into the hundreds, the thousands, and the hundreds of thousands. Should we not expect our ever more refined approximations to draw ever nearer to some particular positive value? We are, after all, carrying out a kind of limit process. We define the area $\mathcal{A}$ to equal the value of this limit process.

In this section we establish some of the basic machinery that, when fully threshed out in sections to come, will allow us to determine many areas under curves with relative ease, as well as solve a host of other problems both practical and theoretical.

Before going forward a choice is offered to the reader. The first choice is to proceed in linear fashion from here up to and including Theorem 5.10 without skipping anything. The second choice is this: from here read up to but excluding Definition 5.1, then skip to §5.2 and read up to and including Theorem 5.10 (only taking that theorem to be a *definition*), then reading the part of §5.1 coming *after* Definition 5.4, omitting the proof of Proposition 5.7. The first choice presents the Riemann integral more as in a mathematical analysis book, while the second approach is more in line with a mainstream calculus book.

A **partition** of a closed, bounded interval $[a, b] \subseteq \mathbb{R}$ is a finite set of points

$$P = \{x_i\}_{i=0}^{n} \subseteq [a, b],$$

called **grid points**, such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

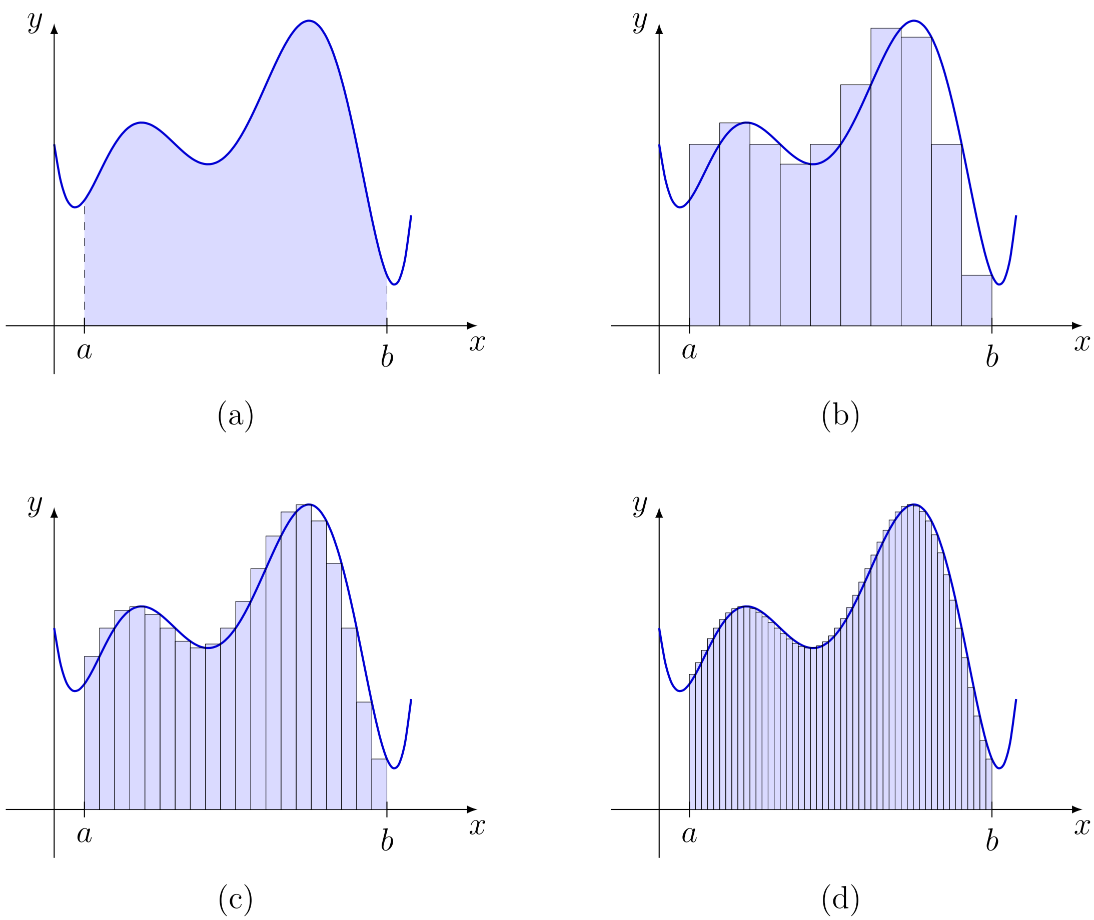


FIGURE 27.

The grid points divide the interval $[a, b]$ into smaller intervals:

$$[a, b] = [a, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-2}, x_{n-1}] \cup [x_{n-1}, b].$$

It is in this sense that the interval $[a, b]$ is "partitioned". For each $1 \leq i \leq n$ we call $I_i = [x_{i-1}, x_i]$ the ith **subinterval** of the partition P, and the **length** of the i subinterval is

$$\Delta x_i = x_i - x_{i-1}.$$

The **mesh** of P, denoted by $\|P\|$, is the length of the longest subinterval of P; that is,

$$\|P\| = \max_{1 \leq i \leq n} \Delta x_i.$$

The collection of all possible partitions of $[a, b]$ we denote by $\mathcal{P}([a, b])$ or $\mathcal{P}[a, b]$.

Notation. It is convenient to denote a partition $\{x_0, x_1, \ldots, x_n\}$ more compactly by $\{x_k\}_{k=0}^n$, and write $\{x_k\}_{k=0}^n \in \mathcal{P}[a, b]$ to specify that $\{x_k\}_{k=0}^n$ is a partition of the interval $[a, b]$.

Definition 5.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded, and let $P = \{x_k\}_{k=0}^n \in \mathcal{P}[a, b]$. Define*

$$M_k = \sup\{f(x) : x_{k-1} \leq x \leq x_k\} \quad \textit{and} \quad m_k = \inf\{f(x) : x_{k-1} \leq x \leq x_k\}$$

for each $1 \leq k \leq n$. Then

$$U(P,f) = \sum_{k=1}^{n} M_k \Delta x_k \quad \text{and} \quad L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k$$

are the ***upper sum of*** f ***with respect to*** P *and* ***lower sum of*** f ***with respect to*** P*, respectively.*

Note that since f is bounded on $[a,b,]$, it must be bounded on each $[x_{k-1}, x_k] \subseteq [a,b]$ and hence M_k and m_k must be real numbers as a consequence of the Completeness Axiom of $\mathbb{R}$.

Definition 5.2. *Let $f : [a,b] \to \mathbb{R}$ be bounded. The* ***upper Riemann integral of*** f ***over*** $[a,b]$ *is*

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}[a,b]\},$$

and the ***lower Riemann integral of*** f ***over*** $[a,b]$ *is*

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}.$$

Proposition 5.3. *If $f : [a,b] \to \mathbb{R}$ is bounded, then*

$$\overline{\int_a^b} f \quad \text{and} \quad \underline{\int_a^b} f$$

exist in $\mathbb{R}$.

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Then there exists some $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in [a,b]$. Let $P = \{x_k\}_{k=0}^{n} \in \mathcal{P}[a,b]$. For each $1 \leq k \leq n$ we have $f(x) \leq M$ for all $x \in [x_{k-1}, x_k]$, and so

$$m_k = \inf\{f(x) : x_{k-1} \leq x \leq x_k\} \leq M$$

for each $1 \leq k \leq n$. Thus

$$L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} M \Delta x_k = M \sum_{k=1}^{n} \Delta x_k = M(b-a),$$

and since $P \in \mathcal{P}[a,b]$ is arbitrary we conclude that the real number $M(b-a)$ is an upper bound for $\{L(P,f) : P \in \mathcal{P}[a,b]\}$. Therefore by the Completeness Axiom the least upper bound of $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ is real-valued, which is to say

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}$$

exists in $\mathbb{R}$.

The proof of the statement concerning the upper Riemann integral of f over $[a,b]$ is similar and so left as an exercise. ∎

Definition 5.4. *A bounded function $f : [a, b] \to \mathbb{R}$ is **Riemann integrable on** $[a, b]$ if*

$$\underline{\int_a^b} f = \overline{\int_a^b} f = I_f \in \mathbb{R}.$$

*We call the real number I_f the **Riemann integral of** f **over** $[a, b]$, and denote it by the symbol*

$$\int_a^b f \quad or \quad \int_a^b f(x)\,dx.$$

The set of all functions that are Riemann integrable on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.

The Riemann integral, also known in these notes as the **definite integral**, is just one of many different kinds of integrals defined in mathematics. In the symbol $\int_a^b f$, which in practice may be read as "the integral of f from a to b," we call a the **lower limit of integration**, b the **upper limit of integration**, and f the **integrand**.

The x in the symbol

$$\int_a^b f(x)\,dx \tag{5.1}$$

given in Definition 5.4 is called the **variable of integration**. It is also called a **dummy variable**, since we could substitute other letters for x and the meaning of the symbol would be unchanged. Thus

$$\int_a^b f(x)\,dx, \quad \int_a^b f(t)\,dt, \quad \int_a^b f(u)\,du$$

and so on are all considered identical Riemann integrals. As the simpler symbol $\int_a^b f$ suggests, only the integrand f and the limits of integration a and b uniquely determine a Riemann integral.

Using the symbol (5.1) is preferred especially when the function $[a, b] \to \mathbb{R}$ in the integrand has no designation. Thus we may write

$$\int_0^9 x^2\,dx$$

to denote the Riemann integral $\int_0^9 f$ with integrand $f(x) = x^2$. The symbol (5.1) is also useful when there is more than one independent variable present in an analysis, as will often be the case from Chapter 13 onward.

Proposition 5.5. *Let $c \in \mathbb{R}$. If $f \equiv c$ on $[a, b]$, then $f \in \mathcal{R}[a, b]$ and $\int_a^b f = c(b - a)$.*

If $f \equiv c$ on $[a, b]$ then it's common practice to write

$$\int_a^b c = c(b - a) \quad \text{or} \quad \int_a^b c\,dx = c(b - a),$$

depending on one's preference.

Example 5.6. Let $r \in \mathbb{R}$ such that $r \neq 0$, and let $c \in [a, b]$. If

$$f(x) = \begin{cases} 0, & x \neq c \\ r, & x = c \end{cases}$$

then $f \in \mathcal{R}[a, b]$ and $\int_a^b f = 0$.

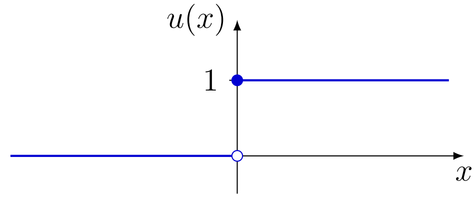


Figure 28. The unit step function.

For the next proposition we introduce the **unit step function** $u : \mathbb{R} \to \mathbb{R}$, defined by

$$u(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

See Figure 28. This function has great importance in, for instance, the study of differential equations and the Laplace transform.

Proposition 5.7. *Let $c \in \mathbb{R}$. If $f(x) = u(x-c)$, then $f \in \mathcal{R}[a,b]$ for any $-\infty < a < b < \infty$.*

Proof. Suppose $f(x) = u(x-c)$, and let $[a,b]$ be any closed, bounded interval. We have

$$f(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases}$$

so if $c \leq a$ then $f(x) = 1$ for all $x \in [a,b]$, which is to say f is a constant function on $[a,b]$ and therefore $f \in \mathcal{R}[a,b]$ by Proposition 5.5. Similarly, if $c > b$ then $f \equiv 0$ on $[a,b]$, and so $f \in \mathcal{R}[a,b]$ follows from Proposition 5.5 once more. If $c = b$, then $f(x) = 0$ for all $a \leq x < b$ and $f(b) = 1$, and thus $f \in \mathcal{R}[a,b]$ obtains from Example 5.6.

It remains to analyze the case when $c \in (a,b)$. Let $P = \{x_i\}_{i=0}^n \in \mathcal{P}[a,b]$ be such that $c \notin P$. Then there exists some $1 \leq k \leq n$ such that $x_{k-1} < c < x_k$. Since $u(x-c) \equiv 0$ on $[a, x_{k-1}]$, for $1 \leq i \leq k-1$ we have

$$m_i = \inf\{u(x-c) : x \in [x_{i-1}, x_i]\} = 0$$

and

$$M_i = \sup\{u(x-c) : x \in [x_{i-1}, x_i]\} = 0.$$

On $[x_{k-1}, x_k]$ we have $u(x-c) = 0$ for $x_{k-1} \leq x < c$ and $u(x-c) = 1$ for $c \leq x \leq x_k$, so $m_k = 0$ and $M_k = 1$. Finally, since $u(x-c) \equiv 1$ on $[x_k, b]$, for $k+1 \leq i \leq n$ we have $m_i = M_i = 1$. Now,

$$\begin{aligned} L(P,f) &= \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^k m_i \Delta x_i + \sum_{i=k+1}^n m_i \Delta x_i = \sum_{i=k+1}^n \Delta x_i \\ &= \sum_{i=k+1}^n (x_i - x_{i-1}) = x_n - x_k = b - x_k < b - c \end{aligned}$$

and

$$U(P,f) = \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^{k-1} M_i \Delta x_i + \sum_{i=k}^n M_i \Delta x_i = \sum_{i=k}^n \Delta x_i$$

$$= \sum_{i=k}^{n}(x_i - x_{i-1}) = x_n - x_{k-1} = b - x_{k-1} > b - c$$

Now suppose $P = \{x_i\}_{i=0}^n \in \mathcal{P}[a,b]$ is such that $c \in P$ and $\|P\| < \epsilon$ for arbitrary $\epsilon > 0$. Then there exists some $1 \leq k \leq n-1$ such that $c = x_k$. Since $u(x-c) \equiv 0$ on $[a, x_{k-1}]$, for $1 \leq i \leq k-1$ we have $m_i = M_i = 0$. On $[x_{k-1}, x_k]$ we have $u(x-c) = 0$ for $x_{k-1} \leq x < x_k = c$ and $u(x-c) = 1$ for $x = x_k = c$, so $m_k = 0$ and $M_k = 1$. Finally, since $u(x-c) \equiv 1$ on $[x_k, b] = [c, b]$, for $k+1 \leq i \leq n$ we have $m_i = M_i = 1$. Nothing has changed except that now $x_k = c$, so

$$L(P,f) = b - x_k = b - c \quad \text{and} \quad U(P,f) = b - x_{k-1} > b - c.$$

We see that for all $P \in \mathcal{P}[a,b]$ we have $L(P,f) \leq b - c$, with $L(P,f) = b - c$ possible if $c \in P$. This shows that $b - c$ is the least upper bound for $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ and therefore

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\} = b - c.$$

As for $U(P,f)$, certainly $b - c$ is a lower bound for $U = \{U(P,f) : P \in \mathcal{P}[a,b]\}$, but is it the greatest lower bound? Since $\|P\| < \epsilon$ we have

$$\Delta x = x_k - x_{k-1} = c - x_{k-1} < \epsilon,$$

so $x_{k-1} > c - \epsilon$ and therefore

$$b - c < U(P,f) = b - x_{k-1} < b - (c - \epsilon) = (b - c) + \epsilon.$$

We see that $b - c$ is a lower bound for the set U, and for every $\epsilon > 0$ there exists some $P \in \mathcal{P}[a,b]$ (so that $U(P,f) \in U$) for which $U(P,f) < (b-c) + \epsilon$. This shows that $b - c$ is indeed the greatest lower bound for U. That is,

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}[a,b]\} = b - c.$$

Since

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

by Definition 5.4 it follows that $f \in \mathcal{R}[a,b]$. ■

From the proof, together with a glance at Proposition 5.5 and Example 5.6, we can see that

$$\int_a^b u(x-c)\,dx = b - c$$

for all $c \in [a,b]$.

5.2 – Riemann Sums

Given a partition $P = \{x_k\}_{k=0}^n$, a **sample point** from $[x_{k-1}, x_k]$ is any point x_k^* chosen from the interval, so that

$$x_{k-1} \le x_k^* \le x_k$$

for each $1 \le k \le n$.

Definition 5.8. *Given a function $f : [a,b] \to \mathbb{R}$, a partition $P = \{x_k\}_{k=0}^n \in \mathcal{P}[a,b]$, and sample points $x_k^* \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$, the sum*

$$S(P,f) = \sum_{k=1}^n f(x_k^*)\Delta x_k$$

is called the ***Riemann sum for f with respect to P on*** *$[a,b]$.*

In this definition as well as the next one it is important to bear in mind that the value of the integer n depends on the choice of partition P. We could write n_P instead of n to emphasize this, but will refrain from doing so to minimize clutter.

Definition 5.9. *Let $L \in \mathbb{R}$. Then we define*

$$\lim_{\|P\| \to 0} S(P,f) = L$$

to mean the following: for every $\epsilon > 0$ there exists some $\delta > 0$ such that if $P = \{x_k\}_{k=0}^n \in \mathcal{P}[a,b]$ with $0 < \|P\| < \delta$, then

$$|S(P,f) - L| < \epsilon$$

for all choice of sample points $x_k^* \in [x_{k-1}, x_k]$, $1 \le k \le n$.

The following theorem provides, at least theoretically, a means of calculating a Riemann integral by evaluating a limit. Further improvements are forthcoming.

Theorem 5.10. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a,b]$ if and only if*

$$\lim_{\|P\| \to 0} S(P,f) = L$$

for some $L \in \mathbb{R}$, in which case $\int_a^b f = L$.

Not every bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$. However if f is continuous on $[a,b]$ then integrability is assured, as we will see in §5.4. Indeed it is a fact that $f \in \mathcal{R}[a,b]$ even if f has a finite number of discontinuities on $[a,b]$, so long as f is bounded on $[a,b]$. We will not need this fact at this time, however.

For each $n \in \mathbb{R}$ let $P_n = \{x_0^n, \ldots, x_{p_n}^n\}$ be some particular partition in $\mathcal{P}[a,b]$ together with chosen sample points $x_k^{n*} \in [x_{k-1}^n, x_k^n]$ for $k = 1, \ldots, p_n$. Given a function $f : [a,b] \to \mathbb{R}$, let $S(P_n, f)$ denote the Riemann sum for f with respect to P_n on $[a,b]$:

$$S(P_n, f) = \sum_{k=1}^{p_n} f(x_k^{n*})\Delta x_k^n,$$

where $\Delta x_k^n = x_k^n - x_{k-1}^n$. Now define

$$\lim_{n\to\infty} S(P_n, f) = L$$

to mean the following: for every $\epsilon > 0$ there exists some integer $N > 0$ such that if $n > N$, then $|S(P_n, f) - L| < \epsilon$. (This kind of limit is called a "sequential limit" and will be developed and analyzed more fully in Chapter 9.)

Proposition 5.11. *Suppose $f \in \mathcal{R}[a,b]$, and let $\{P_n : n \in \mathbb{R}\} \subseteq \mathcal{P}[a,b]$ be a collection of partitions (constructed as above) for which $\|P_n\| \to 0$ as $n \to \infty$. If*

$$\lim_{n\to\infty} S(P_n, f) = L$$

for some $L \in \mathbb{R}$, then $\int_a^b f = L$.

Proof. Suppose that $\lim_{n\to\infty} S(P_n, f) = L$, where $L \in \mathbb{R}$. Since $f \in \mathcal{R}[a,b]$, by Theorem 5.10 there exists some $M \in \mathbb{R}$ such that

$$\int_a^b f = \lim_{\|P\|\to 0} S(P, f) = M. \tag{5.2}$$

Suppose that $M \neq L$. Let $\epsilon > 0$ be sufficiently small so that $L \notin [M - 2\epsilon, M + 2\epsilon]$. By (5.2) there exists some $\delta > 0$ such that, for any $P \in \mathcal{P}[a,b]$ with $0 < \|P\| < \delta$, we have $|S(P, f) - L| < \epsilon$ for all choices of sample points x_k^*. Now, $\|P_n\| \to 0$ as $n \to \infty$ implies that there exists some $N \in \mathbb{R}$ such that $\|P_n\| < \delta$ for all $n > N$, and thus $|S(P_n, f) - M| < \epsilon$ holds for all $n > N$. This yields

$$-|S(P_n, f) - M| > -\epsilon,$$

and since $|M - L| > 2\epsilon$ we obtain

$$|M - L| - |S(P_n, f) - M| > \epsilon,$$

and so by the Triangle Inequality $|x - y| \geq ||x| - |y||$ we find that

$$\begin{aligned} |S(P_n, f) - L| &= \big|(M - L) - (M - S(P_n, f))\big| \\ &\geq \big||M - L| - |M - S(P_n, f)|\big| \\ &\geq |M - L| - |S(P_n, f) - M| > \epsilon \end{aligned}$$

holds for all $n > N$. But this implies that $\lim_{n\to\infty} S(P_n, f) \neq L$, which is a contradiction.

Therefore $M = L$, and we conclude that $\int_a^b f = L$. ■

The idea is that the limit $\lim_{n\to\infty} S(P_n, f)$ can be expected to be relatively easy to evaluate since it acts on a specific sequence of partitions P_n of $[a,b]$, along with specific sample points, that have been chosen for convenience.

Example 5.12. Evaluate $\int_{-1}^2 f$ for $f(x) = 3x^2 - x$.

Solution. For each $n \geq 1$ let $P_n \in \mathcal{P}[-1,2]$ be the partition that subdivides $[-1,2]$ into n subintervals each of length $3/n$ (the length of $[-1,2]$ divided by n). Thus,

$$P_n = \{\, \underbrace{-1+0(3/n)}_{x_0}, \underbrace{-1+1(3/n)}_{x_1}, \underbrace{-1+2(3/n)}_{x_2}, \dots, \underbrace{-1+k(3/n)}_{x_k}, \dots, \underbrace{-1+n(3/n)}_{x_n} \,\},$$

where of course $-1+0(3/n) = -1$ and $-1+n(3/n) = 2$, and so we obtain subintervals

$$[-1,-1+3/n], [-1+3/n, -1+2(3/n)], \dots, [-1+(k-1)(3/n), -1+k(3/n)], \dots$$
$$\dots, [-1+(n-1)(3/n), 2].$$

We need only concern ourselves with the kth subinterval, since it is representative of all the subintervals. A convenient sample point is $x_k^* = -1+k(3/n)$, which is the right endpoint of the kth subinterval. Now, since $\Delta x_k = 3/n$ for all k,

$$\begin{aligned} S(P_n, f) &= \sum_{k=1}^{n} f\left(-1+3k/n\right)(3/n) = \frac{3}{n}\sum_{k=1}^{n}\left[3\left(3k/n-1\right)^2 - (3k/n-1)\right] \\ &= \frac{3}{n}\sum_{k=1}^{n}\left(\frac{27k^2}{n^2} - \frac{21k}{n} + 4\right) = \frac{3}{n}\left[\frac{27}{n^2}\sum_{k=1}^{n}k^2 - \frac{21}{n}\sum_{k=1}^{n}k + \sum_{k=1}^{n}4\right] \\ &= \frac{3}{n}\left[\frac{27}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{21}{n}\cdot\frac{n(n+1)}{2} + 4n\right] = \frac{141n^2+144n+27}{2n^2}, \end{aligned}$$

and so

$$\begin{aligned} \int_{-1}^{2} f &= \int_{-1}^{2}(3x^2 - x)\,dx = \lim_{n\to\infty} S(P_n, f) = \lim_{n\to\infty}\frac{141n^2+144n+27}{2n^2} \\ &= \lim_{n\to\infty}\frac{141+144/n+27/n^2}{2} = \frac{141+0+0}{2} = \frac{141}{2} \end{aligned}$$

by Proposition 5.11. ■

Notice that the sequential limit in the example is handled in exactly the same way as the limit at infinity of a rational function:

$$\lim_{x\to\infty}\frac{141x^2+144x+27}{2x^2} = \lim_{x\to\infty}\frac{141+144/x+27/x^2}{2} = \frac{141}{2}.$$

The full justification for this is given in Section 9.2.[7]

Example 5.13. Use the definition of the definite integral to evaluate

$$\int_4^9 \sqrt{x}\,dx$$

[7] That sequential limits are not developed first in a traditional calculus course is one of the great tragedies of modern civilization, for they are conceptually much easier.

Solution. The strategy of starting out by partitioning the interval $[4, 9]$ into n subintervals of equal length $\Delta x = 5/n$ will not work. Indeed, if we attempt it and let $x_k^* = 4 + 5k/n$ for each $1 \le k \le n$, what results is

$$\int_4^9 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{i=1}^n \sqrt{x_k^*}\,\Delta x_k = \lim_{n\to\infty} \sum_{i=1}^n f\left(4 + \frac{5k}{n}\right)\frac{5}{n} = \lim_{n\to\infty} \frac{5}{n} \sum_{k=1}^n \sqrt{4 + \frac{5k}{n}}, \tag{5.3}$$

which leaves us stymied since we don't have any formula that resolves $\sum_{k=1}^n \sqrt{4 + 5k/n}$ as an expression in terms of n.

The way forward is to give up on having the n subintervals in our partition be of equal length. Instead, we should contrive to have the right endpoint of each subinterval be a perfect square so that the square root in (5.3) can be evaluated.

Let $f(x) = \sqrt{x}$. Referring to Figure 29, what we do is partition $[2, 3]$ on the y-axis in the expected way, choosing $P \in \mathcal{P}[2, 3]$ to be

$$P = \{2, 2 + 1/n, 2 + 2/n, \ldots, 2 + k/n, \ldots, 3\},$$

and setting $y_k^* = 2 + k/n$ for each $1 \le k \le n$. We then choose $Q \in \mathcal{P}[4, 9]$ to be

$$\begin{aligned} Q &= \left\{f^{-1}(2), f^{-1}(2 + 1/n), f^{-1}(2 + 2/n), \ldots, f^{-1}(2 + k/n), \ldots, f^{-1}(3)\right\} \\ &= \left\{4, (2 + 1/n)^2, (2 + 2/n)^2, \ldots, (2 + k/n)^2, \ldots, 9\right\} \end{aligned}$$

and set $x_k^* = f^{-1}(y_k^*) = (2 + k/n)^2$. The length of the kth subinterval in Q is

$$\Delta x_k = \left(2 + \frac{k}{n}\right)^2 - \left(2 + \frac{k-1}{n}\right)^2 = \frac{4}{n} - \frac{1}{n^2} + \frac{2k}{n}.$$

With this new partitioning scheme, we finally compute

$$\begin{aligned} \int_4^9 \sqrt{x}\,dx &= \lim_{n\to\infty} \sum_{i=1}^n \sqrt{x_k^*}\,\Delta x_k = \lim_{n\to\infty} \sum_{i=1}^n \sqrt{(2 + k/n)^2} \cdot (4/n - 1/n^2 + 2k/n) \\ &= \lim_{n\to\infty} \sum_{i=1}^n \left(2 + \frac{k}{n}\right)\left(\frac{4}{n} - \frac{1}{n^2} + \frac{2k}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{8n-2}{n^2} + \frac{8n-1}{n^3}k + \frac{2}{n^3}k^2\right) \\ &= \lim_{n\to\infty} \left[\frac{8n-2}{n^2} \cdot n + \frac{8n-1}{n^3} \cdot \frac{n(n+1)}{2} + \frac{2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right] \end{aligned}$$

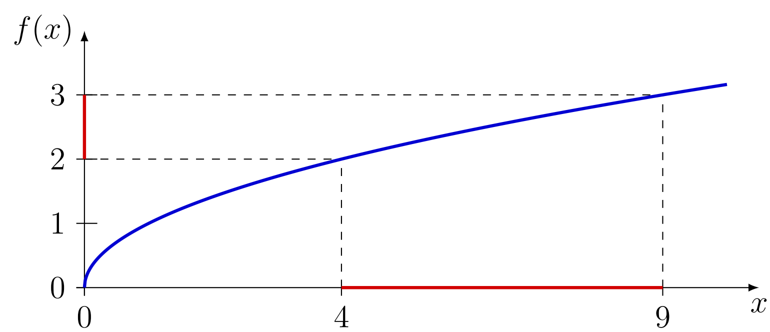


FIGURE 29.

$$= \lim_{n\to\infty} \frac{76n^2 + 15n - 1}{6n^2} = \frac{76}{6} = \frac{38}{3},$$

where the first equality is justified by Proposition 5.11. ■

The problem of approximating areas under curves introduced in Section 5.1 can now be fully resolved using definite integrals.

Definition 5.14. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $a \leq x \leq b$. The **area** of the region R bound by the curves $x = a$, $x = b$, $y = 0$, and $y = f(x)$ is*

$$\mathcal{A}(R) = \int_a^b f.$$

So, in light of Example 5.13, the area of the region R bound by $x = 4$, $x = 9$, $y = 0$, and $y = \sqrt{x}$ is $38/3$ square units.

Proposition 5.15. *If $f(x) = x$, then $f \in \mathcal{R}[a, b]$ for all $a, b \in \mathbb{R}$ such that $a < b$.*

5.3 – Properties of the Riemann Integral

In this section we establish various general properties of the Riemann integral. Most of these properties will make the task of evaluating Riemann integrals easier, and certainly all of them will be useful later on to prove further theoretical results.

Definition 5.16. *For any $f \in \mathcal{R}[a,b]$ we define*

$$\int_b^a f = -\int_a^b f.$$

For any function f for which $f(a) \in \mathbb{R}$ we define

$$\int_a^a f = 0.$$

Proposition 5.17 (Linearity Properties of the Riemann Integral). *Suppose $f, g \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$. Then the following hold.*

1. *$cf \in \mathcal{R}[a,b]$, with*

$$\int_a^b cf = c\int_a^b f.$$

2. *$f + g \in \mathcal{R}[a,b]$, with*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Theorem 5.18. *Suppose $f, g \in \mathcal{R}[a,b]$. If $f \le g$ on $[a,b]$, then*

$$\int_a^b f \le \int_a^b g.$$

Theorem 5.19. *Suppose $c \in (a,b)$. If $f \in \mathcal{R}[a,b]$, then $f \in \mathcal{R}[a,c]$, $f \in \mathcal{R}[c,b]$, and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

A result amounting nearly to a converse of Theorem 5.19 is the following proposition, the proof of which will be the one and only time in these notes that we will present a "direct" approach working with explicit partitions.

Proposition 5.20. *Suppose $c \in (a,b)$. If $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$, then $f \in \mathcal{R}[a,b]$.*

Proof. Suppose $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$. Then f is bounded on $[a,c]$ and $[c,b]$, which means it is bounded on $[a,b]$ and so there exists some $M > 1$ such that $f(x) \le M$ for all $a \le x \le b$. By Theorem 5.10 there exist $L_1, L_2 \in \mathbb{R}$ such that

$$\lim_{\substack{\|P\|\to 0\\ P\in\mathcal{P}[a,c]}} S(P,f) = L_1 \quad \text{and} \quad \lim_{\substack{\|P\|\to 0\\ P\in\mathcal{P}[c,b]}} S(P,f) = L_2.$$

Let $\epsilon > 0$. There exists $\delta_1 > 0$ such that whenever $P_1 = \{x_i\}_{i=0}^n \in \mathcal{P}[a,c]$ with $0 < \|P_1\| < \delta_1$, we have

$$|S(P_1, f) - L_1| < \epsilon/5$$

for any choice of sample points $x_i^* \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$. Also there exists $\delta_2 > 0$ such that whenever $P_2 = \{y_i\}_{i=0}^n \in \mathcal{P}[c,b]$ with $0 < \|P_2\| < \delta_2$, we have

$$|S(P_2, f) - L_2| < \epsilon/5$$

for any choice of sample points $y_i^* \in [y_{i-1}, y_i]$, $i = 1, \ldots, n$.

Let

$$\delta = \min\left\{\delta_1, \delta_2, \frac{\epsilon}{5M}\right\},$$

and suppose $P = \{z_i\}_{i=0}^n \in \mathcal{P}[a,b]$ with $\|P\| < \delta$. Assume $c \notin P$, so $z_{k-1} < c < z_k$ for some $1 \le k \le n$. Let $z_i^* \in [z_{i-1}, z_i]$ for each $i = 1, \ldots, n$. Observe that $P_1 = \{z_i\}_{i=0}^{k-1} \cup \{c\} \in \mathcal{P}[a,c]$ with $\|P_1\| < \delta_1$, and $P_2 = \{c\} \cup \{z_i\}_{i=k}^n \in \mathcal{P}[c,b]$ with $\|P_2\| < \delta_2$. For P_1 take $z_1^* < \cdots < z_{k-1}^* < c$ as sample points, and for P_2 take $c < z_{k+1}^* < \cdots < z_n^*$ as sample points. Then

$$|S(P_1, f) - L_1| < \epsilon/5 \quad \text{and} \quad |S(P_2, f) - L_2| < \epsilon/5,$$

and also

$$|f(c)(c - z_{k-1})|,\ |f(c)(z_k - c)|,\ |f(z_k^*)\Delta z_k| \le M \cdot \frac{\epsilon}{5M} = \frac{\epsilon}{5}$$

since $\|P\| < \epsilon/5M$. Now,

$$S(P_1, f) = \sum_{i=1}^{k-1} f(z_i^*)\Delta z_i + f(c)(c - z_{k-1})$$

and

$$S(P_2, f) = f(c)(z_k - c) + \sum_{i=k+1}^{n} f(z_i^*)\Delta z_i,$$

so that

$$S(P, f) = [S(P_1, f) - f(c)(c - z_{k-1})] + f(z_k^*)\Delta z_k + [S(P_2, f) - f(c)(z_k - c)].$$

We thus have

$$\begin{aligned}
&|S(P,f) - (L_1 + L_2)| \\
&\qquad = \big|[S(P_1, f) - L_1] + [S(P_2, f) - L_2] - f(c)(c - z_{k-1}) - f(c)(z_k - c) + f(z_k^*)\Delta z_k\big| \\
&\qquad \le |S(P_1, f) - L_1| + |S(P_2, f) - L_2| + |f(c)(c - z_{k-1})| + |f(c)(z_k - c)\ \ + |f(z_k^*)\Delta z_k| \\
&\qquad < \epsilon/5 + \epsilon/5 + \epsilon/5 + \epsilon/5 + \epsilon/5 = \epsilon.
\end{aligned}$$

The situation when $c \in P$ is actually simpler to analyze. So for any $\epsilon > 0$ there's some $\delta > 0$ such that any partition P of $[a,b]$ having mesh less than δ will have Riemann sum $S(P, f)$ for which

$$|S(P, f) - (L_1 + L_2)| < \epsilon$$

holds for any choice of sample points. This demonstrates that

$$\lim_{\substack{\|P\|\to 0\\ P\in\mathcal{P}[a,b]}} S(P,f) = L_1 + L_2,$$

where of course $L_1 + L_2 \in \mathbb{R}$. Therefore $f \in \mathcal{R}[a,b]$. ■

In the proof of Proposition 5.20 note that, by Theorem 5.10,

$$L_1 = \int_a^c f \quad \text{and} \quad L_2 = \int_c^b f,$$

and so the proof establishes that

$$\int_a^b f = \int_a^c f + \int_c^b f,$$

which one might expect in light of Theorem 5.19 (whose proof can be done in a very similar fashion).

Theorem 5.21. *Suppose $f \in \mathcal{R}[a,b]$. If*

$$\{x \in [a,b] : g(x) \neq f(x)\}$$

is a finite set, then $g \in \mathcal{R}[a,b]$ and

$$\int_a^b g = \int_a^b f.$$

5.4 – Integrable Functions

In this section we develop tools to help determine whether a given function is Riemann integrable on an interval $[a, b]$.

Theorem 5.22. *If $\varphi \in \mathcal{R}[a, b]$, $\text{Ran}(\varphi) \subseteq [\alpha, \beta]$, and $\psi : [\alpha, \beta] \to \mathbb{R}$ is continuous, then $\psi \circ \varphi \in \mathcal{R}[a, b]$.*

Proposition 5.23. *If f is continuous on $[a, b]$, then $f \in \mathcal{R}[a, b]$.*

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Let $\varphi : [a, b] \to \mathbb{R}$ be the identity function on $[a, b]$, so that $\varphi(x) = x$ for all $x \in [a, b]$. Since $\text{Ran}(\varphi) = [a, b]$, and $\varphi \in \mathcal{R}[a, b]$ by Proposition 5.15, it follows by Theorem 5.22 that $f \circ \varphi \in \mathcal{R}[a, b]$. Now, for any $x \in [a, b]$,

$$(f \circ \varphi)(x) = f(\varphi(x)) = f(x),$$

so we see that $f \circ \varphi = f$ and therefore $f \in \mathcal{R}[a, b]$. ■

Proposition 5.24. *Suppose $f, g \in \mathcal{R}[a, b]$. Then the following properties hold.*

1. $f^2 \in \mathcal{R}[a, b]$,
2. $fg \in \mathcal{R}[a, b]$,
3. $|f| \in \mathcal{R}[a, b]$ *with* $\left|\int_a^b f\right| \le \int_a^b |f|$,
4. $f \vee g, f \wedge g \in \mathcal{R}[a, b]$.

Proof.
Proof of Part (1). Suppose that $f \in \mathcal{R}[a, b]$. Then $f : [a, b] \to \mathbb{R}$ is a bounded function, so that $\text{Ran}(f) \subseteq [\alpha, \beta]$ for some $-\infty < \alpha < \beta < \infty$. Defining the function ψ by $\psi(x) = x^2$, it is clear that ψ is continuous on $[\alpha, \beta]$. By Theorem 5.22 we conclude that $\psi \circ f \in \mathcal{R}[a, b]$. Now, since

$$(\psi \circ f)(x) = \psi(f(x)) = [f(x)]^2 = f^2(x)$$

for any $x \in [a, b]$, we see that $\psi \circ f = f^2$ and therefore $f^2 \in \mathcal{R}[a, b]$.

Proof of Part (2). Suppose that $f, g \in \mathcal{R}[a, b]$. By Proposition 5.17 we have $f + g, f - g \in \mathcal{R}[a, b]$, and then by Part (1) we have $(f + g)^2, (f - g)^2 \in \mathcal{R}[a, b]$. Applying Proposition 5.17 once more, it follows that

$$fg = \frac{1}{4}[(f + g)^2 - (f - g)^2] \in \mathcal{R}[a, b]$$

as was to be shown.

Proof of Part (3). Suppose that $f \in \mathcal{R}[a, b]$. Once again $\text{Ran}(f) \subseteq [\alpha, \beta]$ for some $-\infty < \alpha < \beta < \infty$. If $\psi(x) = |x|$, then ψ is continuous on $[\alpha, \beta]$. By Theorem 5.22 we conclude that $\psi \circ f \in \mathcal{R}[a, b]$. Now, since

$$(\psi \circ f)(x) = \psi(f(x)) = |f(x)| = |f|(x)$$

for any $x \in [a, b]$, we see that $\psi \circ f = |f|$ and therefore $|f| \in \mathcal{R}[a, b]$. ■

5.5 – The Fundamental Theorem of Calculus

Recall that, according to the Extreme Value Theorem, a continuous function on a closed interval $[a, b]$ will attain an absolute maximum value and an absolute minimum value. That is, there will be some $x_1, x_2 \in [a, b]$ such that $f(x_1) \geq f(x)$ for all $x \in [a, b]$ and $f(x_2) \leq f(x)$ for all $x \in [a, b]$. Then we can write

$$f(x_1) = \max\{f(x) : x \in [a, b]\} \quad \text{and} \quad f(x_2) = \min\{f(x) : x \in [a, b]\}.$$

This is essential in what follows.

Lemma 5.25. *Let f be continuous on $[a, b]$ and $c \in [a, b]$. Define $\alpha_c, \beta_c : [a, b] \to \mathbb{R}$ by*

$$\alpha_c(x) = \begin{cases} \max\{f(t) : t \in [c, x]\}, & \textit{if } x \geq c \\ \max\{f(t) : t \in [x, c]\}, & \textit{if } x < c \end{cases}$$

and

$$\beta_c(x) = \begin{cases} \min\{f(t) : t \in [c, x]\}, & \textit{if } x \geq c \\ \min\{f(t) : t \in [x, c]\}, & \textit{if } x < c \end{cases}$$

Then $\lim_{x\to c} \alpha_c(x) = \lim_{x\to c} \beta_c(x) = f(c)$.

Remark. Note α_c and β_c are indeed real-valued functions for any $a \leq c \leq b$ since f is bounded on $[a, b]$ by the Extreme Value Theorem.

Proof. We will assume that $c \in (a, b)$. If $c = a$ or $c = b$ only a right- or left-hand limit, respectively, would need to be considered, but otherwise the argument would be the same.

Let $\epsilon > 0$. Since f is continuous at c, there exists some $\delta_1 > 0$ such that $c \leq x < c+\delta_1$ implies $|f(x) - f(c)| < \epsilon/2$. Suppose that $c < x < c + \delta_1$. For any $t \in [c, x]$ we have $|f(t) - f(c)| < \epsilon/2$, whence

$$f(c) - \epsilon/2 < f(t) < f(c) + \epsilon/2$$

obtains and thus

$$f(c) - \epsilon/2 < \alpha_c(x) = \max\{f(t) : t \in [c, x]\} \leq f(c) + \epsilon/2.$$

From this it can be seen that

$$|\alpha_c(x) - f(c)| \leq \epsilon/2 < \epsilon,$$

and so

$$\lim_{x\to c^+} \alpha_c(x) = f(c).$$

Next, continuity of f at c implies that there is some $\delta_2 > 0$ such that $|f(x) - f(c)| < \epsilon/2$ whenever $c - \delta_2 < x \leq c$. Suppose that $c - \delta_2 < x < c$. For any $t \in [x, c]$ we have $|f(t) - f(c)| < \epsilon/2$, which gives

$$f(c) - \epsilon/2 < f(t) < f(c) + \epsilon/2$$

and thus

$$f(c) - \epsilon/2 < \alpha_c(x) = \max\{f(t) : t \in [x, c]\} \leq f(c) + \epsilon/2.$$

Once again we're led to conclude that

$$|\alpha_c(x) - f(c)| \le \epsilon/2 < \epsilon,$$

and so

$$\lim_{x\to c^-} \alpha_c(x) = f(c).$$

So $\lim_{x\to c} \alpha_c(x) = f(c)$ for any $c \in (a,b)$. The proof runs along similar lines for the function β_c and so it omitted. ■

Theorem 5.26 (The Fundamental Theorem of Calculus, Part 1). *If f is continuous on $[a,b]$, then the function $\Phi : [a,b] \to \mathbb{R}$ given by*

$$\Phi(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is differentiable on $[a,b]$, with $\Phi'(x) = f(x)$ for each $a \le x \le b$.

Remark. As is our custom, it is intended that $\Phi'(a)$ and $\Phi'(b)$ be taken as signifying the one-sided derivatives $\Phi'_+(a)$ and $\Phi'_-(b)$, respectively.

Proof. Suppose that f is continuous on $[a,b]$, so that $f \in \mathcal{R}[a,b]$ by Proposition 5.23. Let $c \in (a,b)$. Using Theorem 5.19, we have

$$\lim_{x\to c^+} \frac{\Phi(x) - \Phi(c)}{x - c} = \lim_{x\to c^+} \frac{1}{x-c}\left(\int_a^x f - \int_a^c f\right) = \lim_{x\to c^+} \frac{1}{x-c}\int_c^x f \tag{5.4}$$

Let $\alpha_c, \beta_c : [a,b] \to \mathbb{R}$ be as defined in Lemma 5.25. For any fixed $x \in (c,b)$ we have $\beta_c(x) \le f(t) \le \alpha_c(x)$ for all $t \in [c,x]$. Thus by Theorem 5.18 we obtain

$$\int_c^x \beta_c(x) \le \int_c^x f \le \int_c^x \alpha_c(x),$$

whence Proposition 5.5 gives

$$\beta_c(x)\cdot(x-c) \le \int_c^x f \le \alpha_c(x)\cdot(x-c)$$

and therefore

$$\beta_c(x) \le \frac{1}{x-c}\int_c^x f \le \alpha_c(x). \tag{5.5}$$

Since the inequality (5.5) holds for all $c < x < b$ and

$$\lim_{x\to c^+} \alpha_c(x) = \lim_{x\to c^+} \beta_c(x) = f(c)$$

by Lemma 5.25, by (5.4) and the Squeeze Theorem we obtain

$$\lim_{x\to c^+} \frac{\Phi(x) - \Phi(c)}{x - c} = \lim_{x\to c^+} \frac{1}{x-c}\int_c^x f = f(c).$$

A similar argument shows that

$$\lim_{x\to c^-} \frac{\Phi(x) - \Phi(c)}{x - c} = \lim_{x\to c^-} \frac{1}{x-c}\int_c^x f = f(c),$$

and thus
$$\Phi'(c) = \lim_{x\to c} \frac{\Phi(x) - \Phi(c)}{x - c} = f(c).$$
Since $a < c < b$ is arbitrary, we conclude that Φ is differentiable on (a, b) with $\Phi'(x) = f(x)$ for all $x \in (a, b)$.

Now let $c = a$. We have
$$\alpha_a(x) = \max\{f(t) : t \in [a, x]\} \quad \text{and} \quad \beta_a(x) = \min\{f(t) : t \in [a, x]\},$$
and Lemma 5.25 gives
$$\lim_{x\to a^+} \alpha_a(x) = \lim_{x\to a^+} \beta_a(x) = f(a). \tag{5.6}$$
Given any $x \in (a, b)$ we have $\beta_a(x) \le f(t) \le \alpha_a(x)$ for all $t \in [a, x]$, implying
$$\int_a^x \beta_a(x) \le \int_a^x f \le \int_a^x \alpha_a(x),$$
and thus
$$\beta_a(x) \le \frac{1}{x - a} \int_a^x f \le \alpha_a(x). \tag{5.7}$$
Since (5.7) holds for all $a < x < b$, by (5.6) and the Squeeze Theorem we obtain
$$\Phi'_+(a) = \lim_{x\to a^+} \frac{\Phi(x) - \Phi(a)}{x - a} = \lim_{x\to a^+} \frac{\Phi(x)}{x - a} = \lim_{x\to a^+} \frac{1}{x - a} \int_a^x f = f(a),$$
where $\Phi(a) = 0$ by Definition 5.18. A similar argument shows that $\Phi'_-(b) = f(b)$.

Therefore Φ is differentiable on $[a, b]$ with $\Phi'(x) = f(x)$ for all $a \le x \le b$. ■

Example 5.27. Given that
$$F(x) = \int_2^{\sin x} (1 - t^2)^7 \, dt,$$
find F'.

Solution. Define $\Phi : \mathbb{R} \to \mathbb{R}$ by
$$\Phi(x) = \int_2^x (1 - t^2)^7 \, dt.$$
Then
$$F(x) = \Phi(\sin(x)) = (\Phi \circ \sin)(x)$$
and the Chain Rule gives
$$F'(x) = (\Phi \circ \sin)'(x) = \Phi'(\sin(x)) \cdot \sin'(x) = \Phi'(\sin(x)) \cdot \cos(x)$$
for all $x \in \mathbb{R}$. Now, by Theorem 5.26 we have $\Phi'(x) = (1 - x^2)^7$, and so
$$F'(x) = \Phi'(\sin x) \cdot \cos x = (1 - \sin^2 x)^7 (\cos x) = (\cos^2 x)^7 (\cos x) = \cos^{15}(x)$$
for all $x \in \mathbb{R}$. ■

Recalling Definition 4.40, what Theorem 5.26 says is that Φ is an antiderivative for f on $[a, b]$. We make use of this fact to prove the following.

Theorem 5.28 (**The Fundamental Theorem of Calculus, Part 2**). *Let f be continuous on $[a, b]$. If F is any antiderivative for f on $[a, b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a) \tag{5.8}$$

Proof. F be an antiderivative for f on $[a, b]$. By Theorem 4.43 there exists some constant c such that $F = \Phi + c$, where Φ is the antiderivative for f on $[a, b]$ that is given in Theorem 5.26. That is,

$$F(x) = \Phi(x) + c = \int_a^x f + c$$

for $x \in [a, b]$, which gives $F(a) = \int_a^a f + c = c$, and so

$$F(b) = \int_a^b f + c = \int_a^b f + F(a).$$

Equation (5.8) now follows. ∎

Remark. Theorem 5.28 applies even in the case when $a = b$, since both sides of equation (5.8) become zero.

The first part of the Fundamental Theorem of Calculus shows how a definite integral can be used to determine an antiderivative for a function f on a closed interval $[a, b]$, while the second part shows how an antiderivative can be used to determine a definite integral for f on $[a, b]$. The symmetry is something to behold, the two parts taken together effectively uniting the differential and integral branches of calculus.

The great utility of Theorem 5.28 can be appreciated by comparing the lengthy calculation in Example 5.13 to the following calculation.

Example 5.29. Evaluate

$$\int_4^9 \sqrt{x}\, dx$$

Solution. An antiderivative for $f(x) = \sqrt{x}$ on $[4, 9]$ is $F(x) = \frac{2}{3}x^{3/2}$. We compute

$$\int_4^9 \sqrt{x}\, dx = F(9) - F(4) = \frac{2}{3}\left(9^{3/2} - 4^{3/2}\right) = \frac{2}{3}(27 - 8) = \frac{38}{3},$$

and we're done. ∎

One should not get swept away by an irrational exuberance when wielding Theorem 5.28 in a crusade to vanquish inimical definite integrals. Consider the integral

$$\int_0^8 \frac{1}{3}x^{-2/3} dx. \tag{5.9}$$

An antiderivative for $f(x) = \frac{1}{3}x^{-2/3}$ is $F(x) = \sqrt[3]{x}$. This is only true for $x \neq 0$, however, since the domain of f is $(-\infty, 0) \cup (0, \infty)$, which certainly includes the interval $(0, 8]$ but not quite the entire interval of integration $[0, 8]$! The integral (5.9) is what is called an "improper integral," which is the subject of §8.7. A direct application of Theorem 5.28 is not possible in this case, even though it may often "accidentally" arrive at the right answer.

We turn to a new matter. Suppose f is continuous on $[a, b]$, and $\Phi : [a, b] \to \mathbb{R}$ is a function given by

$$\Phi(x) = \int_x^b f(t)\,dt$$

for all $x \in [a, b]$. What is $\Phi'(x)$? We might be tempted to use Definition 5.16 to write $\int_x^b f$ as $-\int_b^x f$ (thereby formally restoring x as the upper limit of integration) and then employ Theorem 5.26 to obtain $\Phi'(x) = -f(x)$. The flaw with this approach is that the integral in Theorem 5.26 requires x to be greater than or equal to the lower limit of integration, whereas in our situation this is not so. More must be done in order to rigorously derive an expression for $\Phi'(x)$, but the good news is that it will indeed prove to be $-f(x)$!

Proposition 5.30. *If f is continuous on $[a, b]$, then the function $\Phi : [a, b] \to \mathbb{R}$ given by*

$$\Phi(x) = \int_x^b f(t)\,dt, \quad a \le x \le b$$

is differentiable on $[a, b]$, with $\Phi'(x) = -f(x)$ for each $a \le x \le b$.

Proof. Suppose f is continuous on $[a, b]$. By Theorem 5.26 it is known that f has an antiderivative F on $[a, b]$, and thus F is an antiderivative for f on $[x, b]$ for any $a \le x \le b$. By Theorem 5.28 (and the remark following it) we have, for all $x \in [a, b]$,

$$\Phi(x) = \int_x^b f(t)\,dt = F(b) - F(x),$$

which shows that Φ is differentiable on $[a, b]$, and moreover

$$\Phi'(x) = \big(F(b) - F(x)\big)' = 0 - F'(x) = -f(x)$$

as was to be shown. ∎

The following proposition is something that will be useful in the study of differential equations. What it shows is that the choice of a for the lower limit of integration in Theorem 5.26 is not essential: any $c \in [a, b]$ may be the lower limit of integration, and the resultant function Φ will prove to be an antiderivative of f on $[a, b]$.

Proposition 5.31. *If f is continuous on (a, b) and $c \in (a, b)$, then the function $\Phi : (a, b) \to \mathbb{R}$ given by*

$$\Phi(x) = \int_c^x f(t)\,dt, \quad a < x < b \tag{5.10}$$

is differentiable on (a, b), with $\Phi'(x) = f(x)$ for each $a < x < b$.

If f is continuous on $[a, b]$, then $\Phi' = f$ on $[a, b]$ for any choice of $c \in [a, b]$ in (5.10).

Proof. Suppose f is continuous on (a,b) and $c \in (a,b)$. Let $\epsilon > 0$ be such that $c \in [a+\epsilon, b-\epsilon]$. Since f is continuous on $[c, b-\epsilon]$ and

$$\Phi(x) = \int_c^x f(t)\,dt$$

for $x \in [c, b-\epsilon]$, Theorem 5.26 implies that $\Phi'(x) = f(x)$ for $c < x < b-\epsilon$, and $\Phi'_+(c) = f(c)$. Since f is continuous on $[a+\epsilon, c]$ and

$$\Phi(x) = \int_c^x f(t)\,dt = -\int_x^c f(t)\,dt$$

for $x \in [a+\epsilon, c]$, Proposition 5.30 implies that

$$\Phi'(x) = -\frac{d}{dx}\int_x^c f(t)\,dt = -[-f(x)] = f(x)$$

for $a+\epsilon < x < c$, and $\Phi'_-(c) = f(c)$. Now, $\Phi'_+(c) = \Phi'_-(c) = f(c)$ shows that $\Phi'(c) = f(c)$, and therefore $\Phi'(x) = f(x)$ for all $x \in (a+\epsilon, b-\epsilon)$. Since $\epsilon > 0$ is arbitrary, we conclude that Φ is differentiable on (a,b) with $\Phi'(x) = f(x)$ for all $x \in (a,b)$.

The last statement in the proposition follows from combining the first statement with Theorem 5.26 and Proposition 5.30. ■

5.6 – The Substitution Rule

We develop here an important tool that will enable us to determine a wider class of indefinite integrals and, as a result, evaluate more definite integrals as well.

Theorem 5.32. *Let I be an interval, and let $g : I \to \mathbb{R}$ be such that g is differentiable on I. Suppose F is an antiderivative for f on an open interval J containing $g(I)$. Then*

$$\int (f \circ g)g' = \{F \circ g + c : c \in \mathbb{R}\}. \tag{5.11}$$

As with similar results in §4.8, there are two alternate notations that may be used to denote (5.11): what might be called the "concise" notation,

$$\int (f \circ g)g' = F \circ g + c,$$

and what might be called the "classical" notation,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c. \tag{5.12}$$

In any case c is understood to be an arbitrary real-valued constant. Now for the proof of the theorem.

Proof. Fix $x \in \text{Int}(I)$. Then x is in the interior of $\text{Dom}(g)$, and so g is differentiable at x. Now, $g(x) \in g(I) \subseteq J$, and since J is open and F is differentiable on J, it follows that F is differentiable at $g(x)$. By the Chain Rule as given by Theorem 3.21, $F \circ g$ is differentiable at x with

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Now suppose x is a left endpoint of I. Then g is right-differentiable at x, but as before we find that F is differentiable in the usual (two-sided) sense at $g(x)$. A simple extension of the Chain Rule implies that $F \circ g$ is right-differentiable at x with

$$(F \circ g)'_+(x) = f(g(x))g'_+(x).$$

Similarly, if x is a right endpoint of I we find that $F \circ g$ is left-differentiable at x with

$$(F \circ g)'_-(x) = f(g(x))g'_-(x).$$

We have now shown that $F \circ g$ is an antiderivative for $(f \circ g)g'$ on I, and therefore (5.11) follows from Proposition 4.45. ∎

Example 5.33. Determine

$$\int \sqrt{ax + b}\,dx,$$

where $a \neq 0$ and b are constants, by using Theorem 5.32.

Solution. In light of equation (5.12), which is the essential embodiment of the theorem, it's clear the first task is to find functions f and g such that

$$f(g(x))g'(x) = \sqrt{ax+b}.$$

If we let $g(x) = ax+b$ and $f(x) = \sqrt{x}$ we find that $f(g(x))g'(x) = a\sqrt{ax+b}$, which is almost what we seek save for the factor of a. This is most easily rectified if we adjust f so that $f(x) = \sqrt{x}/a$. Thus,

$$\int \sqrt{ax+b}\,dx = \int g(f(x))g'(x)\,dx$$

for $f(x) = (1/a)x^{1/2}$ and $g(x) = ax+b$. Now, because $F(x) = (2/3a)x^{3/2}$ is an antiderivative for f, equation (5.12) gives

$$\int \sqrt{ax+b}\,dx = F(g(x)) + c = F(ax+b) + c = \frac{2}{3a}(ax+b)^{3/2} + c$$

and we're done. ∎

There's a mechanical procedure known as ***u*-substitution** which can help streamline finding an indefinite integral of the form

$$\int f(g(x))g'(x)\,dx \tag{5.13}$$

over an applicable interval I. Let $u = g(x)$ on I. Substitute u for $g(x)$ and du for $g'(x)\,dx$ in (5.13) to obtain the indefinite integral $\int f(u)\,du$. (Here it's best to write $\int f(u)\,du$ rather than $\int f$ to make it clear that f is regarded as a function of u.) Now, the function F in Theorem 5.32 is an antiderivative for f on the interval $g(I)$, so by Proposition 4.45,

$$\int f(u)\,du = F(u) + c,$$

whereupon we replace u with $g(x)$ on the right-hand side to obtain $F(g(x)) + c$ precisely as in equation (5.12). In this sense we may write

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du,$$

which is generally referred to as the **Substitution Rule** since a new variable u has been substituted for $g(x)$. The procedure is as follows:

- For $\int f(g(x))g'(x)\,dx$, let $u = g(x)$ and $du = g'(x)\,dx$ to obtain
$$\int f(u)\,du.$$
- Determine an antiderivative F for f, so that
$$\int f(u)\,du = F(u) + c.$$
- Substitute $g(x)$ for u in $F(u) + c$ to obtain
$$\int f(g(x))g'(x)\,dx = F(g(x)) + c.$$

Example 5.34. For constants $a \neq 0$ and $b \in \mathbb{R}$, find

$$\int \sqrt{ax+b}\,dx$$

using the Substitution Rule.

Solution. In this approach we make a reasoned guess as to what $g(x)$ ought to be, and then hope the machinery of the procedure will make any corrections that may be necessary. If we guess that $g(x) = ax+b$, then let $u = g(x) = ax+b$ and $du = g'(x)\,dx = a\,dx$. Here is the correction: since $a\,dx$ is not in evidence in our integrand, we formally manipulate $du = a\,dx$ to obtain $dx = (1/a)\,du$. Now,

$$\int \sqrt{ax+b}\,dx = \int \sqrt{u}\cdot\frac{1}{a}\,du = \frac{1}{a}\int\sqrt{u}\,du = \frac{1}{a}\cdot\frac{2}{3}u^{3/2} + c,$$

and when we substitute $ax+b$ for u we obtain

$$\int \sqrt{ax+b}\,dx = \frac{2}{3a}(ax+b)^{3/2} + c$$

precisely as before. ■

Example 5.35. Find

$$\int \sin^{10}\theta\cos\theta\,d\theta.$$

Solution. Let $u = \sin\theta$, so that $du = (\sin\theta)'\,d\theta = \cos\theta\,d\theta$, which happens to be in the integrand so no further manipulations are necessary. We obtain

$$\int \sin^{10}\theta\cos\theta\,d\theta = \int u^{10}\,du = \frac{1}{11}u^{11} + c,$$

and when we substitute $\sin\theta$ for u,

$$\int \sin^{10}\theta\cos\theta\,d\theta = \frac{1}{11}\sin^{11}\theta + c$$

is the end result. ■

Example 5.36. Evaluate

$$\int_0^3 \frac{y^2+1}{\sqrt{y^3+3y+4}}\,dy.$$

Solution. First the Substitution Rule will be employed to determine

$$\int \frac{y^2+1}{\sqrt{y^3+3y+4}}\,dy.$$

Let $u = y^3+3y+4$, so that $du = (y^3+3y+4)'\,dy = (3y^2+3)\,dy$. The integrand does not quite contain this expression, but if we divide by 3 we obtain just what's needed: $(y^2+1)\,dy = \frac{1}{3}\,du$. Now,

$$\int \frac{y^2+1}{\sqrt{y^3+3y+4}}\,dy = \int \frac{1}{\sqrt{y^3+3y+4}}\cdot(y^2+1)\,dy = \int \frac{1}{\sqrt{u}}\cdot\frac{1}{3}\,du$$

$$= \frac{1}{3}\int u^{-1/2}\,du = \frac{2}{3}u^{1/2} + C = \frac{2}{3}\sqrt{y^3+3y+4} + C.$$

Thus an antiderivative for

$$f(y) = \frac{y^2+1}{\sqrt{y^3+3y+4}}$$

is

$$F(y) = \frac{2}{3}\sqrt{y^3+3y+4}.$$

Therefore

$$\int_0^3 \frac{y^2+1}{\sqrt{y^3+3y+4}}\,dy = F(3) - F(0) = \frac{2}{3}\sqrt{40} - \frac{2}{3}\sqrt{4} = \frac{4(\sqrt{10}-1)}{3}$$

by Theorem 5.28. ■

Example 5.36 indicates how the Substitution Rule together with the Fundamental Theorem of Calculus can be used to evaluate certain definite integrals. The next theorem provides a substitution rule that is specialized to handle definite integrals directly. The proof makes no use whatsoever of facts established about indefinite integrals.

Theorem 5.37. *For functions f and g, if g' is continuous on $[a,b]$ and f is continuous on $g([a,b])$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. The continuity of g' on $[a,b]$ implies that g is continuous on $[a,b]$, and so by the Extreme Value Theorem $g([a,b])$ is a closed interval $[m,M]$. Since f is continuous on $[m,M]$, we can find a function φ that is continuous on an open interval J that contains $[m,M]$ such that $\varphi(x) = f(x)$ for all $x \in [m,M]$. Now, let $[\alpha,\beta]$ be such that

$$[m,M] \subseteq (\alpha,\beta) \subseteq [\alpha,\beta] \subseteq J.$$

Since φ is continuous on $[\alpha,\beta]$, by Theorem 5.26 there exists a function Φ, continuous on $[\alpha,\beta]$, such that $\Phi'(u) = \varphi(u)$ for all $u \in (\alpha,\beta)$.

Fix $x \in (a,b)$. Since g is differentiable at x and Φ is differentiable at $g(x) \in [m,M] \subseteq (\alpha,\beta)$, by the Chain Rule we obtain

$$(\Phi\circ g)'(x) = \Phi'(g(x))g'(x) = \varphi(g(x))g'(x) = f(g(x))g'(x) = ((f\circ g)g')(x).$$

Since $\Phi\circ g$ and $(f\circ g)g'$ are continuous on $[a,b]$, and $(\Phi\circ g)' = (f\circ g)g'$ on (a,b), it follows that $\Phi\circ g$ is an antiderivative for $(f\circ g)g'$ on $[a,b]$, and so by Theorem 5.28 we obtain

$$\int_a^b f(g(x))g'(x)\,dx = (\Phi\circ g)(b) - (\Phi\circ g)(a) = \Phi(g(b)) - \Phi(g(a)). \tag{5.14}$$

Now, suppose that $g(a) \le g(b)$, and note that $[g(a),g(b)] \subseteq [m,M]$. For any $u \in (g(a),g(b))$ we have $u \in [m,M] \subseteq (\alpha,\beta)$ and thus $\Phi'(u) = \varphi(u) = f(u)$, and since Φ and f are continuous on $[g(a), g(b)]$ we conclude that Φ is an antiderivative for f on $[g(a), g(b)]$, giving

$$\int_{g(a)}^{g(b)} f(u)\,du = \Phi(g(b)) - \Phi(g(a)). \tag{5.15}$$

On the other hand if $g(b) \leq g(a)$, we have $[g(b), g(a)] \subseteq [m, M]$ and similar arguments lead to the conclusion that Φ is an antiderivative for f on $[g(b), g(a)]$, giving

$$\int_{g(b)}^{g(a)} f(u)\,du = \Phi(g(a)) - \Phi(g(b));$$

but then, by Theorem 5.18(2),

$$\int_{g(a)}^{g(b)} f(u)\,du = -\int_{g(b)}^{g(a)} f(u)\,du = -[\Phi(g(a)) - \Phi(g(b))] = \Phi(g(b)) - \Phi(g(a)), \tag{5.16}$$

as with (5.15). Comparing equations (5.15) and (5.16) with (5.14), it is clear that

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

which completes the proof. ■

6

Applications of Integration

6.1 – The Mean Value Theorem for Integrals

In §4.2 we encountered the traditional Mean Value Theorem, sometimes called the Mean Value Theorem for Derivatives to distinguish it from the following.

Theorem 6.1 (**Mean Value Theorem for Integrals**). *If the function f is continuous on $[a,b]$, then there exists some $c \in (a,b)$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

Proof. Suppose that f is continuous on $[a,b]$. By the Fundamental Theorem of Calculus the function $F : [a,b] \to \mathbb{R}$ given by

$$F(x) = \int_a^x f$$

is continuous on $[a,b]$ and differentiable on (a,b). Thus by the Mean Value Theorem for Derivatives there exists some $c \in (a,b)$ such that

$$F'(c) = \frac{F(b)-F(a)}{b-a}.$$

Since $F'(c) = f(c)$ by the Fundamental Theorem of Calculus, and

$$F(b) = \int_a^b f \quad \text{and} \quad F(a) = \int_a^a f = 0,$$

it follows that

$$f(c) = \frac{1}{b-a}\left(\int_a^b f - \int_a^a f\right) = \frac{1}{b-a}\int_a^b f$$

as was to be shown. ■

6.2 – Regions Between Curves

Definition 6.2. *Let f and g be continuous functions on $[a,b]$. The **area** of the region R bound by the curves $x=a$, $x=b$, $y=f(x)$, and $y=g(x)$ is*

$$\mathcal{A}(R)=\int_a^b |f-g|.$$

If $g(x)=0$ and $f(x)\geq 0$ for all $x\in[a,b]$, then

$$\mathcal{A}(R)=\int_a^b |f(x)-g(x)|\,dx=\int_a^b f(x)\,dx=\int_a^b f,$$

which is just the area under f between $x=a$ and $y=b$ as defined in Section 5.2.

Example 6.3. Find the area of the region bounded by $f(x)=\sqrt{x}$ and $g(x)=x^2$.

Solution. We start by finding the points where the curves intersect, which entails finding all x for which $f(x)=g(x)$. Thus, we solve the equation $\sqrt{x}=x^2$:

$$\sqrt{x}=x^2 \quad\Rightarrow\quad x=x^4 \quad\Rightarrow\quad x^4-x=0 \quad\Rightarrow\quad x(x^3-1) \quad\Rightarrow\quad x=0 \quad\text{or}\quad x=1.$$

The curves intersect at points $(0,0)$ and $(1,1)$, bounding the region R shown in Figure 30. Clearly $f(x)=\sqrt{x}\geq x^2=g(x)$ on $[0,1]$, and so

$$\begin{aligned}\mathcal{A}(R)&=\int_0^1 \left|\sqrt{x}-x^2\right|\,dx=\int_0^1\left(\sqrt{x}-x^2\right)dx=\left[\frac{2}{3}\sqrt{x^3}-\frac{1}{3}x^3\right]_0^1\\&=\left[\frac{2}{3}\cdot\sqrt{1^3}-\frac{1}{3}\cdot 1^3\right]-\left[\frac{2}{3}\cdot\sqrt{0^3}-\frac{1}{3}\cdot 0^3\right]=\frac{1}{3}\end{aligned}$$

is the area of R. ■

Example 6.4. Find the area of the region R that lies between the curves $f(x)=2x-x^3$ and $g(x)=x^2$ for $-2\leq x\leq 2$.

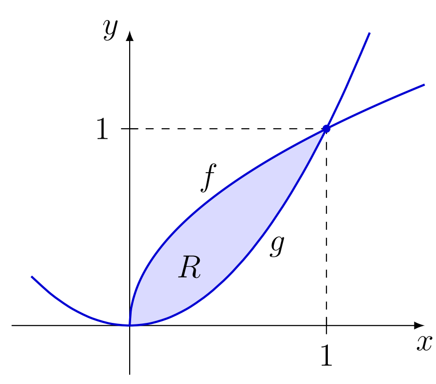


Figure 30.

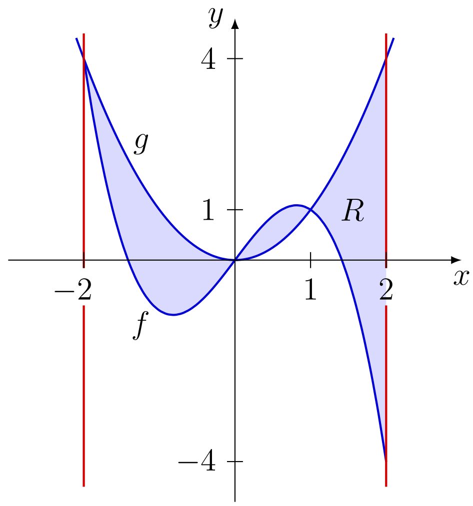


FIGURE 31.

Solution. Again we start by finding where the curves given by f and$g$ intersect, which means finding all x for which $f(x) = g(x)$, and thus we must solve the equation $2x - x^3 = x^2$:

$$2x - x^3 = x^2 \quad \Rightarrow \quad x^3 + x^2 - 2x = 0 \quad \Rightarrow \quad x(x+2)(x-1) = 0 \quad \Rightarrow \quad x = -2, 0, 1.$$

So the curves intersect at points $(-2, f(-2)) = (-2, 4)$, $(0, f(0)) = (0, 0)$, and $(1, f(1)) = (1, 1)$, as shown in Figure 31. Just from the figure it can be seen that $f(x) \geq g(x)$ on the interval $[0, 1]$, and $f(x) \leq g(x)$ on $[-2, 0] \cup [1, 2]$. (The Intermediate Value Theorem could be employed to demonstrate these facts rigorously.) We now calculate:

$$\begin{aligned}
\mathcal{A}(R) &= \int_{-2}^{2} |f(x) - g(x)|\, dx \\
&= \int_{-2}^{0} |f(x) - g(x)|\, dx + \int_{0}^{1} |f(x) - g(x)|\, dx + \int_{1}^{2} |f(x) - g(x)|\, dx \\
&= \int_{-2}^{0} [g(x) - f(x)]\, dx + \int_{0}^{1} [f(x) - g(x)]\, dx + \int_{1}^{2} [g(x) - f(x)]\, dx \\
&= \int_{-2}^{0} \left(x^2 - 2x + x^3\right) dx + \int_{0}^{1} \left(2x - x^3 - x^2\right) dx + \int_{1}^{2} \left(x^2 - 2x + x^3\right) dx \\
&= \left[\frac{1}{3}x^3 - x^2 + \frac{1}{4}x^4\right]_{-2}^{0} + \left[x^2 - \frac{1}{4}x^4 - \frac{1}{3}x^3\right]_{0}^{1} + \left[\frac{1}{3}x^3 - x^2 + \frac{1}{4}x^4\right]_{1}^{2} \\
&= -\left(-\frac{8}{3} - 4 + 4\right) + \left(1 - \frac{1}{4} - \frac{1}{3}\right) + \left[\left(\frac{8}{3} - 4 + 4\right) - \left(\frac{1}{3} - 1 + \frac{1}{4}\right)\right] = \frac{37}{6}.
\end{aligned}$$

That is, $\mathcal{A}(R) = 6\frac{1}{6}$ square units. ■

Example 6.5. Find the area of the region R bounded by $4x + y^2 = 12$ and $x = y$.

Solution. In this case it is easier to set up both curves as functions of y instead of x: from $x = y$ we obtain the function $f(y) = y$; and from $4x+y^2 = 12$ we obtain the function $g(y) = (12-y^2)/4$.

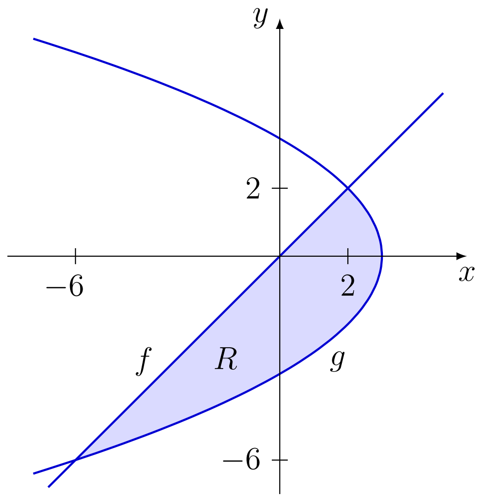


FIGURE 32.

To find where the curves f and g intersect we find all y values for which $f(y) = g(y)$, which means solving the equation $y = (12 - y^2)/4$:

$$y = \frac{12 - y^2}{4} \quad \Rightarrow \quad y^2 + 4y + 12 = 0 \quad \Rightarrow \quad (y+6)(y-2) = 0 \quad \Rightarrow \quad y = -6, 2.$$

Thus the curves intersect at the points $(-6, -6)$ and $(2, 2)$, and the region R appears as in Figure 32, with $g(y) \geq f(y)$ for all $y \in [-6, 2]$. The area is

$$\begin{aligned}
\mathcal{A}(R) &= \int_{-6}^{2} |f(y) - g(y)|\, dy = \int_{-6}^{2} [g(y) - f(y)]\, dy \\
&= \int_{-6}^{2} \left(3 - \frac{1}{4}y^2 - y\right) dy = \left[3y - \frac{1}{12}y^3 - \frac{1}{2}y^2\right]_{-6}^{2} \\
&= \left[3(2) - \frac{1}{12}(2)^3 - \frac{1}{2}(2)^2\right] - \left[3(-6) - \frac{1}{12}(-6)^3 - \frac{1}{2}(-6)^2\right] \\
&= \frac{10}{3} + 18 = \frac{64}{3}.
\end{aligned}$$

■

6.3 – Volumes by Slicing

We wish to find the volume $\mathcal{V}(S)$ of a solid S that exists in a region of space where $-\infty < a \le x \le b < \infty$. Suppose that for each $x \in [a,b]$ the cross sectional area of S is $A(x)$, where $A : [a,b] \to [0,\infty)$ is a continuous function. We use the function A to estimate the volume of S. To start, let $P = \{x_i\}_{i=0}^{\infty}$ be any partition of $[a,b]$, so that $x_0 = a$ and $x_n = b$. Choose a sample point x_i^* in each interval $[x_{i-1}, x_i]$ of the partition. By way of approximation, we assume the cross-sectional area of S is constantly equal to $A(x_i^*)$ for all $x_{i-1} \le x \le x_i$; that is $A(x) \approx A(x_i^*)$ for all $x \in [x_{i-1}, x_i]$. Thus the volume V_i of the "slice" of S that exists in the region of space where $x_{i-1} \le x \le x_i$ we approximate to be $A(x_i^*)\Delta x_i$, where as usual $\Delta x_i = x_i - x_{i-1}$ is the length of the interval $[x_{i-1}, x_i]$. Doing this for all integers $1 \le i \le n$, we estimate the volume of S to be

$$\mathcal{V}(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i. \tag{6.1}$$

We now naturally *define* the *exact* volume of S to be the limiting value that is approached by the sum at right in (6.1) as $\|P\| \to 0$, where $\|P\| = \max_{1 \le i \le n} \Delta x_i$. That is, we define

$$\mathcal{V}(S) = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx.$$

Thus the volume of S is taken to be given by the definite integral $\int_a^b A$, which will exist as a real number since the function A is given to be continuous on $[a,b]$. This is the **Slicing Method** of finding the volume of a solid in space.

Example 6.6. Find the volume of a pyramid with height h and rectangular base with dimensions a and b.

Solution. The pyramid may be oriented so that its apex is at $(0,0)$ and its base at $x = h$, as at left in Figure 33. Most conveniently we consider cross sections of the pyramid that are perpendicular to the x-axis, as they will always be rectangles. In particular we consider a cross section at some $x \in [0,h]$, which will have dimensions a_x and b_x shown at left in Figure 33.

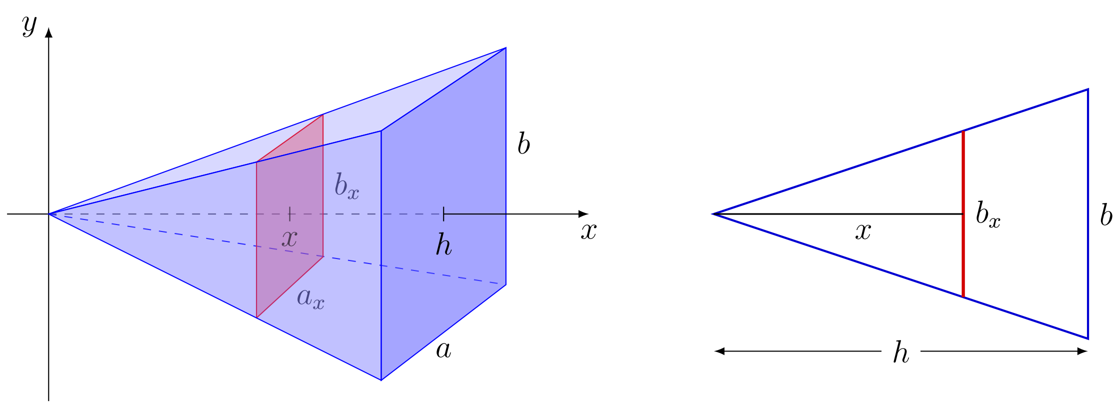


Figure 33.

At right in Figure 33 is the intersection of the situation at left with the xy-plane. By similar triangles

$$\frac{b_x}{x} = \frac{b}{h},$$

and so $b_x = bx/h$. In a similar fashion we find that $a_x = ax/h$, and so the cross section of the pyramid has area

$$A(x) = a_x b_x = \left(\frac{ab}{h^2}\right)x^2.$$

Now we find the volume of the pyramid to be

$$\int_0^h A(x)\,dx = \frac{ab}{h^2}\int_0^h x^2\,dx = \frac{ab}{h^2}\cdot\frac{h^3}{3} = \frac{abh}{3}.$$

In particular a pyramid with height h and a square base with sides of length h has volume $\frac{1}{3}h^3$. ■

6.4 – Lengths of Planar Curves

♪ Somedaaaa-a-a-aaayyyyy, Over the Rainbow.... ♫

6.5 – Physical Applications

Pressure is a scalar quantity defined to be force per unit area, where by "force" we mean here the magnitude of a force vector directed orthogonally down upon some flat surface. The following two examples apply the Riemann integral to answer questions about pressure on a large spherical body such as a moon.

Example 6.7 (**Pressure at Small Depths**)**.** We purpose to determine the pressure P at a depth of x meters below the surface of an airless moon of radius R meters comprised of incompressible material that has uniform density ρ. In this example we assume that x is small compared to R, so that the "sphericalness" of the moon may be safely ignored. The problem is that even though the density is the same throughout the interior of the moon, the gravitational force directed toward the center of the moon will vary with depth. Indeed, at the very center of the moon there will be no gravitational force at all. This does not mean, however, that the pressure at the moon's center will be zero!

To start, we partition the interval $[0, x]$ into n subintervals of equal length x/n. The ith subinterval is

$$I_i = \left[\tfrac{i-1}{n}x, \tfrac{i}{n}x\right],$$

corresponding to a depth between $\frac{i-1}{n}x$ meters and $\frac{i}{n}x$ meters. For $t \in [0, \infty)$, let $g(t)$ be the magnitude of the moon's gravitational force (the "gravity") at a distance of t meters from the surface of the moon. Physics informs us that it is reasonable to assume that g is a continuous function. For each $1 \leq i \leq n$ we approximate the gravity throughout the depth interval I_i to equal $g(ix/n)$, which is in fact the gravity at the bottom end of the depth range from $\frac{i-1}{n}x$ meters to $\frac{i}{n}x$ meters.

Consider a tiny square patch of area A at depth x. Above this square patch is a rectangular column of mass of cross-sectional area A that reaches up to the surface of the moon. For simplicity's sake we will assume for now that the pressure on the square patch is due entirely to the mass within this column. The column is broken into shorter columns of height x/n in accordance with our partition of the interval $[0, x]$. Each of the smaller columns is a rectangular box having volume Ax/n, and so contains a mass of $\rho Ax/n$. In the ith box B_i the force of gravity is assumed to be a constant $g(ix/n)$, and so the weight w_i of the mass inside B_i is

$$w_i = \left(\frac{\rho A}{n}x\right) g\left(\frac{i}{n}x\right)$$

by the usual "mass times force" formula. Since the stuff making up the moon is incompressible, the weight w_i of the mass inside box B_i is conveyed all the way down to the tiny square of area A at depth x. Thus the total force on the square patch is the sum of the weights $w_1, \ldots, w_n$:

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \frac{\rho Axg(ix/n)}{n}.$$

This force we divide by the area A to obtain an estimate of $P(x)$, the pressure at depth x:

$$P(x) \approx \frac{1}{A}\sum_{i=1}^{n}\left(\frac{\rho A}{n}x\right)g\left(\frac{i}{n}x\right) = \sum_{i=1}^{n}\frac{\rho x}{n}g\left(\frac{i}{n}x\right).$$

Since ρg is a continuous function, $P(x)$ will approach some limiting value as $n \to \infty$, and it is this limiting value that we take to be the *exact* pressure at depth x:

$$P(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\rho g\left(\frac{i}{n}x\right)\frac{x}{n}.$$

Letting $\Delta x_i = x/n$ and $x_i^* = ix/n$ for each $1 \le i \le n$, we have

$$P(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\rho g(x_i^*)\Delta x_i.$$

By Proposition 5.11 it follows that

$$P(x) = \int_0^x \rho g(t)\,dt.$$

From physics it is known that the force of gravity r meters from the center of the moon is $GM(r)/r^2$, where here $M(r)$ denotes the mass of the moon that lies within a distance r from the center, and G is the gravitational constant. Since at a depth of t meters the center of the moon is $R - t$ meters away, we have

$$g(t) = \frac{GM(R-t)}{(R-t)^2} = \frac{G}{(R-t)^2}\cdot\frac{4}{3}\pi(R-t)^3\rho = \frac{4}{3}\pi(R-t)G\rho.$$

Thus we have

$$P(x) = \frac{4}{3}\pi G\rho^2\int_0^x (R-t)\,dt = \frac{4}{3}\pi G\rho^2\left(Rx - \frac{1}{2}x^2\right) = 4\pi G\rho^2 x\left(\frac{2R-x}{6}\right).$$

In particular the pressure at the center of the moon is

$$P(R) = \frac{2}{3}\pi G\rho^2 R^2,$$

which certainly is not zero.

More realistically the density of a moon increases with increasing depth, in which case a similar analysis will find that

$$P(x) = \int_0^x \rho(t)g(t)\,dt,$$

where the constant ρ is replaced by a function $\rho(t)$. ■

Example 6.8 (Pressure at Great Depths). In the previous example it was assumed that the pressure upon the square patch of area A at depth x was a box-shaped column with square cross-section of constant area A all the way up to the surface of the moon. This is reasonable at small depths, but at great depths we must account for the spherical shape of the moon.

The gravitational field of the moon is radial in nature, which is to say the field lines of the moon's gravity are straight lines that all converge to a single point which we take to be the center of the moon. Thus the "column" of mass that is exerting pressure upon the aforementioned square patch must broaden toward the moon's surface, to form something of a truncated pyramid.

Our partition of the interval $[0, x]$ chops up this truncated pyramid into smaller chunks that are themselves truncated pyramids. We must calculate the volume of the ith such chunk. ■

7

Transcendental Functions

7.1 – The Inverse Function Theorem

Some functions are capable of assuming the same value at different points in their domain. For instance the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ takes on the value 9 when x is 3 or -3. A function that cannot do this is called one-to-one.

Definition 7.1. *A function $f : X \to Y$ is **one-to-one** if $f(x_1) \neq f(x_2)$ for every $x_1, x_2 \in X$ with $x_1 \neq x_2$.*

An equivalent definition states that f is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Note that while $f(x) = x^2$ is not one-to-one because $f(-3) = 9 = f(3)$, the function $g(x) = x^3$ is one-to-one. We could also obtain a one-to-one function from $f(x) = x^2$ by *restricting its domain* to the nonnegative real numbers (thereby deleting -3 from the domain, for instance); that is, $f(x) = x^2$ is one-to-one if we assume $f : [0, \infty) \to \mathbb{R}$.

Definition 7.2. *A function $f : X \to Y$ is **onto** if, for each $y \in Y$, there exists some $x \in X$ such that $f(x) = y$. In other words $f : X \to Y$ is onto if and only if $\mathrm{Ran}(f) = Y$.*

If a function $f : X \to Y$ is not onto, we can obtain an onto function by *restricting its codomain* to $\mathrm{Ran}(f)$. For example $f(x) = x^2$ is onto if we assume $f : \mathbb{R} \to [0, \infty)$, and it is both one-to-one and onto if we assume $f : [0, \infty) \to [0, \infty)$.

Remark. *By convention we will henceforth always take the codomain of a function to be equal to its range!* Since this will result in all functions being automatically onto, we will make no further reference to the notion.

Theorem 7.3. *If f is a one-to-one function, then there exists a unique function g such that $\mathrm{Dom}(g) = \mathrm{Ran}(f)$, and $f(g(x)) = x$ for all $x \in \mathrm{Dom}(g)$.*

Proof. Suppose f is a one-to-one function. For the existence part of the proof, define the function $g : \mathrm{Ran}(f) \to \mathrm{Dom}(f)$ as follows: for each $x \in \mathrm{Ran}(f)$ let $g(x)$ equal the unique y

value for which $f(y) = x$. Clearly $\mathrm{Dom}(g) = \mathrm{Ran}(f)$, and also

$$f(g(x)) = f(y) = x$$

for all $x \in \mathrm{Ran}(f)$.

For the uniqueness part of the proof, suppose that h is a function such that $\mathrm{Dom}(h) = \mathrm{Ran}(f)$ and $f(h(x)) = x$ for all $x \in \mathrm{Ran}(f)$. Now, $\mathrm{Ran}(f) = \mathrm{Dom}(g)$ immediately implies that $\mathrm{Dom}(h) = \mathrm{Dom}(g)$. Moreover, for every x in the common domain we have

$$f(h(x)) = x = f(g(x)),$$

and since f is one-to-one we obtain $h(x) = g(x)$. Therefore $h = g$. ■

Definition 7.4. *Let f be a one-to-one function. The **inverse** of f, denoted by f^{-1}, is the unique function for which* $\mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)$, *and* $f(f^{-1}(x)) = x$ *for all* $x \in \mathrm{Ran}(f)$.

Theorem 7.5. *If f^{-1} is the inverse of f, then* $\mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)$ *and* $\mathrm{Ran}(f^{-1}) = \mathrm{Dom}(f)$.

Proof. We already have $\mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)$ by Definition 7.4. Let $y \in \mathrm{Ran}(f^{-1})$. Then there exists some $x \in \mathrm{Dom}(f^{-1})$ such that $f^{-1}(x) = y$, and since $\mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)$ it follows that $x \in \mathrm{Ran}(f)$; that is, there exists some $z \in \mathrm{Dom}(f)$ such that $f(z) = x$. Now, Definition 7.4 implies that $f(f^{-1}(x)) = x$, and since

$$f(z) = x = f(f^{-1}(x)) = f(y)$$

and f is one-to-one, we obtain $y = z \in \mathrm{Dom}(f)$. This verifies $\mathrm{Ran}(f^{-1}) \subseteq \mathrm{Dom}(f)$.

Next, let $y \in \mathrm{Dom}(f)$. Setting $x = f(y)$, we have $x \in \mathrm{Ran}(f) = \mathrm{Dom}(f^{-1})$ and so $f(f^{-1}(x)) = x$. Since f is one-to-one and

$$f(f^{-1}(x)) = x = f(y),$$

it follows that $y = f^{-1}(x)$ and thus $y \in \mathrm{Ran}(f^{-1})$. Hence $\mathrm{Dom}(f) \subseteq \mathrm{Ran}(f^{-1})$. ■

Theorem 7.6. *If f^{-1} is the inverse of f, then $f(x) = y$ if and only if $f^{-1}(y) = x$.*

Proof. Suppose that $f(x) = y$. This implies $x \in \mathrm{Dom}(f)$ and $y \in \mathrm{Ran}(f)$, with the latter (together with Definition 7.4) in turn implying $f(f^{-1}(y)) = y$. Since f is one-to-one (this is understood because it has an inverse) and $f(f^{-1}(y)) = y = f(x)$, we conclude that $f^{-1}(y) = x$.

Now suppose that $f^{-1}(y) = x$. Then $y \in \mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)$, whence we obtain $f(f^{-1}(y)) = y$ and therefore $f(x) = y$. ■

The theorem above is often notationally distilled down to

$$f(x) = y \Leftrightarrow f^{-1}(y) = x.$$

An immediate consequence of Theorems 7.5 and 7.6 is that $f^{-1}(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$ and $f(f^{-1}(y)) = y$ for all $y \in \mathrm{Dom}(f^{-1})$, which is commonly how the inverse of a function is defined in textbooks.

Proposition 7.7. *If f is a one-to-one function that is continuous on an interval I, then f is either an increasing or decreasing function on I.*

Lemma 7.8. *If f is a one-to-one function that is continuous on an open interval I, then $f(I)$ is open.*

Proof. By Proposition 7.7, f is either increasing or decreasing on I. For the sake of argument assume that f is increasing (the proof is similar if it is decreasing). Fix $b \in f(I)$. Then there exists some $a \in I$ such that $f(a) = b$. Since I is open there is some $\delta > 0$ such that $[a - \delta, a + \delta] \subseteq I$, and thus $b \in f((a - \delta, a + \delta))$.

Suppose $y \in f((a - \delta, a + \delta))$. Then there is some $x \in (a - \delta, a + \delta)$ such that $f(x) = y$, and since f is increasing we have

$$f(a - \delta) < y < f(a + \delta).$$

Hence $y \in (f(a - \delta), f(a + \delta))$ and we obtain

$$f((a - \delta, a + \delta)) \subseteq (f(a - \delta), f(a + \delta)).$$

Therefore $b \in (f(a - \delta), f(a + \delta))$, an open interval.

Next, suppose $y \in (f(a - \delta), f(a + \delta))$. By the Intermediate Value Theorem, since f is continuous on $[a - \delta, a + \delta]$, there exists some $x \in (a - \delta, a + \delta) \subseteq I$ such that $f(x) = y$, and thus $y \in f(I)$. Therefore

$$b \in (f(a - \delta), f(a + \delta)) \subseteq f(I),$$

so b is an interior point of $f(I)$. Since $b \in f(I)$ is arbitrary, we conclude that $f(I)$ is an open set. ■

With this proposition and lemma we are now in a position to prove the two most momentous calculus results of this section.

Theorem 7.9. *If f is a one-to-one function that is continuous on an open interval I, then f^{-1} is continuous on $f(I)$.*

Proof. Suppose that f is a one-to-one function that is continuous on open interval I. Let $b \in f(I)$ be arbitrary. Then there exists some $a \in I$ such that $f(a) = b$, and so $a = f^{-1}(b)$.

Let $\epsilon > 0$. Since I is open there exists some $\gamma > 0$ such that $[a - \gamma, a + \gamma] \subseteq I$. Set $\hat{\epsilon} = \min\{\epsilon, \gamma\}$, so that $[a - \hat{\epsilon}, a + \hat{\epsilon}] \subseteq I$, and define

$$\delta_1 = |f(a - \hat{\epsilon}) - f(a)| \quad \text{and} \quad \delta_2 = |f(a + \hat{\epsilon}) - f(a)|.$$

Note that since f is one-to-one on $[a - \hat{\epsilon}, a + \hat{\epsilon}]$ and $a - \hat{\epsilon} < a < a + \hat{\epsilon}$, by Proposition 7.7 we must have either

$$f(a - \hat{\epsilon}) < f(a) < f(a + \hat{\epsilon}) \quad \text{or} \quad f(a - \hat{\epsilon}) > f(a) > f(a + \hat{\epsilon}).$$

Choose $\delta = \min\{\delta_1, \delta_2\}$, and suppose $|y - f(a)| < \delta$. Then

$$|y - f(a)| < |f(a - \hat{\epsilon}) - f(a)| \quad \text{and} \quad |y - f(a)| < |f(a + \hat{\epsilon}) - f(a)|$$

both hold, and since $f(a-\hat{\epsilon})$ and $f(a+\hat{\epsilon})$ lie on opposite sides of $f(a)$, it follows that y lies between $f(a-\hat{\epsilon})$ and $f(a+\hat{\epsilon})$. By the Intermediate Value Theorem there is some $x \in (a-\hat{\epsilon}, a+\hat{\epsilon})$ such that $f(x) = y$. Then $|x-a| < \hat{\epsilon}$ for $x = f^{-1}(y)$, which implies that

$$|f^{-1}(y) - f^{-1}(b)| < \hat{\epsilon} \leq \epsilon$$

and therefore f^{-1} is continuous at b.

Since $b \in f(I)$ is arbitrary it's concluded that f^{-1} is continuous on $f(I)$. ∎

Theorem 7.10 (Inverse Function Theorem). *Let f be a one-to-one function that is differentiable on an open interval I, and let $a \in I$. If $f'(a) \neq 0$, then f^{-1} is differentiable at $f(a)$ with*

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Proof. Suppose $f'(a) \neq 0$. Let $\epsilon > 0$. By Theorem 2.12(5), from

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

we obtain

$$\lim_{x \to a} \frac{x-a}{f(x) - f(a)} = \lim_{x \to a} \frac{1}{\dfrac{f(x) - f(a)}{x - a}} = \frac{1}{f'(a)}.$$

Since a is an interior point of I, there is some $\gamma > 0$ such that $0 < |x - a| < \gamma$ implies that

$$\left| \frac{x-a}{f(x) - f(a)} - \frac{1}{f'(a)} \right| < \epsilon. \tag{7.1}$$

Now, by Theorem 7.9, f^{-1} is continuous on $f(I)$. Moreover, $b = f(a) \in f(I)$ and $f(I)$ is an open set by Lemma 7.8, so there is some $\delta > 0$ such that $(b-\delta, b+\delta) \subseteq f(I)$ and $y \in (b-\delta, b+\delta)$ implies that $|f^{-1}(y) - f^{-1}(b)| < \gamma$.

Let y be such that $0 < |y - b| < \delta$. Then $y \in f(I)$, so there is some $x \in I$ such that $f(x) = y$, and thus $x = f^{-1}(y)$. Also

$$f(x) = y \neq b = f(a),$$

so $x \neq a$ since f is one-to-one, which means $f^{-1}(y) \neq f^{-1}(b)$. We now have

$$0 < |f^{-1}(y) - f^{-1}(b)| < \gamma,$$

or equivalently $0 < |x - a| < \gamma$, and so from (7.1) we obtain

$$\left| \frac{f^{-1}(y) - f^{-1}(b)}{y - b} - \frac{1}{f'(a)} \right| < \epsilon.$$

This shows

$$\lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} = \frac{1}{f'(a)},$$

and since $1/f'(a) \in \mathbb{R}$ we conclude that f^{-1} is differentiable at b and $(f^{-1})'(b) = 1/f'(a)$. ∎

An application of this theorem is the following result, which has been assumed to be true since the early days of Chapter 2.

Proposition 7.11. *$\lim_{x\to c} \sqrt[m]{x} = \sqrt[m]{c}$ for any integer $m > 0$, where $c \in (-\infty, \infty)$ if m is odd and $c \in (0, \infty)$ if m is even.*

Proof. First assume that $m > 0$ is odd. Then $f(x) = x^m$ is a one-to-one and differentiable function on $(-\infty, \infty)$, and the inverse of f is $f^{-1}(x) = \sqrt[m]{x}$. Now, for any $c \in (-\infty, 0) \cup (0, \infty)$ there exists a unique $a \in (-\infty, 0) \cup (0, \infty)$ such that $f(a) = x^m = c$, and since

$$f'(a) = ma^{m-1} \neq 0,$$

Theorem 7.10 implies that f^{-1} is differentiable, and hence continuous, at c. Thus

$$\lim_{x\to c} \sqrt[m]{x} = \lim_{x\to c} f^{-1}(x) = f^{-1}(c) = \sqrt[m]{c}$$

for any $c \neq 0$.

The case when $c = 0$ must be handled separately. In doing so, we need the fact that $\sqrt[m]{x}$ is an increasing function on $(-\infty, 0)$ and $(0, \infty)$, which is easily established using the known one-to-oneness and continuity of $\sqrt[m]{\cdot}$ on these intervals: on $(0, \infty)$ note that

$$\sqrt[m]{1} = 1 < 2 = \sqrt[m]{2^m}$$

for $1 < 2^m$ and invoke Proposition 7.7; on $(-\infty, 0)$ note that

$$\sqrt[m]{-2^m} = -2 < -1 = \sqrt[m]{-1}$$

and again appeal to Proposition 7.7.

We now proceed to show that

$$\lim_{x\to 0} \sqrt[m]{x} = \sqrt[m]{0} = 0.$$

Let $\epsilon > 0$. Choose $\delta = \epsilon^m$. Suppose that $0 < |x| < \epsilon^m$. If $x > 0$ we obtain $0 < x < \epsilon^m$, whence $0 < \sqrt[m]{x} < \epsilon$ implies that $|\sqrt[m]{x}\,| < \epsilon$ as desired. If $x < 0$ we obtain $0 < -x < \epsilon^m$, whence $-\epsilon^m < x < 0$ implies that $-\epsilon < \sqrt[m]{x} < 0$, leading again to $|\sqrt[m]{x}\,| < \epsilon$.

We now have established that $\lim_{x\to c} \sqrt[m]{x} = \sqrt[m]{c}$ for all $c \in \mathbb{R}$ when $m > 0$ is odd. The proof is actually more straightforward when $m > 0$ is assumed to be even, since we then have $c \in (0, \infty)$ and need not consider the case when $c = 0$. ■

Example 7.12. Let f be given by $f(x) = 2x^3 + x - 12$. Find $(f^{-1})'(6)$.

Solution. Clearly f is differentiable on $(-\infty, \infty)$, and since

$$f'(x) = 6x^2 + 1 > 0$$

for all $-\infty < x < \infty$, it follows by the Increasing/Decreasing Test that f is increasing on $(-\infty, \infty)$ and therefore is one-to-one. Now, since

$$f(2) = 2(2)^3 + 2 - 12 = 6$$

and

$$f'(2) = 6(2)^2 + 1 = 25 \neq 0,$$

by Theorem 7.10 we conclude that f^{-1} is differentiable at 6 and

$$(f^{-1})'(6) = \frac{1}{f'(2)} = \frac{1}{25}.$$

Note that it is no easy task to find a general expression for $f^{-1}(x)$ by direct algebraic means! ■

With Proposition 7.11 in hand, we are now in a position to at last prove the statement of Theorem 2.12(7) in §2.3.

Proof of Law (7). By hypothesis we have $\lim_{x\to c} f(x) = L$ for some $c, L \in \mathbb{R}$. Let $m > 0$ be an even integer, and suppose there exists some $\gamma > 0$ such that $f(x) \geq 0$ for all $x \in B'_\gamma(c)$. Then we must have $L \geq 0$. Assume that $L > 0$, so L is a point in the interior of $\text{Dom}(\sqrt[m]{\cdot}\,) = [0, \infty)$. By Proposition 7.11 $\sqrt[m]{\cdot}$ is continuous on $(0, \infty)$, and so by Proposition 2.41

$$\lim_{x\to c} \sqrt[m]{f(x)} = \sqrt[m]{\lim_{x\to c} f(x)} = \sqrt[m]{L}, \quad \text{if } L > 0 \tag{7.2}$$

The case when $L = 0$ must be handled separately, since 0 is not an interior point of the domain of $\sqrt[m]{\cdot}$ and so Proposition 2.41 cannot be applied. Thus, suppose that $\lim_{x\to c} f(x) = 0$. Let $\epsilon > 0$. Then there exists some $\beta > 0$ such that

$$0 < |x - c| < \beta \ \text{ implies } \ |f(x)| < \epsilon^m.$$

Choose $\delta = \min\{\beta, \gamma\}$, and suppose $0 < |x - c| < \delta$. Then $f(x) \geq 0$ and $|f(x)| < \epsilon^m$ together yield $0 \leq f(x) < \epsilon^m$, whence we obtain $0 \leq \sqrt[m]{f(x)} < \epsilon$ since $\sqrt[m]{\cdot}$ is an increasing function, and thus

$$\left|\sqrt[m]{f(x)}\right| < \epsilon.$$

We have shown that

$$\lim_{x\to c} \sqrt[m]{f(x)} = 0 = \sqrt[m]{0} = \sqrt[m]{\lim_{x\to c} f(x)},$$

which together with (7.2) proves Law (7) under all possible circumstances when m is even.

The proof is actually easier when $m > 0$ is assumed to be odd, since then $\text{Dom}(\sqrt[m]{\cdot}\,) = (-\infty, \infty)$ and so Proposition 2.41 applies for any $L \in \mathbb{R}$. ■

Example 7.13. Show that the function

$$f(x) = \frac{3x - 1}{2x + 5}$$

is one-to-one, and find its inverse f^{-1}. Also state the domain and range of both f and f^{-1}.

Solution. Suppose that $f(a) = f(b)$. From this we obtain

$$\frac{3a - 1}{2a + 5} = \frac{3b - 1}{2b + 5},$$

and hence

$$(3a - 1)(2b + 5) = (3b - 1)(2a + 5).$$

Multiplying then gives

$$6ab + 15a - 2b - 5 = 6ab + 15b - 2a - 5$$

$$15a - 2b = 15b - 2a$$

$$17a = 17b$$

$$a = b,$$

which shows that f is one-to-one.

Now, let $y = f(x)$, so that $y = (3x-1)/(2x+5)$. We solve this for x,

$$y = \frac{3x-1}{2x+5} \quad \Rightarrow \quad 2xy + 5y = 3x - 1 \quad \Rightarrow \quad 3x - 2xy = 5y + 1$$
$$\Rightarrow \quad x(3-2y) = 5y + 1 \quad \Rightarrow \quad x = \frac{5y+1}{3-2y}.$$

According to Theorem 7.6 $x = f^{-1}(y)$, so we obtain

$$f^{-1}(y) = \frac{5y+1}{3-2y},$$

which can be written $f^{-1}(x) = (5x+1)/(3-2x)$.

Finally,

$$\mathrm{Dom}(f) = \mathrm{Ran}(f^{-1}) = \{x \mid x \neq -5/2\} = (-\infty, -5/2) \cup (-5/2, \infty)$$

and

$$\mathrm{Ran}(f) = \mathrm{Dom}(f^{-1}) = \{x \mid x \neq 3/2\} = (-\infty, 3/2) \cup (3/2, \infty).$$

■

Example 7.14. Find all the inverses associated with $f(x) = (x-4)^2$, and state their domains and ranges.

Solution. Let f_1 be the restriction of f to the interval $[4, \infty)$. That is, $f_1(x) = f(x)$ for $x \geq 4$. Then f_1 is a one-to-one function and thus has an inverse f_1^{-1}. To find f_1^{-1} set $y = f_1(x)$, so that $y = (x-4)^2$ for $x \geq 4$. Then

$$\sqrt{y} = |x - 4| = x - 4,$$

whence $x = 4 + \sqrt{y}$. Since $y = f_1(x)$ if and only if $x = f_1^{-1}(y)$ by Theorem 7.6, we obtain $f_1^{-1}(y) = 4 + \sqrt{y}$.

Next, let f_2 be the restriction of f to the interval $(-\infty, 4]$. That is, $f_2(x) = f(x)$ for $x \leq 4$. Then f_2 is a one-to-one function and has an inverse f_2^{-1}. To find f_2^{-1} set $y = f_2(x)$, so that $y = (x-4)^2$ for $x \leq 4$. Then

$$\sqrt{y} = |x - 4| = -(x-4) = 4 - x,$$

whence $x = 4 - \sqrt{y}$. Since $y = f_2(x)$ if and only if $x = f_2^{-1}(y)$ by Theorem 7.6, we obtain $f_2^{-1}(y) = 4 - \sqrt{y}$.

We have now found that there are two (local) inverses associated with f: the function f_1^{-1} given by

$$f_1^{-1}(y) = 4 + \sqrt{y}$$

with $\mathrm{Dom}(f_1^{-1}) = \mathrm{Ran}(f_1) = [0, \infty)$ and $\mathrm{Ran}(f_1^{-1}) = \mathrm{Dom}(f_1) = [4, \infty)$, and f_2^{-1} given by

$$f_2^{-1}(y) = 4 - \sqrt{y}.$$

with $\mathrm{Dom}(f_2^{-1}) = \mathrm{Ran}(f_2) = [0, \infty)$ and $\mathrm{Ran}(f_2^{-1}) = \mathrm{Dom}(f_2) = (-\infty, 4]$. ■

7.2 – The Natural Logarithm and Exponential Functions

In what follows it is important to recall a few properties of the Riemann integral that were established back in Chapter 5. Let $[a, b]$ be a closed interval (so we assume $a < b$), and let f and g be integrable functions on $[a, b]$.

- For any $c \in [a, b]$, $\int_c^c f(x)dx = 0$
- If $f(x) > 0$ for all $x \in [a, b]$, then$\int_a^b f(x)\,dx > 0$
- If $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Definition 7.15. *The* ***natural logarithm function*** $\ln : (0, \infty) \to \mathbb{R}$ *is given by*

$$\ln(x) = \int_1^x \frac{1}{t}\,dt,$$

for all $x > 0$.

By the Fundamental Theorem of Calculus we have, for each $x > 0$,

$$\ln'(x) = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}. \tag{7.3}$$

Thus the natural logarithm is differentiable on $(0, \infty)$, which immediately implies that it is also continuous on $(0, \infty)$. By the Chain Rule we find that, for any function u differentiable at x with $u(x) \in (0, \infty)$,

$$(\ln \circ u)'(x) = [\ln(u(x))]' = \ln'(u(x)) \cdot u'(x) = \frac{1}{u(x)} \cdot u'(x) = \frac{u'(x)}{u(x)}, \tag{7.4}$$

which in general can be written as $(\ln \circ u)' = u'/u$. This formula enables us to prove the following momentous theorem.

Theorem 7.16. *For all* $a, b > 0$, $\ln(ab) = \ln(a) + \ln(b)$.

Proof. For any $a > 0$ we have by (7.4)

$$[\ln(ax)]' = \frac{1}{ax} \cdot (ax)' = \frac{1}{ax} \cdot a = \frac{1}{x} = \ln'(x).$$

Since $\ln(x)$ and $\ln(ax)$ have the same derivative they must differ by some constant c:

$$\ln(ax) = \ln(x) + c. \tag{7.5}$$

To determine c put $x = 1$ into (7.5) to get $\ln(a) = \ln(1) + c$. But $\ln(1) = 0$, so in fact $c = \ln(a)$ and (7.5) becomes

$$\ln(ax) = \ln(x) + \ln(a)$$

for any $x > 0$. The proof is finished by setting $x = b$ in this last equation. ■

The property exhibited by Theorem 7.16 is actually the defining property of a "logarithmic function." Another wonderful property now follows.

Proposition 7.17. *For all* $a > 0$ *and* $r \in \mathbb{Q}$, $\ln(a^r) = r\ln(a)$.

Proof. Let $r \in \mathbb{Q}$. By (7.4) we obtain, for any $x > 0$,

$$[\ln(x^r)]' = \frac{1}{x^r} \cdot (x^r)' = \frac{1}{x^r} \cdot rx^{r-1} = r \cdot \frac{1}{x} = r \cdot \ln'(x) = [r\ln(x)]',$$

where the second equality follows from the Power Rule for Differentiation as established in Calculus 1. Thus $\ln(x^r)$ and $r\ln(x)$ have the same derivative and must therefore differ only by a constant c:

$$\ln(x^r) = r\ln(x) + c. \tag{7.6}$$

To find c put $x = 1$ into (7.6) to get

$$r\ln(1) + c = \ln(1^r).$$

Since $\ln(1^r) = \ln(1) = 0$ we discover that $c = 0$. Putting this into (7.6) along with $x = a$ then yields $\ln(a^r) = r\ln(a)$ as desired. ■

We would like to extend the property in Proposition 7.17 so that it is applicable when the exponent r is *any real number*, which means entertaining the notion of an exponent that is an irrational number. But right now irrational exponents have no meaning for us. Fantastical beasties like 2^π and $5^{\sqrt{2}}$ are so far just that: fantasies. Don't lose heart, though. The developments of the next few pages will ultimately lead us to a working definition for an irrational exponent.

It's given that the domain of the natural logarithm is $(0, \infty)$, but what is its range?

Theorem 7.18. *The range of the natural logarithm is* $\mathbb{R}$

Proof. Let $M > 0$ be any real number. The functions $f(t) = 1/t$ and $g(t) = 1/2$ are integrable on $[1, 2]$ with $f(t) \geq g(t)$ for all $t \in [1, 2]$, so by the third property of definite integrals given at the beginning of this section we obtain

$$\ln(2) = \int_1^2 \frac{1}{t}\,dt \geq \int_1^2 \frac{1}{2}\,dt = \frac{1}{2}.$$

Let N be an integer such that $N \geq 2(M + 1)$. Now, employing Proposition 7.17,

$$\ln\left(2^N\right) = N\ln(2) \geq 2(M + 1) \cdot \frac{1}{2} = M + 1 > M.$$

Since $\ln(1) = 0 < M$, $\ln\left(2^N\right) > M$, and the natural logarithm function is continuous on $(0, \infty)$, the Intermediate Value Theorem implies that there exists some $c_1 \in \left(1, 2^N\right)$ such that $\ln(c_1) = M$. Thus $\mathrm{Ran}(\ln)$ includes all positive real numbers (along with 0).

The observation that

$$\ln\left(2^{-N}\right) = -N\ln(2) < -M$$

and $\ln(1) = 0 > -M$ enables us to invoke the Intermediate Value Theorem to assert that $\ln(c_2) = -M$ for some $c_2 \in \left(2^{-N}, 1\right)$. Thus $\mathrm{Ran}(\ln)$ includes all negative real numbers.

Therefore $\mathrm{Ran}(\ln) = (-\infty, \infty)$. ■

Notice that since $\ln'(x) = 1/x > 0$ for all $x \in (0, \infty)$, the natural logarithm must be strictly increasing on its domain $(0, \infty)$ and so is one-to-one. Therefore, since $\operatorname{Ran}(f) = \mathbb{R}$ there must be some *unique* positive real number, universally denoted by e, for which $\ln(e) = 1$. Then by Proposition 7.17 we have

$$\ln(e^r) = r\ln(e) = r, \quad \text{for all } r \in \mathbb{Q}.$$

The following definition extends this formulation to the case when r is irrational, again exploiting the fact that ln is one-to-one with range $\mathbb{R}$.

Definition 7.19. *If z is irrational, then e^z is the unique real number for which* $\ln(e^z) = z$.

Therefore we have

$$\ln(e^x) = x \tag{7.7}$$

for all $x \in \mathbb{R}$. What we've now done is given meaning to e^x for *any* real number x, which is our first step toward giving meaning to irrational exponents in general. Notice that for any $x \in \mathbb{R}$ we must have $e^x > 0$, otherwise $\ln(e^x)$ would be undefined.

Definition 7.20. *The* ***exponential function*** $\exp : \mathbb{R} \to (0, \infty)$ *is given by*

$$\exp(x) = e^x$$

for all $x \in \mathbb{R}$.

Since $\operatorname{Dom}(\exp) = \mathbb{R} = \operatorname{Ran}(\ln)$ and

$$\ln(\exp(x)) = \ln(e^x) = x$$

for all $x \in \operatorname{Ran}(\ln)$, it follows from Definition 7.4 that the *exponential function* exp *is the inverse of the natural logarithm function* ln. Of course, we should expect ln to have an inverse since it is one-to-one, and since ln must be the inverse of exp we have

$$\exp(\ln(x)) = e^{\ln(x)} = x \tag{7.8}$$

for all $x \in \operatorname{Ran}(\exp) = (0, \infty)$. Doubters could verify this directly: for any $x > 0$, since $\ln(e^{\ln(x)}) = \ln(x)$ holds as a consequence of (7.7), and since ln is one-to-one, it can only be concluded that $e^{\ln(x)} = x$.

Remark. Since $\ln(x)$ is a continuous one-to-one function, $\exp(x)$ is its inverse, and $\ln((0, \infty)) = \mathbb{R}$, we find from Theorem 7.9 that $\exp(x)$ is continuous on $\mathbb{R}$.

For any $r \in \mathbb{Q}$ and $a > 0$, observe that $\ln(a^r) = r\ln(a)$ by Proposition 7.17, and $\ln(e^{r\ln(a)}) = r\ln(a)$ by (7.7). Hence $a^r = e^{r\ln(a)}$ follows by once again exploiting the one-to-oneness of the natural logarithm. This leads us at last to a natural interpretation of the expression a^z for $a > 0$ and z an *irrational* number.

Definition 7.21. *For all $a > 0$ and z irrational, $a^z = e^{z\ln(a)}$.*

Equipped with this definition and the lovely result of the previous paragraph, we obtain

$$a^x = e^{x\ln(a)}, \quad \text{for all } a > 0 \text{ and } x \in \mathbb{R}. \tag{7.9}$$

Taking the natural logarithm of both sides of (7.9) gives $\ln(a^x) = \ln(e^{x\ln(a)}) = x\ln(a)$, which delivers to us a result that improves on Proposition 7.17.

Theorem 7.22. *For all $a > 0$ and $x \in \mathbb{R}$, $\ln(a^x) = x\ln(a)$.*

We are almost in a position to prove a generalized power rule for differentiation, but first we need one important property of the exponential function which is so wonderful it deserves to be enshrined as a theorem by itself.

Theorem 7.23. *For all $x \in \mathbb{R}$, $\exp'(x) = \exp(x)$.*

Proof. For each $b \in \mathbb{R}$ there exists some $a \in (0, \infty)$ such that $\ln(a) = b$, and since $\ln'(a) = 1/a \neq 0$ it follows from Theorem 7.10 that exp is differentiable at b. Hence exp is differentiable on $\mathbb{R}$, and since ln is differentiable on $(0, \infty)$ (which is the range of exp), the Chain Rule implies that $\ln \circ \exp$ is differentiable on $\mathbb{R}$ and

$$(\ln \circ \exp)'(x) = \ln'(\exp(x)) \exp'(x).$$

Treating the left-hand side we obtain

$$(\ln \circ \exp)'(x) = [\ln(\exp(x))]' = (x)' = 1$$

for all $x \in \mathbb{R}$. As for the right-hand side, (7.3) implies

$$\ln'(\exp(x)) \exp'(x) = \frac{1}{\exp(x)} \cdot \exp'(x) = \frac{\exp'(x)}{\exp(x)}.$$

Equating the two results yields

$$\frac{\exp'(x)}{\exp(x)} = 1$$

for all $x \in \mathbb{R}$, and hence $\exp'(x) = \exp(x)$. ∎

What Theorem 7.23 is saying is that the exponential function is its own derivative: $(e^x)' = e^x$. Are there any other functions that have this property?

As a result of this theorem the Chain Rule gives, for differentiable function u,

$$\left[e^{u(x)}\right]' = (\exp \circ u)'(x) = \exp'(u(x)) \cdot u'(x) = \exp(u(x)) \cdot u'(x) = u'(x)e^{u(x)}. \tag{7.10}$$

Now, at last, we are in a position to obtain the most general power rule of differentiation that we will ever have need of throughout the remainder of our study of calculus.

Theorem 7.24. *For all $x > 0$ and $r \in \mathbb{R}$, $(x^r)' = rx^{r-1}$.*

Proof. Making use of (7.9), as well as (7.10) with $u(x) = r\ln(x)$, we get

$$(x^r)' = \left[e^{r\ln(x)}\right]' = e^{r\ln(x)} \cdot [r\ln(x)]' = e^{r\ln(x)} \cdot \frac{r}{x} = x^r \cdot \frac{r}{x} = rx^{r-1}$$

in one go. ∎

Proposition 7.25. *For* $x \in (-\infty, 0) \cup (0, \infty)$,

$$\int \frac{1}{x}\, dx = \ln|x| + c.$$

Proof. Let $F(x) = \ln|x|$. If $x \in (0, \infty)$, then $F(x) = \ln(x)$ and so

$$F'(x) = \frac{1}{x}$$

by (7.3). If $x \in (-\infty, 0)$, then $F(x) = \ln(-x)$ and so

$$F'(x) = \frac{(-x)'}{-x} = \frac{-1}{-x} = \frac{1}{x}$$

by (7.4). Thus F is an antiderivative for $1/x$ on $(-\infty, 0) \cup (0, \infty)$, and therefore the family of functions of the form $\ln|x| + c$, where $c \in \mathbb{R}$, constitutes all of the antiderivatives of $1/x$ on $(-\infty, 0) \cup (0, \infty)$. ■

The proof of the proposition (or the proposition itself) tells us immediately that

$$\ln'|x| = \frac{1}{x},$$

which is a formula that will be useful in later applications.

Example 7.26. For

$$f(x) = \frac{x^2+1}{x^3+3x+1},$$

determine $\int f(x)dx$, and use the result to evaluate $\int_{-2}^{-1} f(x)dx$.

Solution. Let $u = x^3 + 3x + 1$, so that, by the u-substitution procedure, we obtain

$$du = (3x^2+3)\, dx = 3(x^2+1)\, dx$$

and hence $(x^2+1)\, dx = \frac{1}{3}\, du$. Now,

$$\begin{aligned}\int f(x)dx &= \int \frac{x^2+1}{x^3+3x+1}\, dx = \int \frac{1/3}{u}\, du = \frac{1}{3}\int \frac{1}{u}\, du = \frac{1}{3}\ln|u| + c \\ &= \frac{1}{3}\ln|x^3+3x+1| + c \qquad (7.11)\end{aligned}$$

Now, on the interval $[-2, -1]$ we find that $x^3 + 3x + 1 < 0$, and so from (7.11) we conclude that an antiderivative for f on $[-2, -1]$ is

$$F(x) = \frac{1}{3}\ln(-x^3 - 3x - 1).$$

Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_{-2}^{-1} f(x)dx &= F(x)\Big|_{-2}^{-1} = F(-1) - F(-2) \\ &= \frac{1}{3}\left[\ln\big(-(-1)^3 - 3(-1) - 1\big) - \ln\big(-(-2)^3 - 3(-2) - 1\big)\right]\end{aligned}$$

$$= \frac{1}{3}\left[\ln(3) - \ln(13)\right] = \frac{1}{3}\ln\!\left(\frac{3}{13}\right),$$

as foreseen by the Emperor. ∎

Example 7.27. Evaluate

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx$$

Solution. Let $u = \cos x$, so by the u-substitution procedure we obtain $du = -\sin x\,dx$ and hence $\sin x\,dx = -du$. When $x = 0$ we have $u = \cos 0 = 1$, and when $x = \pi/2$ we have $u = \cos(\pi/2) = 0$. So,

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx = -\int_1^0 \frac{1}{1+u}\,du = -\left[\,\ln|1+u|\,\right]_1^0 = -\left[\ln(1) - \ln(2)\right] = \ln(2).$$

Note that the substitution $u = 1 + \cos x$ would have worked just as well. ∎

Example 7.28. Consider the function $h(x) = x^{\tan(x)}$. In accordance with our customary definition we take the domain of h to be the set

$$\mathrm{Dom}(h) = \{x \in \mathbb{R} : x^{\tan(x)} \in \mathbb{R}\},$$

which in this case is fairly complicated because when $x < 0$ the expression $x^{\tan(x)}$ will only be real-valued if $\tan(x) = m/n$ such that m is an integer and n is an *odd* integer! Happily the domain for h' is easier to apprehend since $h'(x)$ can only be defined for x in the *interior* of $\mathrm{Dom}(h)$. This disqualifies all $x \le 0$, because while there are some negative values for x for which $h(x)$ is real—like $h(-\pi/4) = -4/\pi$, for instance—these values will not form any kind of interval.

Let

$$S = \mathrm{Dom}(\tan) \cap (0,\infty) = (0,\pi/2) \cup \bigcup_{k=0}^{\infty}\left(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi\right).$$

For any $x_0 \in S$ we find that $\tan(x_0)$ is some real number, and since $x_0 > 0$ it follows that

$$h(x_0) = x_0^{\tan(x_0)} > 0.$$

Indeed, because S is an open set there exists some $\gamma > 0$ such that $I = (x_0 - \gamma, x_0 + \gamma) \subseteq S$, and so $x^{\tan(x)} > 0$ for all $x \in I$. This shows not only that x_0 is in the interior of $\mathrm{Dom}(h)$, but also that $x^{\tan(x)}$ is in the domain of the natural logarithm function for all $x \in I$. Thus we may write

$$h(x) = x^{\tan(x)} = \exp(\ln(x^{\tan(x)})) = \exp(\tan(x)\ln(x)) = e^{\tan(x)\ln(x)}$$

for all $x \in I$ using Equation (7.8) and Theorem 7.22. Then by Equation (7.10) we obtain

$$h'(x) = e^{\tan(x)\ln(x)} \cdot [\tan(x)\ln(x)]' = x^{\tan(x)}\left[\frac{\tan(x)}{x} + \sec^2(x)\ln(x)\right]$$

for all $x \in I$, which shows in particular that $h'(x_0)$ is defined and therefore $x_0 \in \mathrm{Dom}(h')$. Since $x_0 \in S$ is arbitrary we conclude that $S \subseteq \mathrm{Dom}(h')$. It can be shown with a little analysis that $\mathrm{Dom}(h') \subseteq S$, and so at last we have $\mathrm{Dom}(h') = S$. ∎

A significant consequence of this section's developments is the following theorem, which the typical reader will have taken as true without proof since it was introduced in elementary algebra.

Theorem 7.29 (**Laws of Exponents**). *Let $a > 0$ and $x, y \in \mathbb{R}$.*

1. $a^x \cdot a^y = a^{x+y}$
2. $a^x / a^y = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(a \cdot b)^x = a^x \cdot b^x$
5. $(a/b)^x = a^x / b^x$

Proof.
Proof of Part (1). Let $a > 0$. For any $x, y \in \mathbb{R}$ we have $a^x, a^y \in (0, \infty) = \mathrm{Dom}(\ln)$, and so by Theorems 7.16 and 7.22 we have

$$\ln(a^x \cdot a^y) = \ln(a^x) + \ln(a^y) = x\ln(a) + y\ln(a) = (x+y)\ln(a) = \ln(a^{x+y}).$$

Since the natural logarithm function is one-to-one, it immediately follows that

$$a^x \cdot a^y = a^{x+y}$$

as desired.

Proof of Part (2). First observe that, for any $y \in \mathbb{R}$,

$$a^{-y} \cdot a^y = a^{-y+y} = a^0 = 1,$$

where the first equality follows from Part (1) and the last from the definition of the zero exponent. Dividing by a^y then yields

$$a^{-y} = \frac{1}{a^y},$$

as we might expect. Now,

$$\frac{a^x}{a^y} = a^x \cdot \frac{1}{a^y} = a^x \cdot a^{-y} = a^{x+(-y)} = a^{x-y}$$

obtains by employing Part (1) once again.

Proof of Part (3). By Theorem 7.22,

$$\ln\left[(a^x)^y\right] = y\ln(a^x) = y \cdot x\ln(a) = (xy)\ln(a) = \ln(a^{xy}),$$

and therefore

$$(a^x)^y = a^{xy}$$

by the one-to-oneness of the natural logarithm function.

Proof of Part (4). We have

$$\ln(a \cdot b)^x = x\ln(a \cdot b) = x\big[\ln(a) + \ln(b)\big] = x\ln(a) + x\ln(b) = \ln(a^x) + \ln(b^x) = \ln(a^x \cdot b^x),$$

and so

$$(a \cdot b)^x = a^x \cdot b^x$$

by the one-to-oneness of the natural logarithm function.

Proof of Part (5). Left as an exercise. ■

The next example should go a long way toward making clear the great utility of L'Hôpital's Rule and the Fundamental Theorem of Calculus, for they help us to evaluate a limit involving an integral of the form

$$\int_a^b e^{t^2} dt,$$

which cannot be evaluated by any technique we have encountered thus far.

Example 7.30. Evaluate

$$\lim_{x \to \infty} \left(\frac{1}{x} \int_0^x e^{t^2} dt \right).$$

Solution. Letting

$$f(x) = \int_0^x e^{t^2} dt \quad \text{and} \quad g(x) = x,$$

the limit may be written as $\lim_{x \to \infty} f(x)/g(x)$. Since the function $t \mapsto e^{t^2}$ is everywhere continuous, Theorem 5.26 implies the function f is differentiable on $(0, b)$ for each $b > 0$, and thus f is differentiable on $(0, \infty)$. Also g is differentiable on $(0, \infty)$, with $g(x) \neq 0$ for all $x \in (0, \infty)$. Now, by Theorem 5.26 we have $f'(x) = e^{x^2}$, and so

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{e^{x^2}}{1} = +\infty.$$

Since $|g(x)| \to +\infty$, it follows that

$$\lim_{x \to \infty} \left(\frac{1}{x} \int_0^x e^{t^2} dt \right) = +\infty$$

by Theorem 4.35. ■

7.3 – The Natural Logarithm and Exponential Functions

Let $b \in (0,1) \cup (1,\infty)$, and define $f : (-\infty,\infty) \to (0,\infty)$ by

$$f(x) = b^x.$$

By (7.9) we may also write

$$f(x) = e^{x\ln(b)} = \exp(x\ln(b)),$$

and since $\exp(x) > 0$ for all $x \in \mathbb{R}$ (and $\ln(b) \in \mathbb{R}$), it follows that $f(x) > 0$ for all $x \in \mathbb{R}$. That is, $\operatorname{Ran}(f) \subseteq (0,\infty)$.

Observe that $\ln(b) \neq 0$. Indeed, $\ln(b) = 0$ implies that $b = \exp(0) = e^0 = 1$, but we are assuming that $b \neq 1$. Let $y \in (0,\infty)$. Since $\operatorname{Ran}(\exp) = (0,\infty)$ there exists some $x \in \mathbb{R}$ such that $\exp(x) = y$, and then

$$f\left(\frac{x}{\ln(b)}\right) = \exp\left(\frac{x}{\ln(b)} \cdot \ln(b)\right) = \exp(x) = y$$

shows that $y \in \operatorname{Ran}(f)$. Hence $(0,\infty) \subseteq \operatorname{Ran}(f)$ and we conclude that $\operatorname{Ran}(f) = (0,\infty)$.

Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$. Then since $\ln(b) \neq 0$ we obtain $x_1\ln(b) \neq x_2\ln(b)$, and so by the one-to-oneness of $\exp(x)$ we find that

$$f(x_1) = \exp(x_1\ln(b)) \neq \exp(x_2\ln(b)) = f(x_2).$$

That is, $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$, and therefore f is one-to-one and so has an inverse function $f^{-1} : (0,\infty) \to (-\infty,\infty)$.

By (7.10) we have, for any $x \in \mathbb{R}$,

$$f'(x) = [e^{x\ln(b)}]' = (x\ln(b))'e^{x\ln(b)} = \ln(b) \cdot e^{x\ln(b)} = b^x\ln(b), \tag{7.12}$$

and thus f is a differentiable function. Note that $\ln(b) \neq 0$ and $e^{x\ln(b)} \neq 0$ imply $f'(x) \neq 0$.

Let $y \in (0,\infty)$. There exists some $x \in \mathbb{R}$ such that $f(x) = b^x = y$. Since $f'(x) \neq 0$ and f is one-to-one and differentiable, by Theorem 7.10 we find that f^{-1} is differentiable at y and

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{b^x\ln(b)} = \frac{1}{y\ln(b)}. \tag{7.13}$$

Definition 7.31. *Let $b \in (0,1) \cup (1,\infty)$. The **base-b exponential function** is the bijection $\exp_b : (-\infty,\infty) \to (0,\infty)$ given by*

$$\exp_b(x) = b^x$$

*for all $x \in \mathbb{R}$. The inverse of $\exp_b$ is the **base-b logarithmic function**, or **base-b logarithm**, $\log_b : (0,\infty) \to (-\infty,\infty)$.*

The functions f and f^{-1} above are thus $\exp_b$ and $\log_b$, respectively. In light of equations (7.12) and (7.13) we have the following result.

Theorem 7.32. *For all* $x \in \mathbb{R}$*,*
$$\exp_b'(x) = b^x \ln(b);$$
and for all $x \in (0, \infty)$*,*
$$\log_b'(x) = \frac{1}{x \ln(b)}.$$

Example 7.33. For the function h given by $h(x) = 7\log_3(4 - \ln(x^5))$, find $\mathrm{Dom}(h)$, and then find h' and $\mathrm{Dom}(h')$.

Solution. Let $f(x) = 4 - \ln(x^5)$ and $g(x) = 7\log_3(x)$, so that
$$f'(x) = -\frac{5}{x} \quad \text{and} \quad g'(x) = \frac{7}{x \ln(3)},$$
and $h = g \circ f$. Then
$$\mathrm{Dom}(h) = \{x : x \in \mathrm{Dom}(f) \text{ and } f(x) \in \mathrm{Dom}(g)\} = \{x : x > 0 \text{ and } 4 - \ln(x^5) > 0\}.$$
Recalling that $\exp(x)$ is an increasing function, so that $x < y$ if and only if $e^x < e^y$, we have
$$4 - \ln(x^5) > 0 \quad \Leftrightarrow \quad 4 - 5\ln(x) > 0 \quad \Leftrightarrow \quad \ln(x) < 4/5 \quad \Leftrightarrow \quad e^{\ln(x)} < e^{4/5} \quad \Leftrightarrow \quad x < e^{4/5},$$
and so
$$\mathrm{Dom}(h) = \{x : x > 0 \text{ and } 4 - \ln(x^5) > 0\} = \{x : x > 0 \text{ and } x < e^{4/5}\} = (0, e^{4/5}).$$

For any $x \in (0, e^{4/5})$ the function f is differentiable at x, and since $f(x) = 4 - \ln(x^5) > 0$ it follows that g is differentiable at $f(x)$. Therefore by the Chain Rule $h = g \circ f$ is differentiable at x, with
$$\begin{aligned} h'(x) &= (g \circ f)'(x) = g'(f(x))f'(x) = g'(4 - \ln(x^5))f'(x) \\ &= \frac{7}{[4 - \ln(x^5)]\ln(3)} \cdot \left(-\frac{5}{x}\right) = \frac{35}{x\ln(3)[\ln(x^5) - 4]} \end{aligned}$$
and $\mathrm{Dom}(h') = (0, e^{4/5})$. ■

7.4 – Hyperbolic Functions

The **hyperbolic sine** is the function $\sinh : \mathbb{R} \to \mathbb{R}$ given by

$$\sinh(x) = \frac{e^x - e^{-x}}{2},$$

and the **hyperbolic cosine** is the function $\cosh : \mathbb{R} \to \mathbb{R}$ given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Both of these functions are everywhere continuous and differentiable. Their graphs are shown in Figure 34.

Let us take a closer look at sinh. Since $\sinh(0) = 0$, $\sinh(x) \to +\infty$ as $x \to +\infty$, and $\sinh(x) \to -\infty$ as $x \to -\infty$, the Intermediate Value Theorem implies that the range of sinh is $\mathbb{R}$. Now, since $(e^x)' = e^x$ and $(e^{-x})' = -e^{-x}$, for any $x \in \mathbb{R}$ we find that

$$\sinh'(x) = \frac{e^x + e^{-x}}{2} > 0,$$

so by Theorem 4.20 the hyperbolic sine is increasing on $\mathbb{R}$, and hence it is one-to-one on $\mathbb{R}$. This implies sinh has an inverse $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$, recalling that the domain of the inverse of a function f equals the range of f. To determine $\sinh^{-1}$ explicitly, let $y = \sinh(x)$. Now,

$$y = \sinh(x) \quad\Rightarrow\quad 2e^x y = 2e^x \sinh(x) \quad\Rightarrow\quad 2e^x y = e^{2x} - 1 \quad\Rightarrow\quad e^{2x} - 2ye^x - 1 = 0.$$

Let $u = e^x$ to obtain $u^2 - 2yu - 1 = 0$. Complete the square:

$$\begin{aligned} u^2 - 2yu - 1 = 0 \quad&\Rightarrow\quad u^2 - 2yu + y^2 = 1 + y^2 \quad\Rightarrow\quad (u-y)^2 = 1 + y^2 \\ &\Rightarrow\quad |u - y| = \sqrt{1+y^2} \quad\Rightarrow\quad u = y \pm \sqrt{y^2+1} \end{aligned}$$

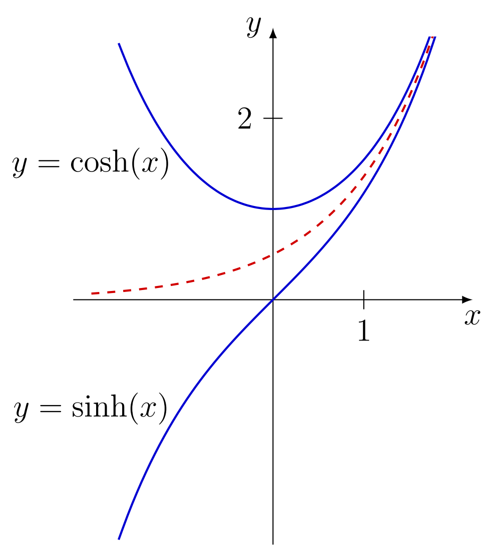


Figure 34. Both $y = \sinh(x)$ and $y = \cosh(x)$ approach $y = e^x/2$ as $x \to +\infty$.

$$\Rightarrow \quad e^x = y \pm \sqrt{y^2+1}.$$

Note that $y - \sqrt{y^2+1}$ is always negative, whereas e^x is never negative. Thus we must have

$$e^x = y + \sqrt{y^2+1},$$

and hence

$$x = \ln\left(y + \sqrt{y^2+1}\right)$$

Since $y = \sinh(x)$ if and only if $\sinh^{-1}(y) = x$, it follows that

$$\sinh^{-1}(y) = \ln\left(y + \sqrt{y^2+1}\right)$$

for all $y \in \mathbb{R}$. We now have an explicit formula for the inverse hyperbolic sine function.

The hyperbolic cosine function, cosh, is not one-to-one on $\mathbb{R}$. In fact we have

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh(x)$$

for any $x \in \mathbb{R}$. The derivative of cosh is

$$\cosh'(x) = \frac{e^x - e^{-x}}{2},$$

so for $x \in (-\infty, 0)$ we have

$$x < 0 \;\Rightarrow\; x < -x \;\Rightarrow\; e^x < e^{-x} \;\Rightarrow\; e^x - e^{-x} < 0 \;\Rightarrow\; \cosh'(x) < 0,$$

and for $x \in (0, \infty)$ we have

$$x > 0 \;\Rightarrow\; x > -x \;\Rightarrow\; e^x > e^{-x} \;\Rightarrow\; e^x - e^{-x} > 0 \;\Rightarrow\; \cosh'(x) > 0.$$

Thus, by Theorem 4.20, cosh is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. In particular cosh is increasing on $[0, \infty)$, and hence is one-to-one there. (If there were to exist some $a > 0$ such that $\cosh(0) \geq \cosh(a)$, then cosh could not be increasing on $(0, a)$—a contradiction.) This means that cosh with domain restricted to the nonnegative real numbers, $\cosh : [0, \infty) \to \mathbb{R}$, *is* one-to-one and so has an inverse $\cosh^{-1}$. The domain of $\cosh^{-1}$ equals the range of cosh, which is $[1, \infty)$ since cosh has a global minimum at 0 and $\cosh(0) = 1$. Now, by the same procedure that found $\sinh^{-1}$, we find that $\cosh^{-1} : [1, \infty) \to \mathbb{R}$ is given by

$$\cosh^{-1}(y) = \ln\left(y + \sqrt{y^2-1}\right)$$

for $y \in [1, \infty)$.

Theorem 7.34. *The functions* $\sinh : \mathbb{R} \to \mathbb{R}$ *and* $\cosh : [0, \infty) \to [1, \infty)$ *are one-to-one and onto, with inverse functions given as follows.*

1. $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ *for all* $x \in \mathbb{R}$.
2. $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$ *for all* $x \in [1, \infty)$.

Theorem 7.35.

1. $(\sinh^{-1})'(x) = \dfrac{1}{\sqrt{x^2+1}}$ *for all* $x \in \mathbb{R}$.
2. $(\cosh^{-1})'(x) = \dfrac{1}{\sqrt{x^2-1}}$ *for all* $x \in (1, \infty)$.

Proof.
Proof of Part (1). Recalling that $\ln'(x) = 1/x$, by Theorem 7.34(1) and the Chain Rule we have

$$\begin{aligned}
(\sinh^{-1})'(x) &= \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}(x^2+1)^{-1/2}(2x)\right] = \frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+(x^2+1)} \\
&= \frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+(x^2+1)} \cdot \frac{x\sqrt{x^2+1}-(x^2+1)}{x\sqrt{x^2+1}-(x^2+1)} \\
&= \frac{x^2\sqrt{x^2+1}-(x^2+1)^{3/2}}{x^2(x^2+1)-(x^2+1)^2} = \frac{x^2\sqrt{x^2+1}-(x^2+1)^{3/2}}{-(x^2+1)} \\
&= \frac{x^2-(x^2+1)}{-\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

for any $x \in \mathbb{R}$. The verification of the second part is accomplished similarly. ■

Example 7.36. Show that

$$\int \frac{1}{\sqrt{ax^2+b}}dx = \frac{1}{\sqrt{a}}\sinh^{-1}\left(\sqrt{\frac{a}{b}}\,x\right) + c$$

for any $a, b > 0$.

Solution. By Theorem 7.35(1) we have

$$\int \frac{1}{\sqrt{x^2+1}}dx = \sinh^{-1}(x) + c.$$

Let $u = (a/b)^{1/2}x$, so $u^2 = (a/b)x^2$ and $dx = (b/a)^{1/2}du$. Now,

$$\begin{aligned}
\int \frac{1}{\sqrt{ax^2+b}}dx &= \frac{1}{\sqrt{b}}\int \frac{1}{\sqrt{(a/b)x^2+1}}dx = \frac{1}{\sqrt{b}}\int \frac{\sqrt{b/a}}{\sqrt{u^2+1}}du = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{u^2+1}}du \\
&= \frac{1}{\sqrt{a}}\sinh^{-1}(u) + c = \frac{1}{\sqrt{a}}\sinh^{-1}\left(\sqrt{\frac{a}{b}}\,x\right) + c.
\end{aligned}$$

■

7.5 – Inverse Trigonometric Functions

The function $f(y) = \sin y$, $y \in [-\pi/2, \pi/2]$ is a one-to-one function, and so has an inverse f^{-1} which has domain $[-1, 1]$. Now, since f is differentiable on $(-\pi/2, \pi/2)$ and for every $y \in (-\pi/2, \pi/2)$ we have $f'(y) = \cos y \neq 0$, Theorem 7.10 implies that f^{-1} is differentiable at $f(y) \in (-1, 1)$. Hence f^{-1} is differentiable on $(-1, 1)$. By Definition 7.4 we obtain

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$$

for all $x \in \operatorname{Ran}(f) = [-1, 1]$, and so by the Chain Rule we obtain, for any $x \in (-1, 1)$,

$$(f \circ f^{-1})'(x) = f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1,$$

and thus

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{7.14}$$

Setting $f^{-1}(x) = y$, we obtain $\sin y = f(y) = x$ and thus

$$\cos y = f'(y) = \sqrt{1 - x^2},$$

or equivalently $f'(f^{-1}(x)) = \sqrt{1 - x^2}$. Putting this result into the equation above then gives

$$(f^{-1})'(x) = \frac{1}{\sqrt{1 - x^2}}$$

The function f^{-1} is of course the inverse sine function from trigonometry, written as either $\sin^{-1}$ or arcsin. This proves the first equation in part (1) of the following theorem.

Theorem 7.37.

1. *For all* $x \in (-1, 1)$,

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \quad \textit{and} \quad \arccos'(x) = -\arcsin'(x).$$

2. *For all* $x \in (-\infty, \infty)$,

$$\arctan'(x) = \frac{1}{1 + x^2} \quad \textit{and} \quad \operatorname{arccot}'(x) = -\arctan'(x).$$

3. *For all* $x \in (-\infty, -1) \cup (1, \infty)$,

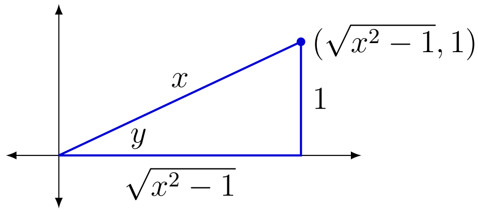
$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2 - 1}} \quad \textit{and} \quad \operatorname{arccsc}'(x) = -\operatorname{arcsec}'(x).$$

We verify the second equation in part (3) of Theorem 7.37. The function $f(y) = \csc y$, $y \in [-\pi/2, 0) \cup (0, \pi/2]$ is one-to-one, and so has an inverse f^{-1} with domain $(-\infty, -1] \cup [1, \infty)$. Recalling that $f'(y) = -\csc y \cot y$ and applying the same arguments that led to (7.14) gives

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

for $x \in (-\infty, -1) \cup (1, \infty)$. Let $y = f^{-1}(x)$ for any $x \in (-\infty, -1) \cup (1, \infty)$, so that $\csc y = f(y) = x$ for $y \in (0, \pi/2) \cup (-\pi/2, 0)$.

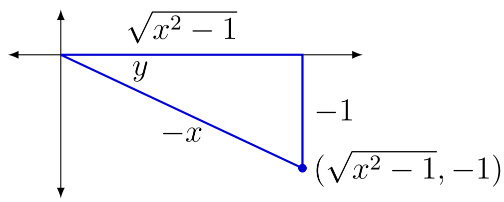
If $y \in (0, \pi/2)$, then $x > 1$ and y can be regarded as an interior angle of a right triangle in Quadrant I as in the figure,



From this it can be seen that $\cot y = \sqrt{x^2-1}$, and so since $|x| = x$ we obtain

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)} = \frac{1}{-\csc y \cot y} = -\frac{1}{x\sqrt{x^2-1}} = -\frac{1}{|x|\sqrt{x^2-1}}.$$

If $y \in (-\pi/2, 0)$, then $x < -1$ and y is an interior angle of a triangle in Quadrant IV,



Now we find that $\cot y = -\sqrt{x^2-1}$, and so since $|x| = -x$ we obtain

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)} = \frac{1}{-\csc y \cot y} = \frac{1}{x\sqrt{x^2-1}} = -\frac{1}{|x|\sqrt{x^2-1}}.$$

By definition f^{-1} is $\csc^{-1}$, which is also written arccsc and so the second half of part (6) is verified.

Example 7.38. If $f(x) = \arccos\left(e^{\sin x}\right)$, then

$$f'(x) = -\frac{1}{\sqrt{1-(e^{\sin x})^2}} \cdot \left(e^{\sin x}\right)' = -\frac{1}{\sqrt{1-e^{2\sin x}}} \cdot e^{\sin x} \cdot (\sin x)' = -\frac{e^{\sin x}\cos x}{\sqrt{1-e^{2\sin x}}}$$

by Theorem 7.37(2) and the Chain Rule. ∎

Example 7.39. If $f(x) = \sin(\sec^{-1}(2x))$, then for $|2x| > 1$ (that is, $|x| > 1/2$) we obtain

$$\begin{aligned} f'(x) &= \cos(\sec^{-1}(2x)) \cdot (\sec^{-1}(2x))' = \cos(\sec^{-1}(2x)) \cdot \frac{1}{|2x|\sqrt{(2x)^2-1}} \cdot (2x)' \\ &= \cos(\sec^{-1}(2x)) \cdot \frac{2}{|2x|\sqrt{(2x)^2-1}} = \cos(\sec^{-1}(2x)) \cdot \frac{1}{|x|\sqrt{(2x)^2-1}}. \end{aligned}$$

However, this can be simplified some more. If we let $\theta = \sec^{-1}(2x)$, then $0 < \theta < \pi$ is such that $\sec\theta = 2x$, and so

$$\cos(\sec^{-1}(2x)) = \cos\theta = 1/2x$$

since $\cos = 1/\sec$. Therefore we have

$$f'(x) = \frac{1}{2x|x|\sqrt{4x^2-1}}$$

for $x \in (-\infty, -1/2] \cup [1/2, \infty)$. ■

An immediate consequence of Theorem 7.37 are the following formulas that allow us to determine various indefinite integrals.

Theorem 7.40.

1. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + c,$ *for all* $a \in (0, \infty)$
2. $\displaystyle\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c,$ *for all* $a \in (-\infty, 0) \cup (0, \infty)$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + c,$ *for all* $a \in (0, \infty)$

Example 7.41. Evaluate

$$\int_0^{5/4} \frac{3}{64x^2+100}dx.$$

Solution. Employ the u-substitution procedure: let $u = 8x$, so that $dx = \frac{1}{8}du$. When $x = 0$ we have $u = 0$; and when $x = 5/4$ we have $u = 10$. Thus,

$$\begin{aligned}\int_0^{5/4} \frac{3}{64x^2+100}dx &= 3\int_0^{10} \frac{1/8}{u^2+10^2}du = \frac{3}{8}\int_0^{10} \frac{1}{10^2+u^2}du = \frac{3}{8}\left[\frac{1}{10}\arctan\left(\frac{u}{10}\right)\right]_0^{10} \\ &= \frac{3}{80}\left[\arctan(1) - \arctan(0)\right] = \frac{3}{80}\left(\frac{\pi}{4} - 0\right) = \frac{3\pi}{320},\end{aligned}$$

using Theorem 7.40(2). ■

Example 7.42. Determine

$$\int \frac{1}{(x+3)\sqrt{x^2+6x}}dx.$$

Solution. Use u-substitution: let $u = x+3$, so $x = u-3$ and $dx = du$, and we obtain

$$\begin{aligned}\int \frac{1}{(x+3)\sqrt{x^2+6x}}dx &= \int \frac{1}{u\sqrt{(u-3)^2+6(u-3)}}du = \int \frac{1}{u\sqrt{u^2-9}}du \\ &= \frac{1}{3}\operatorname{arcsec}\left|\frac{u}{3}\right| + c = \frac{1}{3}\operatorname{arcsec}\left|\frac{x+3}{3}\right| + c,\end{aligned}$$

using Theorem 7.40(3) with $a = 3$. ■

7.6 – L'Hôpital's Rule: Other Indeterminant Forms

Recall the statements of the various theorems known collectively as L'Hôpital's Rule in §4.7, which are used to help evaluate limits that otherwise lead to indeterminate forms of the type $0/0$ or ∞/∞. With some algebraic manipulation we were also able to resolve indeterminate forms such as $0 \cdot \infty$ and $\infty - \infty$.

We now consider indeterminate forms of the type ∞^0, 1^∞ and 0^0. If a limit $\lim_{x\to c} f(x)$ (or a corresponding one-sided limit) leads to ∞^0, 1^∞ or 0^0, the strategy will be to evaluate $\lim_{x\to c} \ln(f(x))$ instead, which will exhibit one of the forms $0/0$ or ∞/∞ and so lend itself to evaluation by L'Hôpital's Rule, and then employ Proposition 2.41 to obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c} \exp\big(\ln(f(x))\big) = \exp\Big(\lim_{x\to c} \ln(f(x))\Big).$$

Example 7.43. Evaluate

$$\lim_{x\to\infty} \left(\frac{1}{5x}\right)^{2/x}.$$

Solution. This limit exhibits the indeterminate form 0^0, so the strategy will be to obtain a limit that lends itself to an application of L'Hôpital's Rule.

For all $x > 0$ we have

$$\left(\frac{1}{5x}\right)^{2/x} = \exp\left[\ln\left(\frac{1}{5x}\right)^{2/x}\right] = \exp\left[\frac{2}{x}\ln\left(\frac{1}{5x}\right)\right] = \exp\left(\frac{-2\ln(5x)}{x}\right).$$

The functions $f(x) = -2\ln(5x)$ and $g(x) = x$ are differentiable on $(0, \infty)$, and $g'(x) = 1 \neq 0$ for all $x \in (0, \infty)$. Since $g(x) \to \infty$ as $x \to \infty$, and

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{-2/x}{1} = 0,$$

by L'Hôpital's Rule (specifically Theorem 4.35) we obtain

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{-2\ln(5x)}{x} = 0$$

as well. Now, since $\exp(x)$ is a continuous function,

$$\lim_{x\to\infty} \left(\frac{1}{5x}\right)^{2/x} = \lim_{x\to\infty} \exp\left(\frac{-2\ln(5x)}{x}\right) = \exp\left(\lim_{x\to\infty} \frac{-2\ln(5x)}{x}\right) = \exp(0) = 1$$

by Proposition 2.41 ■

Recall that for any set $A \subseteq \mathbb{R}$, a point $c \in A$ is said to be in the interior of A if there exists some $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq A$.

Example 7.44. Evaluate

$$\lim_{x\to 0^+} (\cot x)^x.$$

Solution. This limit exhibits the indeterminate form ∞^0. We evaluate

$$\lim_{x\to 0^+} \ln(\cot x)^x = \lim_{x\to 0^+} x\ln(\cot x) = \lim_{x\to 0^+} \frac{\ln(\cot x)}{1/x} \tag{7.15}$$

instead. Certainly $\ln(\cot x)$ and $1/x$ are differentiable on $(0, \pi/2)$, and

$$(1/x)' = -1/x^2 \neq 0$$

for all $x \in (0, \pi/2)$. Moreover we see that $1/x \to \infty$ as $x \to 0^+$. So, we can evaluate (7.15) using L'Hôpital's Rule (specifically Theorem 4.36) if we can evaluate

$$\lim_{x\to 0^+} \frac{(\ln(\cot x))'}{(1/x)'} = \lim_{x\to 0^+} \frac{(1/\cot x)(-\csc^2 x)}{-1/x^2} = \lim_{x\to 0^+} \frac{x^2}{\cos x \sin x} \tag{7.16}$$

But this limit itself gives rise to the indeterminate form $0/0$! So we must attempt to apply L'Hôpital's Rule on (7.16) before we can address (7.15). Clearly x^2 and $\cos x \sin x$ are differentiable on $(0, \pi/4)$ and

$$(\cos x \sin x)' = \cos^2 x - \sin^2 x \neq 0$$

for all $x \in (0, \pi/4)$. Now,

$$\lim_{x\to 0^+} \frac{(x^2)'}{(\cos x \sin x)'} = \lim_{x\to 0^+} \frac{2x}{\cos^2 x - \sin^2 x} = \frac{0}{1-0} = 0,$$

so by Theorem 4.36 the limit (7.16) equals 0, and then by Theorem 4.36 again the limit (7.15) is 0:

$$\lim_{x\to 0^+} \ln(\cot x)^x = 0$$

Now, 0 is in the interior of $\mathrm{Dom}(\exp) = (-\infty, \infty)$, so by (7.8) and Proposition 2.41,

$$\lim_{x\to 0^+} (\cot x)^x = \lim_{x\to 0^+} \exp(\ln(\cot x)^x) = \exp\left(\lim_{x\to 0^+} \ln(\cot x)^x\right) = \exp(0) = e^0 = 1.$$

■

In §7.2 the number e is defined to be that unique real number for which $\ln(e) = 1$. The question naturally arises: How do we determine the value of e? The following provides an answer. Another way of finding e is given in §10.3.

Proposition 7.45.

$$e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$$

Proof. We evaluate the limit

$$\lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + 1/x)}{1/x},$$

which has a $0/0$ indeterminate form and can be treated with L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = \lim_{x\to\infty} \frac{\dfrac{1}{1+1/x}\cdot\left(-\dfrac{1}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+1/x} = \frac{1}{1+0} = 1.$$

Thus we have

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x = 1,$$

and so by (7.8) and Proposition 2.41 it follows that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} \exp\left(\ln\left(1+\frac{1}{x}\right)^x\right) = \exp\left(\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x\right) = \exp(1) = e^1 = e,$$

as was to be shown. ∎

A good approximation to e is

$$e \approx \left(1+\frac{1}{10,000}\right)^{10,000} \approx 2.7181.$$

In actual fact e is an irrational number:

$$e = 2.718281828459045235360287471352662497757247093...$$

Example 7.46. Evaluate

$$\lim_{x\to\infty} (\cosh ax)^{1/bx}$$

for $a > 0$ and $b \neq 0$.

Solution. The limit exhibits an ∞^0 indeterminant form. Applying Theorem 4.35, we obtain

$$\begin{aligned}\lim_{x\to\infty} \ln(\cosh ax)^{1/bx} &= \lim_{x\to\infty} \frac{\ln(\cosh ax)}{bx} = \lim_{x\to\infty} \frac{[\ln(\cosh ax)]'}{(bx)'} = \lim_{x\to\infty} \frac{a\tanh ax}{b}\\ &= \frac{a}{b}\lim_{x\to\infty} \frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}} = \frac{a}{b}\lim_{x\to\infty} \frac{1-e^{-2ax}}{1+e^{-2ax}} = \frac{a}{b}\cdot\frac{1-0}{1+0} = \frac{a}{b}.\end{aligned}$$

Now, by Proposition 2.41,

$$\lim_{x\to\infty} (\cosh ax)^{1/bx} = \lim_{x\to\infty} \exp\left(\ln(\cosh ax)^{1/bx}\right) = \exp\left(\lim_{x\to\infty} \ln(\cosh ax)^{1/bx}\right) = \exp(a/b).$$

That is,

$$\lim_{x\to\infty} (\cosh ax)^{1/bx} = e^{a/b}$$

if $a > 0$ and $b \neq 0$. ∎

Problems

1. Evaluate $\lim_{x\to\infty}(\cosh ax)^{1/bx}$ for $a < 0$ and $b \neq 0$.

2. Evaluate $\lim_{x\to 0^+} x^{\sin x}$.

8

Integration Techniques

8.1 – Integration by Parts

Let I be an interval on which functions u, v are differentiable, and $u'v$ has antiderivative F_0. By the Product Rule of differentiation,

$$(uv)'(x) = u(x)v'(x) + u'(x)v(x) = (uv')(x) + (u'v)(x)$$

for each $x \in I$, and so $uv' = (uv)' - u'v$ on I. It is now clear that uv' has antiderivative $uv - F_0$ on I, and so Proposition 4.45 implies that

$$\begin{aligned}\int uv' &= \{(uv - F_0) + c : c \in \mathbb{R}\} = \{uv - (F_0 + c) : c \in \mathbb{R}\} \\ &= \{uv - F : F \text{ is an antiderivative of } u'v \text{ on } I\} \\ &= uv - \{F : F \text{ is an antiderivative of } u'v \text{ on } I\} = uv - \int u'v.\end{aligned}$$

We have proven the following theorem.

Theorem 8.1 (**Integration by Parts: Indefinite Integrals**). *On an interval I where u, v are differentiable and $u'v$ has an antiderivative,*

$$\int uv' = uv - \int u'v. \tag{8.1}$$

We call equation (8.1) the **integration by parts** formula for indefinite integrals, which can also be written as

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx. \tag{8.2}$$

The integration by parts formula for indefinite integrals extends quite naturally to definite integrals as follows.

Theorem 8.2 (Integration by Parts: Definite Integrals). *For any closed interval $[a,b]$ where u,v are differentiable and $u'v$ has an antiderivative,*

$$\int_a^b uv' = uv\Big|_a^b - \int_a^b u'v.$$

Proof. Let F be an antiderivative for $u'v$ on $[a,b]$. By Theorem 8.1 it follows that $G = uv - F$ is an antiderivative for uv' on $[a,b]$, and so

$$\begin{aligned}\int_a^b uv' &= G(b) - G(a) = (uv - F)(b) - (uv - F)(a)\\ &= [(uv)(b) - (uv)(a)] - [F(b) - F(a)] = uv\Big|_a^b - \int_a^b u'v\end{aligned}$$

by the Fundamental Theorem of Calculus. ■

The conclusion of Theorem 8.2 is often written as

$$\int_a^b u(x)v'(x)\,dx = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\,dx.$$

Some examples will illustrate how the integration by parts technique is put into practice.

Example 8.3. Find

$$\int x\cos 2x\,dx.$$

Solution. Let $u(x) = x$ and $v'(x) = \cos 2x$, so that $u'(x) = 1$ and we can choose $v(x) = \frac{1}{2}\sin 2x$. (In fact any antiderivative for v' will be suited to our purpose, but the one with constant term equal to zero is usually the easiest to work with.) By Theorem 8.1 it follows that

$$\begin{aligned}\int x\cos 2x\,dx &= x\cdot\frac{1}{2}\sin 2x - \int\frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\Big(-\frac{1}{2}\cos 2x + c\Big)\\ &= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x - \frac{c}{2} = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c.\end{aligned}$$

For the last equality we observe that, since c is arbitrary, both $-c/2$ and c can assume any real value, and so replacing the term $-c/2$ with the term c changes nothing. ■

Example 8.4. Find

$$\int \ln(x)\,dx.$$

Solution. Setting $u(x) = \ln(x)$ and $v'(x) = 1$, we obtain $\ln(x) = u(x)v'(x)$. Now, $u'(x) = 1/x$, and a suitable choice for v is $v(x) = x$. (Choosing $v(x) = x + c$ for any constant c will also work for us, but it's natural to simply let $c = 0$.) Now we find that

$$\int \ln(x)\,dx = \ln(x)\cdot x - \int\frac{1}{x}\cdot x\,dx = x\ln(x) - \int dx = x\ln(x) - x + c$$

by Theorem 8.1. ■

Example 8.5. Find

$$\int \tan^{-1}(x)\,dx.$$

Solution. Setting $u(x) = \tan^{-1}(x)$ and $v'(x) = 1$, we obtain $\tan^{-1}(x) = u(x)v'(x)$. Now, $u'(x) = 1/(x^2+1)$, and a suitable choice for v is $v(x) = x$. Employing integration by parts yields

$$\int \tan^{-1}(x)\,dx = \tan^{-1}(x)\cdot x - \int \frac{1}{x^2+1}\cdot x\,dx = x\tan^{-1}(x) - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx. \tag{8.3}$$

Employing u-substitution, we let $u = x^2+1$, so that $du = 2x\,dx$ and we obtain

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}du = \ln|u| + c = \ln|x^2+1| + c = \ln(x^2+1) + c.$$

Putting this result into (8.3) gives

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + c = x\tan^{-1}(x) - \ln\sqrt{x^2+1} + c,$$

where $-\frac{1}{2}c$ is written simply as c since it is an arbitrary constant anyway. ■

The following example illustrates how integration by parts may be used twice in order to obtain an equation that can be solved for an indefinite integral.

Example 8.6. Find

$$\int e^x \sin(x)\,dx.$$

Solution. Let $u(x) = \sin(x)$ and $v'(x) = e^x$, so that $u'(x) = \cos(x)$ and we can choose $v(x) = e^x$ to obtain

$$\int e^x \sin(x)\,dx = e^x\sin(x) - \int e^x\cos(x)\,dx. \tag{8.4}$$

The new integral on the right-hand side of the equation is no better than the one on the left-hand side. Before despairing, however, let us see what happens if we apply integration by parts to this new integral.

Let $u(x) = \cos(x)$ and $v'(x) = e^x$, so $u'(x) = -\sin(x)$ and $v(x) = e^x$, and we obtain

$$\int e^x\cos(x)\,dx = e^x\cos(x) - \int -e^x\sin(x)\,dx = e^x\cos(x) + \int e^x\sin(x)\,dx. \tag{8.5}$$

Substituting (8.5) into (8.4) yields

$$\int e^x\sin(x)\,dx = e^x\sin(x) - \left(e^x\cos(x) + \int e^x\sin(x)\,dx\right)$$

Now the integral we are to determine appears on both sides of the equation, which leaves us to merely solve for it and be done:

$$\int e^x\sin(x)\,dx = \frac{e^x(\sin x - \cos x)}{2} + c.$$

Do not forget to append an arbitrary constant c in the final result. ■

Question: would the approach in the example above have worked equally well if we had chosen $u(x) = e^x$ and $v'(x) = \cos(x)$ in the second integration by parts procedure?

Example 8.7. Evaluate

$$\int_1^4 \sqrt{t}\ln(t)\,dt.$$

Solution. Let $u(t) = \ln(t)$ and $v'(t) = \sqrt{t}$, so that $u'(t) = 1/t$ and we can choose $v(t) = \frac{2}{3}t^{3/2}$. Now,

$$\begin{aligned}
\int_1^4 \sqrt{t}\ln(t)\,dt &= \left[\frac{2}{3}t^{3/2}\ln(t)\right]_1^4 - \int_1^4 \frac{1}{t}\cdot\frac{2t^{3/2}}{3}dt = \frac{2}{3}\left[t^{3/2}\ln(t)\right]_1^4 - \frac{2}{3}\int_1^4 \sqrt{t}\,dt \\
&= \frac{2}{3}\left[4^{3/2}\ln(4) - 1^{3/2}\ln(1)\right] - \frac{2}{3}\left[\frac{2}{3}t^{3/2}\right]_1^4 \\
&= \frac{2}{3}\left[8\ln(4) - 0\right] - \frac{4}{9}\left[4^{3/2} - 1^{3/2}\right] \\
&= \frac{16}{3}\cdot\ln(4) - \frac{4}{9}\cdot 7 = \frac{48\ln(4) - 28}{9},
\end{aligned}$$

by Theorem 8.2. ■

Example 8.8. Let R be the region in the first quadrant bounded by the coordinate axes, the line $x = 1$, and the curve $y = e^{-x}$. Find the volume $\mathcal{V}$ of the solid generated by revolving R about the y-axis.

Solution. By the Shell Method we have

$$\mathcal{V} = \int_0^1 2\pi x e^{-x}\,dx.$$

Let $u(x) = 2\pi x$ and $v'(x) = e^{-x}$, so that $u'(x) = 2\pi$ and $v(x) = -e^{-x}$. By Theorem 8.2,

$$\begin{aligned}
\mathcal{V} &= \left[-2\pi x e^{-x}\right]_0^1 - \int_0^1 -2\pi e^{-x}\,dx = -2\pi e^{-1} + 2\pi\left[-e^{-x}\right]_0^1 \\
&= -2\pi e^{-1} - 2\pi(e^{-1} - 1) = 2\pi - \frac{4\pi}{e}.
\end{aligned}$$

■

8.2 – Trigonometric Integrals

A **trigonometric integral** is an integral whose integrand consists entirely of trigonometric functions, such as those given in the next two theorems.

Theorem 8.9 (Reduction Formulas). *If n is a positive integer, then the following hold.*

1. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

2. $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

3. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx,\ n \neq 1$

4. $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx,\ n \neq 1$

The proofs for these various reduction formulas involve the method of induction. A demonstration of this for the first formula will be supplied at a later date, as will a proof for at least one part of the following theorem.

Theorem 8.10.

1. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$

2. $\displaystyle\int \cot x\,dx = \ln|\sin x| + c$

3. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + c$

4. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + c$

Example 8.11. Determine

$$\int \tan^2\theta\sec\theta\,d\theta$$

Solution. We start with the identity $\tan^2\theta = \sec^2\theta - 1$,

$$\begin{aligned}
\int \tan^2\theta\sec\theta\,d\theta &= \int (\sec^2\theta - 1)\sec\theta\,d\theta = \int \sec^3\,d\theta - \int \sec\theta\,d\theta \\
&= \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta\right) - \int \sec\theta\,d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int \sec\theta\,d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + c,
\end{aligned}$$

using Theorem 8.9(4) in the second line, and Theorem 8.10(3) in the last line. ■

Example 8.12. Determine

$$\int \sin^7(x)\cos^3(x)\,dx$$

Solution. We have

$$\begin{aligned}\int \sin^7(x)\cos^3(x)\,dx &= \int [\sin^2(x)]^3\cos^3(x)\sin(x)\,dx\\ &= \int [1-\cos^2(x)]^3\cos^3(x)\sin(x)\,dx,\end{aligned}$$

and so if we let $u = \cos(x)$, so that $\sin(x)\,dx$ is replaced by $-du$ by the Substitution Rule, we obtain

$$\begin{aligned}\int \sin^7(x)\cos^3(x)\,dx = -\int (1-u^2)^3u^3\,du &= \int (u^3 - 3u^5 + 3u^7 - u^9)\,du\\ &= \frac{1}{4}u^4 - \frac{1}{2}u^6 + \frac{3}{8}u^8 - \frac{1}{10}u^{10} + c\\ &= \frac{1}{4}\cos^4 - \frac{1}{2}\cos^6 + \frac{3}{8}\cos^8 - \frac{1}{10}\cos^{10} + c.\end{aligned}$$

■

8.3 – Trigonometric Substitution

A **trigonometric substitution** is a variable substitution that is made that converts an integral to a trigonometric integral.

Example 8.13. Determine

$$\int \sqrt{121 - x^2}\, dx.$$

Solution. Let $x = 11 \sin\theta$ for $\theta \in [-\pi/2, \pi/2]$, so that dx is replaced with $11\cos\theta\, d\theta$ as part of the substitution. Observe that $-\pi/2 \le \theta \le \pi/2$ implies $\cos\theta \ge 0$, so that

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

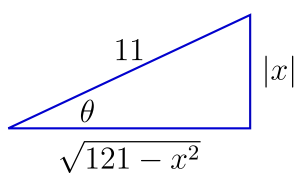
Now,

$$\begin{aligned}\int \sqrt{121 - x^2}\, dx &= \int \sqrt{121 - 121\sin^2\theta} \cdot 11\cos\theta\, d\theta = \int 121\cos\theta\sqrt{1 - \sin^2\theta}\, d\theta \\ &= 121\int \cos\theta\sqrt{\cos^2\theta}\, d\theta = 121\int \cos^2\theta\, d\theta,\end{aligned}$$

and with the deft use of the given formula for $\int \cos^n\theta\, d\theta$ we obtain

$$\int \sqrt{121 - x^2}\, dx = 121\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int (1)\, d\theta\right) = \frac{121}{2}\cos\theta\sin\theta + \frac{121}{2}\theta + c.$$

From $x = 11\sin\theta$ comes $\sin\theta = x/11$, so $\theta = \sin^{-1}(x/11)$ and θ may be characterized as an angle in the right triangle



Note that $x \ge 0$ if $\theta \in [0, \pi/2]$, and $x < 0$ if $\theta \in [-\pi/2, 0)$. From this triangle we see that $\cos\theta = \sqrt{121 - x^2}/11$, and therefore

$$\begin{aligned}\int \sqrt{121 - x^2}\, dx &= \frac{121}{2} \cdot \frac{\sqrt{121 - x^2}}{11} \cdot \frac{x}{11} + \frac{121}{2}\sin^{-1}\left(\frac{x}{11}\right) + c \\ &= \frac{x\sqrt{121 - x^2}}{2} + \frac{121}{2}\sin^{-1}\left(\frac{x}{11}\right) + c.\end{aligned}$$

■

Example 8.14. Determine

$$\int \frac{1}{\sqrt{x^2 - 49}}\, dx, \quad x > 7$$

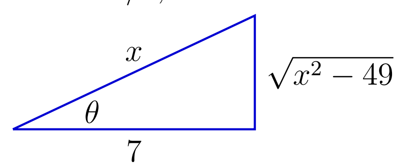
Solution. Let $x = 7\sec\theta$. Then dx is replaced with $7\sec\theta\tan\theta\,d\theta$ as part of the substitution, and we obtain

$$\int \frac{1}{\sqrt{x^2-49}}\,dx = \int \frac{7\sec\theta\tan\theta}{\sqrt{49\sec^2\theta-49}}\,d\theta = \int \frac{7\sec\theta\tan\theta}{7\sqrt{\tan^2\theta}}\,d\theta = \int \frac{\sec\theta\tan\theta}{|\tan\theta|}\,d\theta. \tag{8.6}$$

Now, since we're given $x > 7$, we have $\sec\theta = x/7 > 1$, and so $0 < \theta < \pi/2$ holds. It follows that $\tan\theta > 0$, so then $|\tan\theta| = \tan\theta$ and from (8.6) we obtain

$$\int \frac{1}{\sqrt{x^2-49}}\,dx = \int \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + c.$$

Given that $0 < \theta < \pi/2$ and $\sec\theta = x/7$, we find θ to be an interior angle of the triangle



and so $\tan\theta = \sqrt{x^2-49}/7$ and we obtain

$$\int \frac{1}{\sqrt{x^2-49}}\,dx = \ln\left(\frac{x}{7} + \frac{\sqrt{x^2-49}}{7}\right) + c,$$

where the absolute value signs can be removed since $x > 7$. ■

Other answers are possible in the example above. For instance we can write

$$\begin{aligned}\ln\left(\frac{x}{7} + \frac{\sqrt{x^2-49}}{7}\right) + c &= \ln\left(x + \sqrt{x^2-49}\right) + \ln\left(\frac{1}{7}\right) + c \\ &= \ln\left(x + \sqrt{x^2-49}\right) + c,\end{aligned}$$

where $\ln(1/7)$ is "absorbed" by the arbitrary constant c to yield an arbitrary constant that can just as well be represented by c.

Example 8.15. Determine

$$\int \frac{1}{\sqrt{1-1/x}}\,dx.$$

Solution. First we need $1 - 1/x > 0$, which implies that $x \in (-\infty, 0) \cup (1, \infty)$. Assume that $x > 1$. Now,

$$I := \int \frac{1}{\sqrt{1-1/x}}\,dx = \int \frac{x}{\sqrt{x^2-x}}\,dx = \int \frac{x}{\sqrt{(x-\frac{1}{2})^2-\frac{1}{4}}}\,dx.$$

Letting $u = x - \frac{1}{2}$ (so $u > 1/2$), we obtain

$$I = \int \frac{u+1/2}{\sqrt{u^2-1/4}}\,du = \int \frac{u}{\sqrt{u^2-1/4}}\,du + \frac{1}{2}\int \frac{1}{\sqrt{u^2-1/4}}\,du. \tag{8.7}$$

For the first integral, let $v = u^2 - 1/4$ to get

$$\int \frac{u}{\sqrt{u^2-1/4}}\,du = \frac{1}{2}\int v^{-1/2}\,dv = v^{1/2} = \sqrt{u^2-1/4} = \sqrt{x^2-x} \tag{8.8}$$

(we suppress the arbitrary constant for the time being).

For the second integral make the substitution $u = \frac{1}{2}\sec\theta$. Note that since $u > 1/2$ implies $\sec\theta > 1$, the new variable θ will satisfy $0 < \theta < \pi/2$ and thus

$$\sqrt{u^2 - 1/4} = \sqrt{\frac{1}{4}\sec^2\theta - \frac{1}{4}} = \frac{1}{2}\sqrt{\tan^2\theta} = \frac{1}{2}|\tan\theta| = \frac{1}{2}\tan\theta. \tag{8.9}$$

Now, since $\sec\theta = 2u$ implies $\tan\theta = \sqrt{4u^2 - 1}$,

$$\begin{aligned} \int \frac{1}{\sqrt{u^2 - 1/4}}\,du &= \int \frac{2}{\tan\theta}\cdot\frac{1}{2}\tan\theta\sec\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| \\ &= \ln\left(2u + \sqrt{4u^2 - 1}\right) = \ln\left(2x - 1 + 2\sqrt{x^2 - x}\right) \end{aligned} \tag{8.10}$$

Finally, combining (8.8) and (8.10), we obtain from (8.7)

$$\int \frac{1}{\sqrt{1 - 1/x}}\,dx = \sqrt{x^2 - x} + \frac{1}{2}\ln\left(2x - 1 + 2\sqrt{x^2 - x}\right) + c, \quad \text{if } x > 1.$$

The analysis is similar if we assume that $x < 0$, only (8.9) leads to

$$\sqrt{u^2 - 1/4} = -\frac{1}{2}\tan\theta$$

since $u < -1/2$ implies $\pi/2 < \theta < \pi$; then, noting that $2u + \sqrt{4u^2 - 1} < 0$ for $u < -1/2$, the manipulations in (8.10) take a slightly different tack:

$$\begin{aligned} \int \frac{1}{\sqrt{u^2 - 1/4}}\,du &= \int \frac{-2}{\tan\theta}\cdot\frac{1}{2}\tan\theta\sec\theta\,d\theta = -\int \sec\theta\,d\theta = -\ln|\sec\theta + \tan\theta| \\ &= -\ln\left(-2u - \sqrt{4u^2 - 1}\right) = -\ln\left(1 - 2x - 2\sqrt{x^2 - x}\right). \end{aligned}$$

Therefore

$$\int \frac{1}{\sqrt{1 - 1/x}}\,dx = \begin{cases} \sqrt{x^2 - x} + \ln\sqrt{2x - 1 + 2\sqrt{x^2 - x}} + c, & \text{if } x > 1 \\ \sqrt{x^2 - x} - \ln\sqrt{1 - 2x - 2\sqrt{x^2 - x}} + c, & \text{if } x < 0 \end{cases}$$

which necessarily is piecewise defined. ■

Recalling the definition of an antiderivative given in Chapter 4, what we've found in the example above is that an antiderivative for $f(x) = 1/\sqrt{1 - 1/x}$ on any interval $I \subseteq (1, \infty)$ is given by

$$F_1(x) = \sqrt{x^2 - x} + \ln\sqrt{2x - 1 + 2\sqrt{x^2 - x}} + c$$

for an arbitrarily chosen constant c, which implies that $F_1'(x) = f(x)$ for all $x > 1$. On the other hand, an antiderivative for f on any $I \subseteq (-\infty, 0)$ is given by

$$F_2(x) = \sqrt{x^2 - x} - \ln\sqrt{1 - 2x - 2\sqrt{x^2 - x}} + c,$$

so that $F_2'(x) = f(x)$ for all $x < 0$.

8.4 – Partial Fraction Decomposition

Recall that, formally, a rational function is a function f for which $f(x)$ is a rational expression. Thus, $f(x) = p(x)/q(x)$ for polynomial functions p and q, where q is not the zero function. Given

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

recall that the "degree" of p, denoted by $\deg(p)$, is the highest power of x in the polynomial; thus, $\deg(p) = n$.

Suppose $\deg(p) < \deg(q)$, there are polynomials $q_1(x), \ldots, q_n(x)$ such that each $q_i(x)$ is a factor of $q(x)$ but not all $q_i(x)$ equal $q(x)$, and there are polynomials $p_1(x), \ldots, p_n(x)$ such that $\deg(p_i) \le 1$ for each i and

$$\frac{p(x)}{q(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} + \cdots + \frac{p_n(x)}{q_n(x)}. \tag{8.11}$$

Then the right-hand side of (8.11) is a **partial fraction decomposition** of $p(x)/q(x)$.

Example 8.16. Find a partial fraction decomposition for

$$\frac{7}{2x^2 + 5x - 12}.$$

Solution. First, $2x^2 + 5x - 12 = (2x-3)(x+4)$, so it's reasonable to suppose that there must be fractions $A/(2x-3)$ and $B/(x+4)$ such that

$$\frac{A}{2x-3} + \frac{B}{x+4} = \frac{7}{(2x-3)(x+4)}.$$

for all $x \neq -4, \frac{3}{2}$. What must be done is to determine the appropriate values of A and B. First observe that, for $x \neq -4, \frac{3}{2}$,

$$\begin{aligned}
(2x-3)(x+4) \cdot \left(\frac{A}{2x-3} + \frac{B}{x+4}\right) &= \frac{7}{(2x-3)(x+4)} \cdot (2x-3)(x+4) \\
A(x+4) + B(2x-3) &= 7 \\
(A+2B)x + (4A-3B) &= 0x + 7,
\end{aligned}$$

and so it stands to reason that $A + 2B = 0$ and $4A - 3B = 7$. That is, we have a system of two equations with two unknowns A and B. From $A + 2B = 0$ we obtain $A = -2B$, which we put into $4A - 3B = 7$ to get $4(-2B) - 3B = 7$. Now,

$$4(-2B) - 3B = 7 \quad \Rightarrow \quad -11B = 7 \quad \Rightarrow \quad B = -7/11,$$

and thus $A = -2(-7/11) = 14/11$.

Therefore the partial fraction decomposition we seek is

$$\frac{14/11}{2x-3} + \frac{-7/11}{x+4} = \frac{14}{11(2x-3)} - \frac{7}{11(x+4)},$$

which is easily verified by combining the fractions. ■

The utility of partial fraction decomposition lies in its potential to help determine otherwise intractable integrals.

Example 8.17. Determine

$$\int \frac{7}{2x^2+5x-12}\,dx.$$

Solution. Using the result of the previous example, we obtain

$$\begin{aligned}\int \frac{7}{2x^2+5x-12}\,dx &= \int\left(\frac{14}{11(2x-3)} - \frac{7}{11(x+4)}\right)dx \\ &= \frac{14}{11}\int \frac{1}{2x-3}\,dx - \frac{7}{11}\int \frac{1}{x+4}\,dx \\ &= \frac{7}{11}\ln|2x-3| - \frac{7}{11}\ln|x+4| + c.\end{aligned}$$

■

A general characterization of the partial fraction decomposition methodology is furnished by the following theorem.

Theorem 8.18 (Partial Fraction Decomposition). *Let p and q be polynomial functions such that* $\deg(p) < \deg(q)$, *and suppose q can be factored as a product of polynomials of degree at most* 2. *Then one of the following cases must hold.*

1. *$q(x)$ has form*

$$q_1(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

$a_ix+b_i \neq a_jx+b_j$ whenever $i \neq j$. So $q_1(x)$ is a product of distinct linear factors. Then there exist constants $A_1, A_2, \ldots, A_n$ for which

$$\frac{p(x)}{q_1(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}. \tag{8.12}$$

2. *$q(x)$ has form*

$$q_2(x) = (ax+b)^n$$

for some integer $n \geq 2$. So $q_2(x)$ is a product of repeated linear factors. Then there exist constants $B_1, B_2, \ldots, B_n$ for which

$$\frac{p(x)}{q_2(x)} = \frac{B_1}{ax+b} + \frac{B_2}{(ax+b)^2} + \cdots + \frac{B_n}{(ax+b)^n}. \tag{8.13}$$

3. *$q(x)$ has form*

$$q_3(x) = (a_1x^2+b_1x+c_1)\cdots(a_nx^2+b_nx+c_n),$$

with $b_i^2 - 4a_ic_i < 0$ for each i, and $a_ix^2+b_ix+c_i \neq a_jx^2+b_jx+c_j$ if $i \neq j$. So $q_3(x)$ is a product of distinct irreducible quadratic factors. Then there exist constants $C_1, \ldots, C_n$ and $D_1, \ldots, D_n$ for which

$$\frac{p(x)}{q_3(x)} = \frac{C_1x+D_1}{a_1x^2+b_1x+c_1} + \frac{C_2x+D_2}{a_2x^2+b_2x+c_2} + \cdots + \frac{C_nx+D_n}{a_nx^2+b_nx+c_n}. \tag{8.14}$$

4. *$q(x)$ has form*
$$q_4(x) = (ax^2 + bx + c)^n$$
with $b^2 - 4ac < 0$ and $n \geq 2$. So $q_4(x)$ is a product of repeated irreducible quadratic factors. Then there exist constants $C_1, \ldots, C_n$ and $D_1, \ldots, D_n$ for which
$$\frac{p(x)}{q_4(x)} = \frac{C_1 x + D_1}{ax^2 + bx + c} + \frac{C_2 x + D_2}{(ax^2 + bx + c)^2} + \cdots + \frac{C_n x + D_n}{(ax^2 + bx + c)^n}. \tag{8.15}$$

5. *$q(x)$ has form*
$$q_5(x) = q_1(x)q_2(x)q_3(x)q_4(x).$$
Then the decomposition is given by
$$\frac{p(x)}{q_5(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} + \frac{p_3(x)}{q_3(x)} + \frac{p_4(x)}{q_4(x)}, \tag{8.16}$$
where $p_1(x)/q_1(x)$, $p_2(x)/q_2(x)$, $p_3(x)/q_3(x)$, and $p_4(x)/q_4(x)$ are given by the right-hand sides of equations (8.12), (8.13), (8.14), *and* (8.15), *respectively.*

The best understanding of the overall strategy can be attained by examining an abundance of examples. Example 8.16 illustrates Case (1) of Theorem 8.18, as does the next example.

Example 8.19. Evaluate
$$\int_2^3 \frac{6x^2 + 5x - 3}{x^3 + 2x^2 - 3x}\, dx.$$

Solution. Factoring the denominator yields $x(x+3)(x-1)$, which are three distinct linear factors and so Case (1) applies here:
$$\frac{6x^2 + 5x - 3}{x(x+3)(x-1)} = \frac{A_1}{x} + \frac{A_2}{x+3} + \frac{A_3}{x-1}.$$
Multiplying both sides by $x(x+3)(x-1)$, we obtain
$$\begin{aligned} 6x^2 + 5x - 3 &= A_1(x+3)(x-1) + A_2 x(x-1) + A_3 x(x+3) \\ &= (A_1 x^2 + 2A_1 x - 3A_1) + (A_2 x^2 - A_2 x) + (A_3 x^2 + 3A_3 x) \\ &= (A_1 + A_2 + A_3)x^2 + (2A_1 - A_2 + 3A_3)x - 3A_1 \end{aligned}$$
Equating coefficients of x^2, coefficients of x, and constant terms, we obtain a system of equations,
$$\begin{array}{rcrcrcl} A_1 & + & A_2 & + & A_3 & = & 6 \\ 2A_1 & - & A_2 & + & 3A_3 & = & 5 \\ 3A_1 & & & & & = & 3 \end{array}$$
From the third equation we obtain $A_1 = 1$. Putting this into the first equation yields $1 + A_2 + A_3 = 6$, and so $A_2 = 5 - A_3$. Now from the second equation we have
$$2(1) - (5 - A_3) + 3A_3 = 5 \;\Rightarrow\; 4A_3 - 3 = 5 \;\Rightarrow\; A_3 = 2,$$
and thus $A_2 = 5 - A_3 = 3$.

Turning our attention to the integral, we obtain

$$\begin{aligned}\int_2^3 \frac{6x^2+5x-3}{x^3+2x^2-3x}\,dx &= \int_2^3 \left(\frac{1}{x}+\frac{3}{x+3}+\frac{2}{x-1}\right)dx\\ &= \Big[\ln|x|+3\ln|x+3|+2\ln|x-1|\Big]_2^3\\ &= \ln 3+3\ln 6+2\ln 2-\ln 2-3\ln 5-2\ln 1\\ &= \ln 3+3\ln 6+\ln 2-3\ln 5\\ &= 4\ln 6-3\ln 5\end{aligned}$$

as the answer. ■

Example 8.20. Determine

$$\int \frac{x^2}{(x+1)^3}\,dx.$$

Solution. Here we have a repeated linear factor in the denominator of the integrand, and so in accordance with Case (2) of Theorem 8.18 we obtain

$$\frac{x^2}{(x+1)^3}=\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}+\frac{B_3}{(x+1)^3}.$$

Multiplying both sides by $(x+1)^3$ yields

$$x^2=B_1(x+1)^2+B_2(x+1)+B_3,$$

whence we obtain

$$x^2=B_1x^2+(2B_1+B_2)x+(B_1+B_2+B_3).$$

Equating coefficients of matching powers of x produces the system of equations

$$\begin{array}{rcrcrcl} B_1 & & & & & = & 1\\ 2B_1 & + & B_2 & & & = & 0\\ B_1 & + & B_2 & + & B_3 & = & 0\end{array}$$

Putting $B_1=1$ from the first equation into the second equation gives $2+B_2=0$, or $B_2=-2$. Now the third equation becomes $1-2+B_3=0$, or $B_3=1$. We return to the integral to obtain

$$\begin{aligned}\int \frac{x^2}{(x+1)^3}\,dx &= \int\left(\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}+\frac{B_3}{(x+1)^3}\right)dx\\ &= \int\left(\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}\right)dx \qquad (8.17)\\ &= \ln|x+1|+\frac{2}{x+1}-\frac{1}{2(x+1)^2}+C\end{aligned}$$

■

In Example 8.20 we could also have determined the integral by using a u-substitution approach: let $u=x+1$, so that $x=u-1$ and

$$\int \frac{x^2}{(x+1)^3}\,dx=\int\frac{(u-1)^2}{u^3}\,du=\int\left(\frac{1}{u}-\frac{2}{u^2}+\frac{1}{u^3}\right)du,$$

which can be seen to be the same integral as (8.17).

Example 8.21. Determine

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\, dx.$$

Solution. The denominator is a product of distinct irreducible quadratic factors, and so in accordance with Case (4) of Theorem 8.18 we obtain

$$\frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \frac{C_1 x + D_1}{x^2+1} + \frac{C_2 x + D_2}{x^2+2}.$$

Multiplying both sides by $(x^2+1)(x^2+2)$ yields

$$3x^3 - x^2 + 6x - 4 = (C_1 x + D_1)(x^2+2) + (C_2 x + D_2)(x^2+1),$$

whence we obtain

$$3x^3 - x^2 + 6x - 4 = (C_1 + C_2)x^3 + (D_1 + D_2)x^2 + (2C_1 + C_2)x + (2D_1 + D_2).$$

Equating coefficients of matching powers of x produces the system of equations

$$\begin{array}{rcrcr} C_1 & + & C_2 & = & 3 \\ D_1 & + & D_2 & = & -1 \\ 2C_1 & + & C_2 & = & 6 \\ 2D_1 & + & D_2 & = & -4 \end{array}$$

Solving the system gives $C_1 = 3$, $C_2 = 0$, $D_1 = -3$, and $D_2 = 2$. Returning to the integral, we obtain

$$\begin{aligned} \int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\, dx &= \int \left(\frac{C_1 x + D_1}{x^2+1} + \frac{C_2 x + D_2}{x^2+2} \right) dx \\ &= \int \left(\frac{3x-3}{x^2+1} + \frac{2}{x^2+2} \right) dx \\ &= \int \frac{3x}{x^2+1}\, dx - \int \frac{3}{x^2+1}\, dx + 2 \int \frac{1}{x^2 + (\sqrt{2})^2}\, dx \\ &= \frac{3}{2} \ln|x^2+1| - 3\tan^{-1}(x) + 2 \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}} \right) + c, \end{aligned}$$

where the first integral can be determined by using the substitution $u = x^2 + 1$. Therefore

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\, dx = \frac{3}{2} \ln(x^2+1) - 3\tan^{-1}(x) + \sqrt{2} \tan^{-1}\left(\frac{x}{\sqrt{2}} \right) + c,$$

noting that $|x^2+1| = x^2+1$. ■

Example 8.22. Determine

$$\int \frac{5x^2 + 3x - 2}{x^4 + x^3 - 2x^2}\, dx.$$

Solution. Factoring the denominator, the integrand is

$$\frac{5x^2+3x-2}{x^2(x+2)(x-1)},$$

so $x+2$ and $x-1$ are distinct linear factors, and x is a repeated factor. According to Case (5) of Theorem 8.18 we have

$$\frac{5x^2+3x-2}{x^2(x+2)(x-1)} = \frac{P_1(x)}{(x+2)(x-1)} + \frac{P_2(x)}{x^2} = \left(\frac{A_1}{x+2} + \frac{A_2}{x-1}\right) + \left(\frac{B_1}{x} + \frac{B_2}{x^2}\right),$$

employing the prescribed decompositions for Cases (1) and (2). Multiplying the left and right sides of the equation by $x^2(x+2)(x-1)$ yields

$$5x^2+3x-2 = A_1x^2(x-1) + A_2x^2(x+2) + B_1x(x+2)(x-1) + B_2(x+2)(x-1),$$

and thus

$$5x^2+3x-2 = (A_1+A_2+B_1)x^3 + (-A_1+2A_2+B_1+B_2)x^2 + (-2B_1+B_2)x - 2B_2.$$

Equating coefficients of matching powers of x produces the system of equations

$$\begin{array}{rcrcrcrcl} A_1 & + & A_2 & + & B_1 & & & = & 0 \\ -A_1 & + & 2A_2 & + & B_1 & + & B_2 & = & 5 \\ & & & & -2B_1 & + & B_2 & = & 3 \\ & & & & & & 2B_2 & = & 2 \end{array}$$

The solution to the system is $A_1 = -1$, $A_2 = 2$, $B_1 = -1$, $B_2 = 1$. We now return to the integral,

$$\begin{aligned} \int \frac{5x^2+3x-2}{x^4+x^3-2x^2}\,dx &= \int \left(\frac{-1}{x+2} + \frac{2}{x-1} + \frac{-1}{x} + \frac{1}{x^2}\right) dx \\ &= -\ln|x+2| + 2\ln|x-1| - \ln|x| - \frac{1}{x} + C, \end{aligned}$$

a relatively easy resolution. ■

Example 8.23. Determine

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx.$$

Solution. The denominator of the integrand consists of a distinct linear factor x, and also a repeated irreducible quadratic factor x^2+1. In accordance with Case (5) of Theorem 8.18 we obtain

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{P_1(x)}{x} + \frac{P_4(x)}{(x^2+1)^2} = \frac{A}{x} + \left(\frac{C_1x+D_1}{x^2+1} + \frac{C_2x+D_2}{(x^2+1)^2}\right),$$

employing the prescribed decompositions of Cases (1) and (4). Multiplying the left and right sides of the equation by $x(x^2+1)^2$ yields

$$\begin{aligned} x^4+1 &= A(x^2+1)^2 + (C_1x+D_1)x(x^2+1) + (C_2x+D_2)x \\ &= (A+C_1)x^4 + D_1x^3 + (2A+C_1+C_2)x^2 + (D_1+D_2)x + A. \end{aligned}$$

Equating coefficients gives $A + C_1 = 1$, $D_1 = 0$, $2A + C_1 + C_2 = 0$, $D_1 + D_2 = 0$, and $A = 1$. Thus $D_2 = 0$, and

$$A + C_1 = 1 \quad \Rightarrow \quad 1 + C_1 = 1 \quad \Rightarrow \quad C_1 = 0$$

and

$$2A + C_1 + C_2 = 0 \quad \Rightarrow \quad 2(1) + 0 + C_2 = 0 \quad \Rightarrow \quad C_2 = -2.$$

Turning to the integral, we obtain

$$\begin{aligned}
\int \frac{x^4+1}{x(x^2+1)^2}\,dx &= \int \left(\frac{A}{x} + \frac{C_1 x + D_1}{x^2+1} + \frac{C_2 x + D_2}{(x^2+1)^2} \right) dx \\
&= \int \left(\frac{1}{x} - \frac{2x}{(x^2+1)^2} \right) dx = \ln|x| - \int \frac{2x}{(x^2+1)^2}\,dx \\
&= \ln|x| + \frac{1}{x^2+1} + C
\end{aligned}$$

where the last integral can be determined by letting $u = x^2 + 1$ to get $\int (1/u^2)\,du = -1/u$. ■

8.5 – Improper Riemann Integrals

In this section we undertake a study of improper integrals. Simply put, an **improper Riemann integral** is any sort of "integral" that does not conform to the definition of a Riemann integral as given by Definition 5.4, which requires an interval of integration $[a, b]$ that is closed and bounded, and also a bounded real-valued function f that is defined at every point in the interval of integration so that $[a, b] \subseteq \mathrm{Dom}(f)$. If $f : [0, 2] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} x^3, & \text{if } 0 \leq x < 2 \\ 10, & \text{if } x = 2 \end{cases}$$

then the integral $\int_0^2 f$ is a completely "proper" Riemann integral which can be evaluated using either Definition 5.2 or Theorem 5.21, even though f has a discontinuity at 2. This is therefore not the kind of integral we're concerned with at present.

Suppose f is Riemann integrable on $[a, x]$ for all $x \geq a$. By definition we have

$$\int_a^x f \in \mathbb{R}$$

for each $x \geq a$. This observation leads us to ask whether $\int_a^x f$ tends to some limiting value $L \in \mathbb{R}$ as $x \to \infty$; that is, does the limit

$$\lim_{x\to\infty} \int_a^x f$$

exist in $\mathbb{R}$? Such a question arises frequently in applications, and so motivates the following definition.

Definition 8.24. *If $f \in \mathcal{R}[a, b]$ for all $b \geq a$, then we define*

$$\int_a^\infty f = \lim_{b\to\infty} \int_a^b f$$

*and say that $\int_a^\infty f$ **converges to L** if $\int_a^\infty f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_a^\infty f$ **diverges**.*

If $f \in \mathcal{R}[a, b]$ for all $a \leq b$, then we define

$$\int_{-\infty}^b f = \lim_{a\to-\infty} \int_a^b f$$

*and say that $\int_{-\infty}^b f$ **converges to L** if $\int_{-\infty}^b f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_{-\infty}^b f$ **diverges**.*

If an improper integral converges to some real number L then it is customary to say simply that the integral "converges" or is "convergent." An integral that "diverges" is also said to be "divergent." Any integral of the form $\int_a^\infty f$, $\int_{-\infty}^b f$, or $\int_{-\infty}^\infty f$ (see below) is called an **improper integral of the first kind**.

The next proposition establishes linearity properties specifically for integrals of the form $\int_a^\infty f$ that are identical in form to the linearity properties of the Riemann integral given in §5.3. There are similar linearity properties for all types of improper integrals.

Proposition 8.25. *Suppose* $\int_a^\infty f$ *and* $\int_a^\infty g$ *are convergent and* $c \in \mathbb{R}$. *Then the following hold.*

1. $\int_a^\infty cf$ *is convergent, with*

$$\int_a^\infty cf = c\int_a^\infty f.$$

2. $\int_a^\infty (f+g)$ *is convergent, with*

$$\int_a^\infty (f \pm g) = \int_a^\infty f \pm \int_a^\infty g.$$

The proof is a routine application of relevant laws of limits established back in Chapter 2, and so left as an exercise.

Example 8.26. Determine whether

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. It will be easier to first determine the indefinite integral

$$\int \frac{\ln(x)}{x^2}\,dx.$$

We start with a substitution: let $w = \ln(x)$, so that $dw = (1/x)dx$ and $e^w = e^{\ln(x)} = x$; now,

$$\int \frac{\ln(x)}{x^2}\,dx = \int we^{-w}\,dw.$$

Next, we employ integration by parts, letting $u'(w) = e^{-w}$ and $v(w) = w$ to obtain

$$\int we^{-w}\,dw = -we^{-w} + \int e^{-w}\,dw = -we^{-w} - e^{-w} + C.$$

Hence,

$$\int \frac{\ln(x)}{x^2}\,dx = -\ln(x)\cdot\frac{1}{x} - \frac{1}{x} + C = -\frac{\ln(x)+1}{x} + C.$$

Now we turn to the improper integral,

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln(x)+1}{x}\right]_1^b \\ &= \lim_{b\to\infty}\left[-\frac{\ln(b)+1}{b} + \frac{\ln(1)+1}{1}\right] = \lim_{b\to\infty}\left(\frac{b-\ln(b)+1}{b}\right) \\ &\overset{LR}{=} \lim_{b\to\infty}\left(\frac{1-1/b}{1}\right) = 1,\end{aligned}$$

using L'Hôpital's Rule where indicated.

Therefore the improper integral is convergent, and its value is 1. ∎

Given an improper integral such as $\int_a^\infty f(x)\,dx$, if $f(x) \geq 0$ for all $x \in [a, \infty)$, then the value of the integral can be naturally interpreted as being the area under the curve $y = f(x)$ for $x \geq a$. If the integral is divergent (in this case it will equal ∞), then the area is said to be infinite; and if the integral is convergent, then the area is set equal to the real number the integral converges to. Thus the area under the curve $y = \ln(x)/x^2$, illustrated in Figure 35, is considered to be 1. Thus, the shaded region has an infinite "perimeter" and yet a finite area!

Proposition 8.27. *Suppose that $f \in \mathcal{R}[s,t]$ for all $-\infty < s < t < \infty$. If $\int_{-\infty}^c f$ and $\int_c^\infty f$ converge for some $c \in \mathbb{R}$, then for any $\hat{c} \neq c$ the integrals $\int_{-\infty}^{\hat{c}} f$ and $\int_{\hat{c}}^\infty f$ also converge, and*

$$\int_{-\infty}^{\hat{c}} f + \int_{\hat{c}}^\infty f = \int_{-\infty}^c f + \int_c^\infty f$$

Proof. Suppose $\int_{-\infty}^c f$ and $\int_c^\infty f$ converge for some $c \in \mathbb{R}$, meaning the limits

$$\lim_{a\to-\infty} \int_a^c f \quad \text{and} \quad \lim_{b\to\infty} \int_c^b f$$

both exist. Let $\hat{c} < c$.

For all $b > c$ we have

$$\int_{\hat{c}}^b f = \int_{\hat{c}}^c f + \int_c^b f,$$

where $\int_{\hat{c}}^c f, \int_c^b f \in \mathbb{R}$ since f is integrable on $[\hat{c}, c]$ and $[c, b]$, and so

$$\int_{\hat{c}}^\infty f = \lim_{b\to\infty} \int_{\hat{c}}^b f = \lim_{b\to\infty} \left(\int_{\hat{c}}^c f + \int_c^b f \right) = \int_{\hat{c}}^c f + \lim_{b\to\infty} \int_c^b f = \int_{\hat{c}}^c f + \int_c^\infty f. \tag{8.18}$$

Observing that $\int_{\hat{c}}^c f, \int_c^\infty f \in \mathbb{R}$, we readily conclude that $\int_{\hat{c}}^\infty f \in \mathbb{R}$ and hence $\int_{\hat{c}}^\infty f$ converges.

For all $a < \hat{c}$ we have

$$\int_a^{\hat{c}} f = \int_a^c f - \int_{\hat{c}}^c f,$$

where $\int_a^c f, \int_{\hat{c}}^c f \in \mathbb{R}$ since f is integrable on $[a, c]$ and $[\hat{c}, c]$, and so

$$\int_{-\infty}^{\hat{c}} f = \lim_{a\to-\infty} \int_a^{\hat{c}} f = \lim_{a\to-\infty} \left(\int_a^c f - \int_{\hat{c}}^c f \right) = \lim_{a\to-\infty} \int_a^c f - \int_{\hat{c}}^c f = \int_{-\infty}^c f - \int_{\hat{c}}^c f. \tag{8.19}$$

Observing that $\int_{-\infty}^c f, \int_{\hat{c}}^c f \in \mathbb{R}$, we readily conclude that $\int_{-\infty}^{\hat{c}} f \in \mathbb{R}$ and hence $\int_{-\infty}^{\hat{c}} f$ converges.

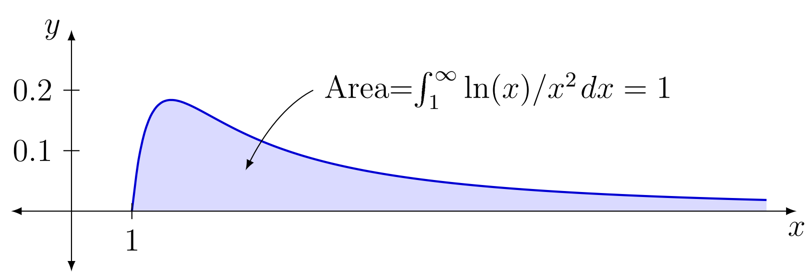


FIGURE 35. The area under the curve $y = \ln(x)/x^2$.

Finally, combining (8.18) and (8.19), we obtain

$$\int_{-\infty}^{\hat{c}} f + \int_{\hat{c}}^{\infty} f = \left(\int_{-\infty}^{c} f - \int_{\hat{c}}^{c} f\right) + \left(\int_{\hat{c}}^{c} f + \int_{c}^{\infty} f\right) = \int_{-\infty}^{c} f + \int_{c}^{\infty} f,$$

as desired. ■

Due to Proposition 8.27 we can unambiguously define an improper integral of the first kind whose interval of integration is $(-\infty, \infty)$.

Definition 8.28. *Suppose that $f \in \mathcal{R}[s,t]$ for all $-\infty < s < t < \infty$. If $\int_{-\infty}^{c} f$ and $\int_{c}^{\infty} f$ both converge for some $-\infty < c < \infty$, then we define*

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{c} f + \int_{c}^{\infty} f.$$

and say that $\int_{-\infty}^{\infty} f$ ***converges****. Otherwise we say $\int_{-\infty}^{\infty} f$* ***diverges****.*

It should be stressed that $\int_{-\infty}^{\infty} f$ can not be reliably evaluated simply by computing the limit

$$\lim_{b\to\infty} \int_{-b}^{b} f,$$

as the next example illustrates.

Example 8.29. Show that

$$\int_{-\infty}^{\infty} \frac{2x}{1+x^2}\,dx$$

diverges, and yet

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{1+x^2}\,dx = 0.$$

Solution. Letting $u = 1 + x^2$ gives $du = 2x\,dx$. Then

$$\int_{0}^{b} \frac{2x}{1+x^2}\,dx = \int_{1}^{1+b^2} \frac{1}{u}\,du = [\ln|u|]_1^{1+b^2} = \ln(1+b^2) - \ln(1) = \ln(1+b^2),$$

and so

$$\int_{0}^{\infty} \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty} \int_{0}^{b} \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty} \ln(1+b^2) = \infty.$$

Thus

$$\int_{0}^{\infty} \frac{2x}{1+x^2}\,dx$$

diverges, and therefore

$$\int_{-\infty}^{\infty} \frac{2x}{1+x^2}\,dx$$

diverges as well.

On the other hand, again employing the substitution $u = 1 + x^2$ we find that

$$\int_{-b}^{b} \frac{2x}{1+x^2}\,dx = \int_{1+b^2}^{1+b^2} \frac{1}{u}\,du = 0,$$

and so

$$\lim_{b\to\infty}\int_{-b}^{b}\frac{2x}{1+x^2}\,dx=\lim_{b\to\infty}(0)=0.$$

■

An **improper integral of the second kind** is an integral of the form

$$\int_a^b f,$$

where $-\infty < a < b < \infty$, for which there exists some $p \in [a,b]$ such that $p \notin \mathrm{Dom}(f)$. The following definition establishes how such an integral is to be evaluated, if it can be evaluated at all, in the case when $p = a$ or $p = b$.

Definition 8.30. *If $f \in \mathcal{R}[c,b]$ for all $c \in (a,b]$ and $a \notin \mathrm{Dom}(f)$, then we define*

$$\int_a^b f = \lim_{c\to a^+}\int_c^b f$$

and say that $\int_a^b f$ ***converges to L*** *if $\int_a^b f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_a^b f$* ***diverges****.*

If $f \in \mathcal{R}[a,c]$ for all $c \in [a,b)$ and $b \notin \mathrm{Dom}(f)$, then we define

$$\int_a^b f = \lim_{c\to b^-}\int_a^c f$$

and say that $\int_a^b f$ ***converges to L*** *if $\int_a^b f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_a^b f$* ***diverges****.*

Very often if f is continuous on, say, $(a,b]$ and $a \notin \mathrm{Dom}(f)$, then f has a vertical asymptote at a; that is, $\lim_{x\to a^+} f(x) = \pm\infty$. However, it could just be that a value for f is simply not specified at a by construction. For example for the function

$$\varphi(x) = \begin{cases} 3x^2, & \text{if } x < 5 \\ 4 - 8x, & \text{if } x > 5 \end{cases}$$

it's seen that $\varphi(5)$ is left undefined, and so the integral $\int_5^9 \varphi$ is an improper integral of the second kind. By Definition 8.30 we obtain

$$\begin{aligned}\int_5^9 \varphi &= \lim_{c\to 5^+}\int_c^9 (4-8x)\,dx = \lim_{c\to 5^+}\left[4x-4x^2\right]_c^9 = \lim_{c\to 5^+}\left[\left(4(9)-4(9)^2\right)-\left(4c-4c^2\right)\right] \\ &= \left(4(9)-4(9)^2\right)-\left(4(5)-4(5)^2\right) = -208,\end{aligned}$$

which shows that $\int_5^9 \varphi$ is convergent.

Example 8.31. Determine whether

$$\int_{-1}^{0}\frac{1}{x^2}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. The function $f(x) = 1/x^2$ being integrated has a vertical asymptote at $x = 0$, which is the right endpoint of the interval of integration $[-1, 0]$. By Definition 8.30 we obtain

$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{c\to 0^-} \int_{-1}^{c} \frac{1}{x^2}\,dx = \lim_{c\to 0^-} \left[-\frac{1}{x}\right]_{-1}^{c} = \lim_{c\to 0^-} \left(-\frac{1}{c} - 1\right) = \infty,$$

which shows that the improper integral is divergent. ■

Example 8.32. Determine whether

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. Here $x/\sqrt{4-x^2}$ has a vertical asymptote at $x = 2$, the right endpoint of the interval of integration $[0, 2]$. By Definition 8.30

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{c\to 2^-} \int_0^c \frac{x}{\sqrt{4-x^2}}\,dx,$$

and so, letting $u = 4 - x^2$ so that $x\,dx = -\frac{1}{2}du$, we obtain

$$\begin{aligned}\lim_{c\to 2^-} \int_0^c \frac{x}{\sqrt{4-x^2}}\,dx &= \lim_{c\to 2^-} \int_4^{4-c^2} \frac{-1/2}{\sqrt{u}}\,du = \lim_{c\to 2^-} \left(-\frac{1}{2}\left[2\sqrt{u}\,\right]_4^{4-c^2}\right)\\ &= \lim_{c\to 2^-} \left(2 - \sqrt{4-c^2}\right) = 2 - \sqrt{4-2^2} = 2.\end{aligned}$$

Hence the improper integral is convergent, and its value is 2. ■

The next definition addresses the circumstance when a function f is not defined at some point p in the interior of an interval of integration. Again, this is commonly due to f having a vertical asymptote at p, so that

$$\lim_{x\to p^+} |f(x)| = \infty \quad \text{or} \quad \lim_{x\to p^-} |f(x)| = \infty,$$

but other scenarios are possible.

Definition 8.33. *Suppose that $f \in \mathcal{R}[a,c]$ for all $c \in [a,p)$, $f \in \mathcal{R}[c,b]$ for all $c \in (p,b]$, and $p \notin \mathrm{Dom}(f)$. If $\int_a^p f$ and $\int_p^b f$ both converge, then we define*

$$\int_a^b f = \int_a^p f + \int_p^b f.$$

and say that $\int_a^b f$ ***converges****. Otherwise we say $\int_a^b f$* ***diverges****.*

Example 8.34. Determine whether

$$\int_{-2}^{3} \frac{1}{x^4}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. Here $1/x^4$ has a vertical asymptote at $x = 0$, an interior point of the interval of integration $[-2, 3]$. Now, by Definition 8.30

$$\int_0^3 \frac{1}{x^4}\,dx = \lim_{c\to 0^+} \int_c^3 \frac{1}{x^4}\,dx = \lim_{c\to 0^+} \left[-\frac{1}{x^3}\right]_c^3 = \lim_{c\to 0^+} \left(-\frac{1}{27} + \frac{1}{c^3}\right) = \infty,$$

which shows that $\int_0^3 x^{-4}\,dx$ is divergent. Thus, since

$$\int_0^3 x^{-4}\,dx \quad \text{and} \quad \int_{-2}^0 x^{-4}\,dx$$

cannot both be convergent, by Definition 8.33 it's concluded that $\int_{-2}^3 x^{-4}\,dx$ is divergent. ■

The integral treated in Example 8.34, like all improper integrals of the second kind, does not look improper at first glance. If one is careless and undertakes to evaluate the integral by conventional means, one is likely to arrive at a reasonable-looking answer without ever suspecting that something is amiss:

$$\int_{-2}^3 \frac{1}{x^4}\,dx = \left[-\frac{1}{x^3}\right]_{-2}^3 = -\frac{1}{27} + \frac{1}{-8} = -\frac{35}{216},$$

which is incorrect! So, before attempting to evaluate a definite integral, it is necessary to check that the integral is not improper in some way.

It is possible to have an integral that is improper in more than one sense, such as

$$\int_0^\infty \frac{1}{x^2}\,dx.$$

Here we have an integral of f over an unbounded interval $[0, \infty)$, so it's an improper integral of the first kind, and also f is undefined at 0, so it's an improper integral of the second kind. Such an integral is called a **mixed improper integral**.

Definition 8.35. *If $f \in \mathcal{R}[s,t]$ for all $a < s < t < \infty$, $a \notin \operatorname{Dom}(f)$, and $\int_a^c f$ and $\int_c^\infty f$ both converge for some $c \in (a, \infty)$, then we define*

$$\int_a^\infty f = \int_a^c f + \int_c^\infty f$$

*and say $\int_a^\infty f$ **converges**. Otherwise we say $\int_a^\infty f$ **diverges**.*

If $f \in \mathcal{R}[s,t]$ for all $-\infty < s < t < b$, $b \notin \operatorname{Dom}(f)$, and $\int_{-\infty}^c f$ and $\int_c^b f$ both converge for some $c \in (-\infty, b)$, then we define

$$\int_{-\infty}^b f = \int_{-\infty}^c f + \int_c^b f$$

*and say $\int_{-\infty}^b f$ **converges**. Otherwise we say $\int_{-\infty}^b f$ **diverges**.*

Example 8.36. Determine whether the mixed improper integral

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. We start by determining the indefinite integral

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx.$$

Let $u = \sqrt{x}$, so that $1+u^2 = 1+x$ and we replace dx with $2u\,du$ to obtain

$$\begin{aligned}\int \frac{1}{\sqrt{x}(1+x)}\,dx = \int \frac{2u}{u(u^2+1)}\,du &= 2\int \frac{1}{u^2+1}\,du \\ &= 2\arctan(u) + c = 2\arctan(\sqrt{x}\,) + c.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{a\to 0^+}\big[2\arctan(\sqrt{x}\,)\big]_a^1 \\ &= \lim_{a\to 0^+} 2[\arctan(1) - \arctan(a)] = 2[\arctan(1) - \arctan(0)] \\ &= 2\Big(\frac{\pi}{4} - 0\Big) = \frac{\pi}{2},\end{aligned}$$

and

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{b\to\infty}\big[2\arctan(\sqrt{x}\,)\big]_1^b \\ &= \lim_{b\to\infty} 2\Big[\arctan(\sqrt{b}\,) - \arctan(1)\Big] = 2\Big(\frac{\pi}{2} - \frac{\pi}{4}\Big) = \frac{\pi}{2}.\end{aligned}$$

Since

$$\int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx \quad\text{and}\quad \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

both converge, we conclude that

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

by Definition 8.35. ■

8.6 – Convergence Tests for Integrals

It can be difficult to determine by direct means whether an improper integral is convergent or not, largely because definite integrals themselves can be difficult to evaluate. One tool to remedy this is the Comparison Test for Integrals. Before stating the theorem, we establish a lemma that will be needed to prove the theorem.

Lemma 8.37. *Suppose the function f is monotone increasing on (a, ∞). If*

$$\lim_{x\to\infty} f(x) = M,$$

then $f(x) \leq M$ for all $x > a$.

Proof. Suppose there exists some $x_0 > a$ such that $f(x_0) > M$. Thus $f(x_0) = M + \epsilon$ for some $\epsilon > 0$. Now, for any $\beta > 0$ we can let $x_1 = \max\{x_0, \beta\} + 1$. Since $x_1 > x_0$ and f is monotone increasing, we have

$$f(x_1) \geq f(x_0) = M + \epsilon \quad\Rightarrow\quad f(x_1) - M \geq \epsilon \quad\Rightarrow\quad |f(x_1) - M| \geq \epsilon.$$

Observing that $x_1 > \beta$ also, we conclude that for any $\beta > 0$ there exists some $x > \beta$ for which $|f(x) - M| \geq \epsilon$, and therefore

$$\lim_{x\to\infty} f(x) \neq M.$$

The contrapositive of the statement of the lemma is proven. ■

Theorem 8.38 (Comparison Test for Integrals). *Suppose $f \in \mathcal{R}[a, x]$ for all $x \geq a$, and $0 \leq f \leq g$ on $[a, \infty)$. If $\int_a^\infty g$ is convergent, then $\int_a^\infty f$ is convergent.*

Proof. Suppose $\int_a^\infty g$ is convergent. By definition it follows that $g \in \mathcal{R}[a, x]$ for all $x \geq a$, and so we may define $\psi : [a, \infty) \to \mathbb{R}$ by $\psi(x) = \int_a^x g$. Similarly we define $\varphi : [a, \infty) \to \mathbb{R}$ by $\varphi(x) = \int_a^x f$.

Now, $g \geq 0$ on $[a, \infty)$ implies that

$$\int_x^y g \geq 0$$

for any $a \leq x < y$. Thus, for any $x, y \in [a, \infty)$ such that $x < y$ we have

$$\psi(y) = \int_a^y g = \int_a^x g + \int_x^y g \geq \int_a^x g = \psi(x),$$

which shows that ψ is monotone increasing on $[a, \infty)$. Since $f \geq 0$ on $[a, \infty)$, a similar argument establishes that φ also is monotone increasing on $[a, \infty)$.

Since $\int_a^\infty g$ converges, there exists some $M \in \mathbb{R}$ such that

$$\lim_{x\to\infty} \int_a^x g = M.$$

That is, ψ is monotone increasing on (a, ∞) and

$$\lim_{x\to\infty} \psi(x) = M,$$

so $\psi(x) \leq M$ for all $x > a$ by Lemma 8.37.

Finally, from the hypothesis that $f \leq g$ on $[a, \infty)$, we have

$$\varphi(x) = \int_a^x f \leq \int_a^x g = \psi(x) \leq M$$

for all $x > a$. It is now established that φ is both monotone increasing and bounded above on (a, ∞), and therefore $\lim_{x\to\infty} \varphi(x)$ exists in $\mathbb{R}$ by Proposition 2.31. Since

$$\int_a^\infty f = \lim_{x\to\infty} \int_a^x f = \lim_{x\to\infty} \varphi(x),$$

it follows that $\int_a^\infty f$ is convergent. ■

Proposition 8.39. *Suppose $f \in \mathcal{R}[a, x]$ for all $x \geq a$. If $\int_a^\infty |f|$ is convergent, then $\int_a^\infty f$ is convergent.*

Proof. Suppose that $\int_a^\infty |f|$ is convergent. Then $|f| \in \mathcal{R}[a, x]$ for all $x \geq a$, and since the same holds true for f by hypothesis, we have $f + |f| \in \mathcal{R}[a, x]$ for all $x \geq a$ by Proposition 5.17. Now, since $0 \leq f + |f| \leq 2|f|$ on $[a, \infty)$, and

$$\int_a^\infty 2|f|$$

is convergent by Proposition 8.25, the Comparison Test for Integrals implies that

$$\int_a^\infty (f + |f|)$$

is convergent. Then, because $\int_a^\infty -|f|$ is convergent by Proposition 8.25, it follows by Proposition 8.25 that

$$\int_a^\infty \big[(f + |f|) + (-|f|)\big]$$

is convergent. Of course $(f + |f|) + (-|f|) = f$ on $[a, \infty)$, and thus we conclude that $\int_a^\infty f$ is convergent. ■

Proposition 8.40 ($\boldsymbol{p}$-Test for Integrals). *Let $a > 1$. Then*

$$\int_a^\infty \frac{1}{x^p} dx$$

is convergent if and only if $p > 1$.

Example 8.41. Show that

$$\int_1^\infty \frac{\sin x}{x} dx$$

converges.

Solution. Let $t \in [1, \infty)$ be arbitrary. Employing integration by parts with $u(x) = 1/x$ and $v'(x) = \sin x$, we have

$$\int_1^t \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x}\right]_1^t - \int_1^t \frac{\cos x}{x^2} dx = \cos(1) - \frac{\cos t}{t} - \int_1^t \frac{\cos x}{x^2} dx. \tag{8.20}$$

We wish to show that the integral at right in (8.20) is convergent. Let

$$\varphi(x) = \frac{\cos x}{x^2} \quad \text{and} \quad \psi(x) = \frac{1}{x^2}.$$

For any $x \in [1, \infty)$,

$$0 \leq |\varphi(x)| = \frac{|\cos x|}{x^2} \leq \frac{1}{x^2} = \psi(x),$$

and thus $0 \leq |\varphi| \leq \psi$ on $[1, \infty)$. Also, since $|\varphi|$ is continuous on $[1, \infty)$, Proposition 5.23 implies that $|\varphi| \in \mathcal{R}[1, x]$ for all $x \geq 1$. Observing that

$$\int_1^\infty \psi = \int_1^\infty \frac{1}{x^2} dx$$

converges by the p-Test, it follows by the Comparison Test that

$$\int_1^\infty |\varphi| = \int_1^\infty \frac{|\cos x|}{x^2} dx$$

likewise converges. Now, since $\varphi \in \mathcal{R}[1, x]$ for all $x \geq 1$, by Proposition 8.39

$$\int_1^\infty \varphi = \int_1^\infty \frac{\cos x}{x^2} dx$$

also converges. That is, the limit

$$\lim_{t \to \infty} \int_1^\infty \frac{\cos x}{x^2} dx$$

exists in $\mathbb{R}$; and since

$$\lim_{t \to \infty} \left(\cos(1) - \frac{\cos t}{t} \right) = \cos(1)$$

also exists in $\mathbb{R}$, from (8.20) we conclude that

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{t \to \infty} \int_1^t \frac{\sin x}{x} dx = \lim_{t \to \infty} \left(\cos(1) - \frac{\cos t}{t} \right) - \lim_{t \to \infty} \int_1^t \frac{\cos x}{x^2} dx$$

exists in $\mathbb{R}$. That is,

$$\int_1^\infty \frac{\sin x}{x} dx$$

is convergent. ∎

Example 8.42. Show that

$$\int_0^1 \frac{\sin x}{x} dx$$

converges, and then conclude that the mixed improper integral

$$\int_0^\infty \frac{\sin x}{x} dx$$

is convergent.

Solution. The integral is an improper integral of the second kind since the integrand is undefined at 0. By definition,

$$\int_0^1 \frac{\sin x}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{\sin x}{x}dx.$$

Making the substitution $u = x^{-1}$, so that formally $-x^{-2}dx$ is replaced with du, we have, for any $t \in (0,1)$,

$$\int_t^1 \frac{\sin x}{x}dx = -\int_t^1 \frac{-x^{-2}\sin x}{x^{-1}}dx = -\int_{1/t}^1 \frac{\sin(1/u)}{u}du = \int_1^{1/t} \frac{\sin(1/u)}{u}du,$$

and thus

$$\int_0^1 \frac{\sin x}{x}dx = \lim_{t\to 0^+}\int_1^{1/t} \frac{\sin(1/u)}{u}du = \lim_{t\to\infty}\int_1^{t} \frac{\sin(1/u)}{u}du = \int_1^{\infty} \frac{\sin(1/u)}{u}du. \tag{8.21}$$

Let

$$f(u) = \frac{\sin(1/u)}{u}.$$

Since f is continuous on $[1,\infty)$, Proposition 5.23 implies that $f \in \mathcal{R}[1,x]$ for all $x \geq 1$. In the proof of Proposition 2.43 we derived the inequality $0 < \sin\theta < \theta$ for all $\theta \in (0,\pi/2)$, and since $1/u \in (0,\pi/2)$ for all $u \in [1,\infty)$, it follows that $0 < \sin(1/u) \leq 1/u$ holds for all $u \in [1,\infty)$, and hence

$$0 < \frac{\sin(1/u)}{u} \leq \frac{1}{u^2}$$

on $[1,\infty)$. By the p-Test it is known that $\int_1^\infty x^{-2}dx$ converges, and therefore by the Comparison Test

$$\int_1^\infty \frac{\sin(1/u)}{u}du$$

likewise converges. Recalling (8.21), we conclude that

$$\int_0^1 \frac{\sin x}{x}dx$$

converges as desired.

Finally, because

$$\int_1^\infty \frac{\sin x}{x}dx$$

was found to converge in the previous example, by Definition 8.35 it follows that

$$\int_0^\infty \frac{\sin x}{x}dx$$

converges also, with

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx.$$

■

Example 8.43. Show that

$$\int_{-\infty}^\infty \frac{\sin x}{x}dx \tag{8.22}$$

is convergent.

Solution. Let $t > 0$. Making the substitution $u = -x$, and recalling that $\sin(-\theta) = -\sin\theta$ in general, we have

$$\int_{-t}^{0} \frac{\sin x}{x} dx = -\int_{t}^{0} \frac{\sin(-u)}{-u} du = -\int_{t}^{0} \frac{\sin u}{u} du = \int_{0}^{t} \frac{\sin u}{u} du.$$

Hence

$$\int_{-\infty}^{0} \frac{\sin x}{x} dx = \lim_{t\to\infty} \int_{-t}^{0} \frac{\sin x}{x} dx = \lim_{t\to\infty} \int_{0}^{t} \frac{\sin x}{x} dx = \int_{0}^{\infty} \frac{\sin x}{x} dx,$$

and since the integral at right is known to converge by Example 8.41 (which is to say it equals a real number), it follows readily that

$$\int_{-\infty}^{0} \frac{\sin x}{x} dx$$

likewise converges. Thus, by Definition 8.28, the integral (8.22) converges, with

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \int_{0}^{\infty} \frac{\sin x}{x} dx + \int_{-\infty}^{0} \frac{\sin x}{x} dx = 2\int_{0}^{\infty} \frac{\sin x}{x} dx.$$

in particular. ■

9

Sequences and Series

9.1 – Numerical Sequences

In mathematics what's called a "sequence" is a function with domain consisting strictly of some subset of integers, most commonly the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ or whole numbers $\{0, 1, 2, \ldots\}$. Some references may insist that the domain of a sequence must always be $\mathbb{N}$, but this is neither necessary nor necessarily desirable.

Definition 9.1. *A* ***sequence*** *is a function f for which $\text{Dom}(f) = \{n \in \mathbb{Z} : n \geq m\}$ for some $m \in \mathbb{Z}$. If $f(n) = a_n$ for each $n \geq m$, then f may be denoted by the symbols*

$$(a_n)_{n=m}^{\infty} \quad \textit{or} \quad (a_m, a_{m+1}, a_{m+2}, \ldots).$$

Here n is the ***index*** *of the sequence, and a_n is the sequence's* ***nth term****.*

Letting $I = \{n \in \mathbb{Z} : n \geq m\}$, a sequence $(a_n)_{n=m}^{\infty}$ may also be denoted by $(a_n)_{n \in I}$. Thus the symbols $(a_n)_{n=1}^{\infty}$ and $(a_n)_{n \in \mathbb{N}}$ are equivalent. Frequently the simpler symbol (a_n) may be used in the interests of brevity or generality.

Given a nonempty set S, if $a_n \in S$ for all n, then we say (a_n) is a **sequence in** S. For our purposes a **numerical sequence** is a sequence $\mathbb{R}$; that is, a sequence (a_n) in which $a_n \in \mathbb{R}$ for all n.

The most common way to define a sequence is to give an **explicit formula** for its nth term. An example of an explicit formula for a sequence is

$$(n^3)_{n=1}^{\infty} = (1, 8, 27, 81, 125, \ldots),$$

where it is explicitly given that $a_n = n^3$ for each $n \geq 1$.

Another example of a sequence given by an explicit formula is

$$\left(\frac{n+1}{n-1}\right)_{n=2}^{\infty} = \left(3, 2, \frac{5}{3}, \frac{3}{2}, \frac{7}{5}, \ldots\right),$$

which starts with $n = 2$. The formula results in division by zero when $n = 1$. We may **reindex** the sequence, so that the index starts at 1 instead of 2, if we replace each n with $k + 1$:

$$\left(\frac{n+1}{n-1}\right)_{n=2}^{\infty} = \left(\frac{(k+1)+1}{(k+1)-1}\right)_{k+1=2}^{\infty} = \left(\frac{k+2}{k}\right)_{k=1}^{\infty} = \left(3, 2, \frac{5}{3}, \frac{3}{2}, \frac{7}{5}, \ldots\right).$$

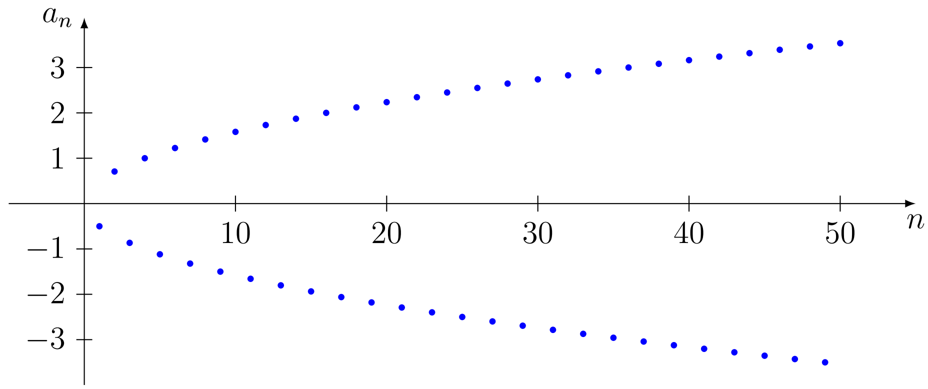


FIGURE 36. The first 50 terms of the alternating sequence $a_n = \frac{1}{2}(-1)^n\sqrt{n},\ n \geq 1$.

Yet another example of an explicit formula for a sequence is

$$\big((-1)^n(3n+1)\big)_{n=0}^{\infty} = (1, -4, 7, -11, 13, -16, \ldots),$$

while in Figure 36 the sequence

$$\left(\frac{1}{2}(-1)^n\sqrt{n}\right)_{n=1}^{\infty}$$

is illustrated.

Another way to define a sequence is by giving a **recurrence relation**, also known as an **implicit formula**. A recurrence relation starts off by specifying the first one or more terms of a sequence (called the **initial values**), then provides a general rule for computing the next term using the values of previous ones. An example would be

$$a_{n+1} = 4a_n - 7, \quad a_1 = 2,$$

where we obtain

$$a_2 = 4a_1 - 7 = 4(2) - 7 = 1,$$

$$a_3 = 4a_2 - 7 = 4(1) - 7 = -3,$$

$$a_4 = 4a_3 - 7 = 4(-3) - 7 = -19,$$

and so on, generating the sequence $(2, 1, -3, -19, \ldots)$.

A rather famous sequence is the **Fibonacci sequence**, which is defined recursively as

$$a_n = a_{n-1} + a_{n-2}, \quad a_0 = 0, a_1 = 1,$$

so that

$$a_2 = a_1 + a_0 = 1 + 0 = 1,$$

$$a_3 = a_2 + a_1 = 1 + 1 = 2,$$

$$a_4 = a_3 + a_2 = 2 + 1 = 3,$$

and in general each term in the sequence is determined to be the sum of the previous two terms, giving

$$(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots).$$

This sequence turns up in nature in all manner of disparate ways, and the terms of the sequence are called "Fibonacci numbers."

Example 9.2. Consider the sequence $(1, 4, 9, 16, 25, \ldots)$. Find an explicit formula for the nth term of the sequence, and also find a recurrence relation that generates the sequence.

Solution. We can rewrite the sequence as

$$(1^2, 2^2, 3^2, 4^2, 5^2, \ldots),$$

from which it becomes clear that the explicit formula is $a_n = n^2$.

Finding a suitable recurrence relation may take a bit more reflection, but the idea is that each term in the sequence is the square of a number 1 greater than the number squared in the previous term, and so

$$a_{n+1} = \left(\sqrt{a_n} + 1\right)^2, \quad a_1 = 1$$

is the recurrence relation. ∎

We now define various properties that sequences may possess that can sometimes be useful in the course of studying them.

Definition 9.3. *A sequence $(a_n)_{n=m}^{\infty}$ is **decreasing** if $a_{n+1} < a_n$ for all $n \geq m$, and **nonincreasing** (or **monotone decreasing**) if $a_{n+1} \leq a_n$ for all $n \geq m$. Similarly, the sequence is **increasing** if $a_{n+1} > a_n$ for all $n \geq m$, and **nondecreasing** (or **monotone increasing**) if $a_{n+1} \geq a_n$ for all $n \geq m$. A sequence that is either nonincreasing or nondecreasing is called **monotone**.*

Note that a decreasing sequence is also necessarily nonincreasing, and an increasing sequence must also be nondecreasing.

Definition 9.4. *A sequence $(a_n)_{n=m}^{\infty}$ is **bounded** if there exists some $\alpha > 0$ such that $|a_n| \leq \alpha$ for all $n \geq m$.*

A sequence (a_n) is **unbounded** if it is not bounded, meaning for every $\alpha > 0$ there can be found some integer N for which $|a_N| > \alpha$. If there exists some $\alpha > 0$ such that $a_n \leq \alpha$ for all n, then α is an **upper bound** for (a_n), and we say (a_n) is **bounded above**. If $a_n \geq \alpha$ for all n, then α is a **lower bound** for (a_n), and we say (a_n) is **bounded below**. Note that a sequence is bounded if and only if it is both bounded above and bounded below.

Example 9.5. Show that the sequence

$$a_n = 2n^2 - 3n + 4, \quad n \geq 1.$$

is increasing and unbounded.

Solution. For any $n \geq 1$ we have

$$a_{n+1} = 2(n+1)^2 - 3(n+1) + 4 = (2n^2 - 3n + 4) + (4n - 1) = a_n + (4n - 1) > a_n,$$

where the inequality is justified since $n \geq 1$ implies that $4n - 1 > 0$. Therefore $a_{n+1} > a_n$ for all $n \geq 1$, and the sequence is increasing.

To show unboundedness it may help to perform a completing the square maneuver to obtain

$$a_n = 2\left(n - \frac{3}{4}\right)^2 + \frac{23}{8}.$$

Now, the sequence $b_n = n - \frac{3}{4}$ is clearly unbounded for $n \geq 1$. We show this nonetheless. Let $\alpha > 0$ be arbitrarily large, and suppose N is an integer such that $N > \alpha + \frac{3}{4}$. Then

$$b_N = N - \frac{3}{4} > \left(\alpha + \frac{3}{4}\right) - \frac{3}{4} = \alpha,$$

and we conclude that for any $\alpha > 0$ there exists some integer N such that $b_N > \alpha$. Finally, since

$$a_n = 2b_n^2 + \frac{23}{8},$$

it is clear that (a_n) is itself unbounded. ■

Example 9.6. Show that the sequence

$$a_n = \frac{n+1}{n-1}, \quad n \geq 2$$

is decreasing and bounded.

Solution. We demonstrate that the sequence is decreasing with a chain of equivalencies: for $n \geq 2$,

$$\begin{aligned} a_{n+1} < a_n \quad &\Leftrightarrow \quad \frac{n+2}{n} < \frac{n+1}{n-1} \quad \Leftrightarrow \quad (n-1)(n+2) < n(n+1) \\ &\Leftrightarrow \quad n^2 + n - 2 < n^2 + n \quad \Leftrightarrow \quad -2 < 0. \end{aligned}$$

The argument is made by running through the steps backwards.

For boundedness we first note that, since the sequence is decreasing, we have

$$a_2 > a_3 > a_4 > \cdots,$$

and hence $a_2 = 3$ is an upper bound for $(a_n)_{n=2}^{\infty}$. As for a lower bound, it is clear that $a_n > 0$ for all $n \geq 2$, and so 0 is a lower bound. In fact, 1 is also a lower bound since

$$a_n > 1 \quad \Leftrightarrow \quad \frac{n+1}{n-1} > 1 \quad \Leftrightarrow \quad n + 1 > n - 1 \quad \Leftrightarrow \quad 1 > -1$$

for all $n \geq 2$. The sequence is thus both bounded below and bounded above, and therefore is bounded. ■

In the example above we found that 3 is an upper bound for

$$a_n = \frac{n+1}{n-1}, \quad n \geq 2,$$

and since one of the terms in the sequence is 3, there can be no upper bound that is lower than 3. For this reason we call 3 the **least upper bound** for the sequence. Somewhat less obvious is that 1 is the **greatest lower bound** for the sequence. The notions of least upper bound (or **supremum**) and greatest lower bound (or **infimum**) were introduced in the context of sets of real numbers in Chapter 1. In the present context the set in question is the range of the function $n \mapsto a_n$ for $n \geq 2$.

9.2 – The Limit of a Sequence

Sequences, like functions, can have a limiting value—which should hardly be surprising since sequences *are* functions of a specific kind.

Definition 9.7. *A sequence (a_n) is said to be **convergent** if there is some $L \in \mathbb{R}$ with the following property: For each $\epsilon > 0$ there exists some $N \in \mathbb{Z}$ such that $|a_n - L| < \epsilon$ whenever $n > N$. We write $a_n \to L$ or*

$$\lim_{n\to\infty} a_n = L,$$

*and say (a_n) **converges** to L (or has **limit** L). A sequence that is not convergent is **divergent**.*

Definition 9.8. *We define $\lim_{n\to\infty} a_n = \infty$ to mean that, for any $\alpha > 0$, there exists some $N \in \mathbb{Z}$ such that $a_n > \alpha$ whenever $n > N$. Similarly, $\lim_{n\to\infty} a_n = -\infty$ means that, for any $\alpha > 0$, there exists some $N \in \mathbb{Z}$ such that $a_n < -\alpha$ whenever $n > N$.*

It's easy to see that if $\lim_{n\to\infty} a_n$ equals ∞ or $-\infty$, then (a_n) cannot be convergent. There is a natural connection between limits of functions of a real variable (defined on intervals in $\mathbb{R}$) and limits of sequences (defined on subsets of $\mathbb{Z}$) which the following theorem makes clear.

Theorem 9.9. *Let f be a function, and suppose there exists some $N \in \mathbb{Z}$ such that $[N, \infty) \subseteq \mathrm{Dom}(f)$ and $f(n) = a_n$ for all integers $n \geq N$. If $\lim_{x\to\infty} f(x) = L$ for some $L \in [-\infty, \infty]$, then $\lim_{n\to\infty} a_n = L$.*

Proof. Suppose that $\lim_{x\to\infty} f(x) = L$, and assume for now that $-\infty < L < \infty$. Let $\epsilon > 0$. Then there exists some $\beta > 0$ such that $x > \beta$ implies $|f(x) - L| < \epsilon$. Let N_2 be any integer greater than β, and choose $N = \max\{N_1, N_2\}$. Suppose that $n > N$. Then

$$|a_n - L| = |f(n) - L| < \epsilon$$

since $f(n) = a_n$ for $n \geq N_1$ and $|f(n) - L| < \epsilon$ for $n \geq N_2 > \beta$. Therefore $\lim_{n\to\infty} a_n = L$.

Now assume $L = \infty$, so $\lim_{x\to\infty} f(x) = \infty$. Let $\alpha > 0$. Then there exists some $\beta > 0$ such that $f(x) > \alpha$ for all $x > \beta$. Let N_2 be any integer greater than β, and choose $N = \max\{N_1, N_2\}$. Then for any $n > N$ we have $a_n = f(n) > \alpha$ and we conclude that $\lim_{n\to\infty} a_n = \infty$. The argument is similar if $L = -\infty$. ∎

Example 9.10. Find the limit of the sequence

$$\left(\frac{\ln(1/n)}{n}\right)_{n=1}^{\infty}$$

or determine that the sequence diverges.

Solution. Here we have $a_n = \ln(1/n)/n$ for $n \in \mathbb{N}$. Define a function f by $f(x) = \ln(1/x)/x$ for all $x \in (0, \infty)$.

We now endeavor to evaluate the limit $\lim_{x\to\infty} \ln(1/x)/x$. First notice that the limit has the form ∞/∞. Since $\ln(1/x)$ and x are differentiable on $(0,\infty)$, and $(x)' = 1 \neq 0$ for all $x \in (0,\infty)$, we can use the left-hand version of L'Hôpital's Rule here. From

$$\lim_{x\to\infty} \frac{[\ln(1/x)]'}{(x)'} = \lim_{x\to\infty} \frac{-1/x}{1} = 0$$

we obtain

$$\lim_{x\to\infty} \frac{\ln(1/x)}{x} = 0$$

by Theorem 7.34, and so since $a_n = f(n)$ for all integers $n \geq 1$, by Theorem 9.9 we determine that $\lim_{n\to\infty} a_n = 0$ as well. ■

Because of Theorem 9.9 we can in practice work out many sequential limits just as if they were limits of functions defined on intervals in $\mathbb{R}$. As long as there can be found an integer N and a function f defined on $[N,\infty)$ such that $f(n) = a_n$ for all $n \geq N$!

Example 9.11. Find the limit of the sequence

$$a_n = \sqrt{3n^2 + 19n + 20} - \sqrt{3n^2 + 4n},$$

or determine that the sequence diverges.

Solution. It's readily seen that the function

$$f(x) = \sqrt{3x^2 + 19x + 20} - \sqrt{3x^2 + 4x}$$

is such that $f(n) = a_n$ for all $n \geq 1$, so we can treat the sequential limit $\lim_{n\to\infty} a_n$ in the same manner as the limit $\lim_{x\to\infty} f(x)$:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(\frac{\sqrt{3n^2+19n+20}-\sqrt{3n^2+4n}}{1}\cdot\frac{\sqrt{3n^2+19n+20}+\sqrt{3n^2+4n}}{\sqrt{3n^2+19n+20}+\sqrt{3n^2+4n}}\right)\\
&= \lim_{n\to\infty}\frac{(3n^2+19n+20)-(3n^2+4n)}{\sqrt{3n^2+19n+20}+\sqrt{3n^2+4n}} = \lim_{n\to\infty}\frac{15n+20}{\sqrt{3n^2+19n+20}+\sqrt{3n^2+4n}}\\
&= \lim_{n\to\infty}\frac{15n+20}{\sqrt{3n^2+19n+20}+\sqrt{3n^2+4n}}\cdot\frac{1/n}{1/n}\\
&= \lim_{n\to\infty}\frac{15+20/n}{\sqrt{3+19/n+20/n^2}+\sqrt{3+4/n}} = \frac{15}{2\sqrt{3}} = \frac{5\sqrt{3}}{2}
\end{aligned}$$

Therefore (a_n) converges to $5\sqrt{3}/2$. ■

One might wonder whether Theorem 9.9 is the final word when it comes to evaluating sequential limits. The difficulty is that there are some sequences (a_n) for which there cannot be found a function f and an integer N such that $[N,\infty) \subseteq \mathrm{Dom}(f)$ and $f(n) = a_n$ for all $n \geq N$. So it is worthwhile developing more tools that should help with evaluating sequential limits directly.

The next proposition makes clear that the limiting value of a convergent sequence is not altered by making changes to a finite number of its terms.

Proposition 9.12. *If a sequence (a_n) converges to L, then adding or subtracting a finite number of terms to or from (a_n) will result in a new sequence that also converges to L.*

Proof. Suppose (a_n) converges to L. Adding m terms to (a_n), or subtracting m terms from (a_n), results in a new sequence (b_n) which, with proper reindexing, is such that $b_n = a_n$ for all sufficiently large n, and thus

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = L.$$

■

Remark. The above proposition, together with the technique of reindexing demonstrated in the previous section, implies that *just about any general result found to be true for a sequence of the form $(a_n)_{n=1}^{\infty}$ is also true for a sequence of the form $(a_n)_{n=m}^{\infty}$ with $m \neq 1$.* In these notes the approach will be to prove results for (a_n)—where the first value of the index n is unspecified—whenever feasible. But sometimes a definitive first value of n is necessary to make clear arguments, in which case working with sequences of the form $(a_n)_{n=m}^{\infty}$ will be preferred. Occasionally, however, we will set $m = 1$ to slightly simplify the narrative.

There are laws of sequential limits that look much the same as the laws of limits for functions defined on an interval.

Theorem 9.13. *Let (a_n) and (b_n) be sequences, and let $L, M \in \mathbb{R}$. If $a_n \to L$ and $b_n \to M$, then*

1. $\displaystyle\lim_{n\to\infty} ca_n = cL = c\lim_{n\to\infty} a_n$ *for any* $c \in \mathbb{R}$
2. $\displaystyle\lim_{n\to\infty} (a_n \pm b_n) = L \pm M = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
3. $\displaystyle\lim_{n\to\infty} a_n b_n = LM = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
4. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, *provided that* $M \neq 0$

Also there is a Squeeze Theorem for sequential limits.

Theorem 9.14 (Squeeze Theorem). *For sequences (a_n), (b_n), and (c_n), suppose there exists some $N \in \mathbb{Z}$ such that $a_n \le b_n \le c_n$ for all $n \ge N$. If*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

for some $L \in [-\infty, \infty]$, then $\lim_{n\to\infty} b_n = L$ as well.

Note that the L mentioned in the theorem may be $-\infty$ or ∞. The proof of the theorem will consider $L \in \mathbb{R}$ and $L = \infty$, and leave the case when $L = -\infty$ to the reader.

Example 9.15. Find the limit of the sequence

$$b_n = \frac{3 + (-1)^n}{n^2},$$

or determine that the sequence diverges.

Solution. Let $a_n = 2/n^2$ and $c_n = 4/n^2$. For all $n \geq 1$ we have

$$a_n = \frac{2}{n^2} \leq b_n = \frac{3+(-1)^n}{n^2} \leq \frac{4}{n^2} = c_n,$$

where

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{n^2} = 0 \quad \text{and} \quad \lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{4}{n^2} = 0.$$

Thus we conclude that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3+(-1)^n}{n^2} = 0$$

by Theorem 9.14. That is, (b_n) converges to 0. ■

The factorial function

$$n \mapsto n! = (1)(2)(3)\cdots(n-1)(n)$$

arises frequently in problems involving sequences. It grows very fast. Indeed "factorial growth" is faster than any kind of exponential growth, so that in particular $n!/n^p \to \infty$ as $n \to \infty$ for any constant exponent $p > 0$. To show this we first prove the following.

Proposition 9.16. *If* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} b_n = L$ *for some* $L \in (0,\infty)$, *then*

$$\lim_{n\to\infty} a_n b_n = \infty.$$

Proof. Suppose $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = L$ for some $L \in (0,\infty)$. Let $\alpha > 0$. Since $L > 0$ and $a_n \to \infty$, there exists some integer N_1 such that $a_n > 2\alpha/L$ for all $n > N_1$. Since $L/2 > 0$ and $b_n \to L$, there exists N_2 such that $|b_n - L| < L/2$ for all $n > N_2$, which implies that $b_n > L/2$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$, and suppose $n > N$. Both $a_n > 2\alpha/L$ and $b_n > L/2$ hold, and therefore

$$a_n b_n > \frac{2\alpha}{L} \cdot \frac{L}{2} = \alpha.$$

By Definition 9.8 we conclude that $a_n b_n \to \infty$ as $n \to \infty$. ■

Example 9.17. Let $p \geq 1$ be an integer. Show that

$$\lim_{n\to\infty} \frac{n!}{n^p} = \infty.$$

Solution. We have

$$\frac{n!}{n^p} = (n-p)! \cdot \left(\frac{n-(p-1)}{n}\right)\left(\frac{n-(p-2)}{n}\right)\cdots\left(\frac{n-2}{n}\right)\left(\frac{n-1}{n}\right).$$

Let

$$a_n = \left(\frac{n-(p-1)}{n}\right)\left(\frac{n-(p-2)}{n}\right)\cdots\left(\frac{n-2}{n}\right)\left(\frac{n-1}{n}\right)$$

and $b_n = (n-p)!$. Since $b_n \geq n-p$ for all n, and $\lim_{n\to\infty}(n-p) = \infty$, the Squeeze Theorem implies that $\lim_{n\to\infty} b_n = \infty$. On the other hand an easy extension of Theorem 9.13(3) gives

$$\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty} \frac{n-(p-1)}{n}\right)\left(\lim_{n\to\infty} \frac{n-(p-2)}{n}\right)\cdots\left(\lim_{n\to\infty} \frac{n-2}{n}\right)\left(\lim_{n\to\infty} \frac{n-1}{n}\right) = 1.$$

Therefore

$$\lim_{n\to\infty} \frac{n!}{n^p} = \lim_{n\to\infty} a_n b_n = \infty$$

by Proposition 9.16. ∎

Theorem 9.18. *Every convergent sequence is bounded.*

Proof. Suppose $(a_n)_{n=1}^{\infty}$ converges, so $\lim_{n\to\infty} a_n = L$ for some real L. Employing Definition 9.7 with $\epsilon = 1$, it follows there exists some integer N such that $|a_n - L| < 1$ for all $n > N$, and thus by the Reverse Triangle Inequality (see §1.6) we find $|a_n| < |L| + 1$ holds for $n > N$. Now, if we set

$$\alpha = \max\{|a_1|, |a_2|, \ldots, |a_N|\},$$

then we have $|a_n| \leq \alpha$ for $1 \leq n \leq N$, and hence $|a_n| < \alpha + |L| + 1$ for all $n \geq 1$. Therefore (a_n) is bounded. ∎

Thus, as we saw in Example 9.17, if a sequence is not bounded then it cannot be convergent. The converse of this theorem is not true in general; that is, if a sequence is bounded, it does not necessarily follow that it is convergent. However, a bounded sequence that is also monotone has a happier ending.

Theorem 9.19 (Monotone Convergence Theorem). *Every bounded monotone sequence is convergent.*

Proof. Suppose that $(a_n)_{n=m}^{\infty}$ is a bounded monotone increasing sequence. Then the set $\{a_n : n \geq m\} \subseteq \mathbb{R}$ has an upper bound, and so by the Completeness Axiom it has a least upper bound $\alpha \in \mathbb{R}$. Now, let $\epsilon > 0$. Then there exists some $N \geq m$ such that $a_N > \alpha - \epsilon$. Since (a_n) is monotone increasing with upper bound α, it follows that $\alpha - \epsilon < a_n \leq \alpha$ for all $n \geq N$, and therefore $|a_n - \alpha| < \epsilon$ for all $n \geq N$. This shows that $\lim_{n\to\infty} a_n = \alpha$ and therefore (a_n) converges.

Next suppose that $(a_n)_{n=m}^{\infty}$ is a bounded monotone decreasing sequence. Then $(-a_n)_{n=m}^{\infty}$ is a bounded monotone increasing sequence, so that $\lim_{n\to\infty}(-a_n) = L$ for some $L \in \mathbb{R}$ by the first part of the proof, and then $\lim_{n\to\infty} a_n = -L$ by Theorem 9.13(1). Therefore (a_n) converges. ∎

The details of the proof of the Monotone Convergence Theorem actually deliver a stronger result which is sometimes useful.

Corollary 9.20. *Let $(a_n)_{n=m}^{\infty}$ be a bounded sequence. If (a_n) is monotone increasing, then $a_n \to \sup\{a_n : n \geq m\}$; and if (a_n) is monotone decreasing, then $a_n \to \inf\{a_n : n \geq m\}$.*

Example 9.21. Let $(a_n)_{n=0}^{\infty}$ be the sequence given by the recurrence relation

$$a_{n+1} = \frac{3}{4}a_n + 1, \quad a_0 = 0.$$

Show that the sequence is bounded and monotone, and find its limit.

Solution. By inspection it can be seen that the terms of the sequence must all be nonnegative; that is, $a_n \geq 0$ for all $n \geq 0$. Also, since $a_0 = 0$, $a_1 = 1$, $a_2 = \frac{7}{4}$, $a_3 = \frac{37}{16}$, it seems plausible that the sequence is monotone increasing (i.e. nondecreasing). In fact, since

$$a_{n+1} \geq a_n \quad \Leftrightarrow \quad \frac{3}{4}a_n + 1 \geq a_n \quad \Leftrightarrow \quad a_n \leq 4 \tag{9.1}$$

for any $n \geq 0$, we can conclude that (a_n) is monotone increasing if we can show that $a_n \leq 4$ for all n.

Clearly $0 \leq a_0 \leq 4$, since $a_0 = 0$ is given. For arbitrary $n \geq 0$ suppose that $0 \leq a_n \leq 4$. Then

$$a_{n+1} = \frac{3}{4}a_n + 1 \leq \frac{3}{4}(4) + 1 = 4$$

and

$$a_{n+1} = \frac{3}{4}a_n + 1 \geq \frac{3}{4}(0) + 1 = 1 > 0,$$

so that $0 \leq a_{n+1} \leq 4$. By the principle of induction we conclude that $0 \leq a_n \leq 4$ for all $n \geq 0$, and therefore (a_n) is a bounded sequence. Then, since $a_n \leq 4$ is true for all n, by (9.1) we conclude that $a_{n+1} \geq a_n$ is also true for all n, and hence (a_n) is monotone increasing.

Next, since (a_n) is a bounded monotone sequence, the Monotone Convergence Theorem implies that (a_n) converges to some $\alpha \in \mathbb{R}$. This is to say that $\lim_{n\to\infty} a_n = \alpha$, and thus $\lim_{n\to\infty} a_{n+1} = \alpha$ also, and so

$$a_{n+1} = \frac{3}{4}a_n + 1 \quad \Rightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{3}{4}a_n + 1\right) \quad \Rightarrow \quad \alpha = \frac{3}{4}\alpha + 1 \quad \Rightarrow \quad \alpha = 4.$$

That is, the sequence (a_n) has limit 4. ∎

Definition 9.22. *A* ***subsequence*** *of* $(a_n)_{n=m}^{\infty}$ *is a sequence* $(a_{n_k})_{k=1}^{\infty}$ *such that*

$$m \leq n_1 < n_2 < n_3 < \cdots$$

holds. If $\lim_{k\to\infty} a_{n_k} = L$ *for some* $L \in \mathbb{R}$, *then* L *is called a* ***subsequential limit*** *of* $(a_n)_{n=m}^{\infty}$.

If $m = 1$ (as it usually is), then the definition requires the values n_k to be such that $n_1 \geq 1$ and $n_{k+1} > n_k$ for all $k \geq 1$. This readily implies that $n_k \geq k$ for all k, which is a fact we'll make use of shortly.

Consider the sequence $a_n = n^2$, $n \geq 1$,

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots) = (1, 4, 9, 16, 25, 36, 49, 64, \ldots).$$

We can define a subsequence $(a_{n_k})_{k=1}^{\infty}$ by setting $n_k = 2k$ for all $k \geq 1$. This gives

$$a_{n_k} = a_{2k} = (2k)^2, \quad k \geq 1;$$

that is,

$$(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots) = (a_2, a_4, a_6, \ldots) = (4, 16, 36, 64, \ldots),$$

which is the subsequence consisting of the "even-numbered terms" of (a_n). Now, if we set $m_k = 2k - 1$ we obtain

$$a_{m_k} = a_{2k-1} = (2k-1)^2, \quad k \geq 1;$$

that is,

$$(a_{m_k})_{k=1}^{\infty} = (a_{m_1}, a_{m_2}, a_{m_3}, \ldots) = (a_1, a_3, a_5, \ldots) = (1, 9, 25, 49, \ldots),$$

which consists of the "odd-numbered terms" of (a_n).

Proposition 9.23. *If (a_n) is a sequence that converges to $L \in \mathbb{R}$, then every subsequence $(a_{n_k})_{k=1}^{\infty}$ also converges to L.*

Proof. Suppose $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. Let $(a_{n_k})_{k=1}^{\infty}$ be a subsequence. It must be shown that $\lim_{k\to\infty} a_{n_k} = L$.

Let $\epsilon > 0$. There exists some positive integer N such that $|a_n - L| < \epsilon$ for all $n > N$. Suppose that $k > N$. Since $n_k \geq k > N$ we obtain $|a_{n_k} - L| < \epsilon$.

Therefore $\lim_{k\to\infty} a_{n_k} = L$. ■

Recall that because of Proposition 9.12 and the possibilities of reindexing a sequence, the above proposition applies to sequences of the form $(a_n)_{n=m}^{\infty}$ for any integer m.

Corollary 9.24. *If (a_n) has subsequences (a_{m_k}) and (a_{n_k}) such that*

$$\lim_{k\to\infty} a_{m_k} = L_1 \quad \text{and} \quad \lim_{k\to\infty} a_{n_k} = L_2$$

for $L_1, L_2 \in \mathbb{R}$ with $L_1 \neq L_2$, then (a_n) diverges.

Proof. If (a_n) does not diverge, then it must converge to some $L \in \mathbb{R}$, and then by Proposition 9.23 we must have

$$\lim_{k\to\infty} a_{m_k} = L = \lim_{k\to\infty} a_{n_k},$$

a contradiction. ■

It should also be clear that (a_n) must diverge if it has a subsequence that diverges. The next example should serve to illustrate the utility of these results.

Example 9.25. Show that the sequence

$$a_n = (-1)^n \frac{3n}{n+1}$$

diverges.

Solution. One subsequence is $(a_{n_k})_{k=1}^{\infty}$ with $n_k = 2k$,

$$(a_{n_k})_{k=1}^{\infty} = (a_{2k})_{k=1}^{\infty} = \left((-1)^{2k} \frac{3(2k)}{2k+1}\right)_{k=1}^{\infty} = \left(\frac{6k}{2k+1}\right)_{k=1}^{\infty},$$

where

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \frac{6k}{2k+1} = \frac{6}{2} = 3.$$

Another subsequence is $(a_{m_k})_{k=1}^{\infty}$ with $m_k = 2k-1$,

$$(a_{m_k})_{k=1}^{\infty} = (a_{2k-1})_{k=1}^{\infty} = \left((-1)^{2k-1} \frac{3(2k-1)}{(2k-1)+1}\right)_{k=1}^{\infty} = \left(-\frac{6k-3}{2k}\right)_{k=1}^{\infty},$$

where

$$\lim_{k\to\infty} a_{m_k} = \lim_{k\to\infty} -\frac{6k-3}{2k} = -\frac{6}{2} = -3.$$

Since $a_{n_k} \to 3$ and $a_{m_k} \to -3$ as $k \to \infty$, we conclude that (a_n) must diverge by Corollary 9.24. ■

9.3 – Infinite Series

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Let $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, and in general

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k,$$

called the ***n*th partial sum** of (a_n). With these values we can construct the **sequence of partial sums** associated with (a_n), which is the sequence $(s_n)_{n=1}^{\infty}$.

Definition 9.26. *We define*

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n,$$

*and call $\sum_{k=1}^{\infty} a_k$ the **infinite series** associated with $(a_n)_{n=1}^{\infty}$ (or simply a **series**). The expression a_k is referred to as the **kth term** of the series.*

*If $\sum_{k=1}^{\infty} a_k = s$ for some $s \in \mathbb{R}$ (i.e. $s_n \to s$), then we say the series $\sum_{k=1}^{\infty} a_k$ **converges** and call s the **sum** of the series.*

*If $\sum_{k=1}^{\infty} a_k$ does not exist in $\mathbb{R}$ (i.e. the sequence $(s_n)_{n=1}^{\infty}$ diverges), then we say the series $\sum_{k=1}^{\infty} a_k$ **diverges**.*

So the symbol $\sum_{k=1}^{\infty} a_k$ represents the limit of a sequence of partial sums (which may or may not exist in $\mathbb{R}$), and not actually a "sum" in the conventional sense. More generally a series can have an index that starts at an integer other than 1: for any $m \in \mathbb{Z}$ we define

$$\sum_{k=m}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=m}^{n} a_k,$$

which is the infinite series associated with $(a_n)_{n=m}^{\infty}$. The terminology changes little: defining $s_n = \sum_{k=m}^{n} a_k$ for $n \geq m$, if $\lim_{n\to\infty} s_n = s$ for some $s \in \mathbb{R}$, we say the series $\sum_{k=m}^{\infty} a_k$ converges to s and write $\sum_{k=m}^{\infty} a_k = s$; otherwise the series diverges. If there is no chance for confusion it's often convenient to write $\sum_{k=m}^{\infty} a_k$ as $\sum a_k$ in the interests of saving space. Other times the symbol $\sum a_k$ will be used in general statements, such as those found in theorems, when the initial index value (whether it's 0, 1, or something else) is simply not important.

When we speak here of "adding a term" to a series $\sum_{k=m}^{\infty} a_k$, this will mean designating some $a_{m-1} \in \mathbb{R}$ and passing to a new series $\sum_{k=m-1}^{\infty} a_k$ associated with $(a_n)_{n=m-1}^{\infty}$. Similarly, "subtracting a term" from $\sum_{k=m}^{\infty} a_k$ will mean removing a_m and passing to the series $\sum_{k=m+1}^{\infty} a_k$ associated with $(a_n)_{n=m+1}^{\infty}$.

Proposition 9.27. *If a series $\sum a_k$ converges, then adding (resp. subtracting) a finite number of real-valued terms to (resp. from) $\sum a_k$ will result in a new series that also converges.*

Thus, for any $m > 0$, a series $\sum_{k=0}^{\infty} a_k$ is convergent if and only if $\sum_{k=m}^{\infty} a_k$ is convergent, assuming that $a_k \in \mathbb{R}$ for all $k \geq 0$. This proposition is an immediate consequence of Proposition 9.12 and so the proof is left as an exercise for those interested.

Remark. Proposition 9.27 immediately implies that altering a divergent series by a finite number of terms yields a series that is again divergent. Indeed, if it were possible to add a term to a divergent series $\sum b_k$ to obtain a convergent series $\sum a_k$, then subtracting the same term from the convergent series $\sum a_k$ would result in the divergent series $\sum b_k$—contradicting the proposition!

An immediate consequence of Theorem 9.13 is the following.

Proposition 9.28. *Let $\sum a_k$ and $\sum b_k$ be convergent series, and let $c \in \mathbb{R}$. Then the following hold.*

1. *$\sum ca_k$ is convergent, with*
$$\sum ca_k = c\sum a_k.$$
2. *$\sum(a_k + b_k)$ is convergent, with*
$$\sum(a_k + b_k) = \sum a_k + \sum b_k.$$

It is often difficult determining whether a series converges or not, but one kind of series that can be considered immediately is a so-called **telescoping series**. This is a series associated with a sequence (a_n) whose partial sums have terms that largely cancel out.

Example 9.29. Determine whether the series
$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k+2}\right)$$
converges or diverges.

Solution. For any integer $n \geq 1$ we have
$$\begin{aligned}\sum_{k=1}^{n} \ln\left(\frac{k+1}{k+2}\right) &= \sum_{k=1}^{n} [\ln(k+1) - \ln(k+2)] \\ &= [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + [\ln(4) - \ln(5)] + \cdots \\ &\qquad \cdots + [\ln(n) - \ln(n+1)] + [\ln(n+1) - \ln(n+2)] \\ &= \ln(2) - \ln(n+2).\end{aligned}$$
Thus the nth partial sum of the associated sequence collapses, or "telescopes," to become $\ln(2) - \ln(n+2)$. Now,
$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k+2}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{k+1}{k+2}\right) = \lim_{n\to\infty} [\ln(2) - \ln(n+2)] = -\infty,$$
which shows that the series diverges. ■

Another kind of series that can be easily analyzed is a **geometric series**, which has the form $\sum_{k=0}^{\infty} ar^k$ for some nonzero constants $a, r \in \mathbb{R}$. (Observe that if either $a = 0$ or $r = 0$ then the series would simply converge to 0.)

Lemma 9.30. *For any $r \neq 1$ and integer $n \geq 0$,*

$$\sum_{k=0}^{n} r^k = 1 + r + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}. \tag{9.2}$$

Proof. Let $r \neq 1$. Clearly the equation holds when $n = 0$. Let $n \geq 0$ be arbitrary and suppose that (9.2) holds. Then

$$\begin{aligned} 1 + r + \cdots + r^{n+1} &= (1 + r + \cdots + r^n) + r^{n+1} \\ &= \frac{r^{n+1} - 1}{r - 1} + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + \frac{r^{n+1}(r - 1)}{r - 1} \\ &= \frac{r^{n+1} - 1}{r - 1} + \frac{r^{n+2} - r^{n+1}}{r - 1} \\ &= \frac{r^{n+2} - 1}{r - 1}, \end{aligned}$$

and thus by induction we conclude that (9.2) holds for all n. ■

Note that (9.2) holds even when $r = 0$, in which case the summation on the right becomes $\sum_{k=0}^{n} 0^k$ and it is understood that 0^0 represents 1.

Proposition 9.31. *Let $a, r \in \mathbb{R}$ such that $a, r \neq 0$. If $|r| < 1$, then*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r};$$

and if $|r| \geq 1$, then the series diverges.

Proof. Suppose that $0 < |r| < 1$, which implies that $r \neq 1$. Using Lemma 9.30,

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k = \lim_{n\to\infty} \frac{r^{n+1} - 1}{r - 1}.$$

Now, since $-1 < r < 1$ we have $r^{n+1} \to 0$ as $n \to \infty$, and so by Theorem 9.13(4)

$$\sum_{k=0}^{\infty} r^k = \frac{\lim_{n\to\infty}(r^{n+1} - 1)}{\lim_{n\to\infty}(r - 1)} = \frac{0 - 1}{r - 1} = \frac{1}{1 - r}.$$

Hence the series $\sum_{k=0}^{\infty} r^k$ converges, so by Proposition 9.28(1) we have

$$\sum_{k=0}^{\infty} ar^k = a \sum_{k=0}^{\infty} r^k = a \cdot \frac{1}{1 - r} = \frac{a}{1 - r}$$

as desired.

Finally, if $|r| \geq 1$, then since $a \neq 0$ we have

$$\lim_{n\to\infty} ar^n \neq 0,$$

and so $\sum_{k=0}^{\infty} ar^k$ diverges by the Divergence Test in §9.4. ■

Example 9.32. Determine whether the series

$$\sum_{k=1}^{\infty} 3^{-k} 8^{k+1} \tag{9.3}$$

converges or diverges.

Solution. We have

$$\sum_{k=1}^{\infty} 3^{-k} 8^{k+1} = \sum_{k=1}^{\infty} \frac{8^{k+1}}{3^k} = \sum_{k=1}^{\infty} 8 \left(\frac{8}{3}\right)^k. \tag{9.4}$$

Now, since

$$\sum_{k=0}^{\infty} 8 \left(\frac{8}{3}\right)^k \tag{9.5}$$

is a divergent geometric series by Proposition 9.31, it follows from Proposition 9.27 that the series on the right-hand side of (9.4)—which is the series (9.5) with the zeroth term removed—must also diverge. Therefore the series (9.3) diverges. ■

9.4 – Divergence and Integral Tests

We now develop two tests that can be used to determine whether a series converges or diverges. First there is the Divergence Test, which is often a quick way to spot a divergent series.

Theorem 9.33 (**Divergence Test**). *If* $\lim_{k\to\infty} a_k \neq 0$*, then the series* $\sum a_k$ *diverges.*

Proof. Suppose that $\sum a_k$ converges. If s_n is the nth partial sum of $\sum a_k$, then there exists some $s \in \mathbb{R}$ such that $\lim_{n\to\infty} s_n = s$. We also have $\lim_{n\to\infty} s_{n-1} = s$. Now, observing that $a_n = s_n - s_{n-1}$, by Theorem 9.13(2) we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

Thus, if $\sum a_k$ converges, then $\lim_{n\to\infty} a_n = 0$. This implies that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_k$ diverges. ■

Example 9.34. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{10 + 3^{-k}}$$

converges or diverges.

Solution. Here $a_k = 1/(10 + 3^{-k})$, and so since $3^{-k} \to 0$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{10 + 3^{-k}} = \frac{1}{10 + 0} = 0.1.$$

Since $\lim_{k\to\infty} a_k = 0.1 \neq 0$, by the Divergence Test we conclude that the series diverges. ■

This test will not detect all divergent series, of course. There are plenty of series $\sum a_k$ that diverge despite the fact that $a_k \to 0$, such as the harmonic series $\sum_{k=1}^{\infty} 1/k$.

Recall that a function f is **nonnegative** on an interval I if $f(x) \geq 0$ for all $x \in I$, and **nonincreasing** on I if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in I$ are such that $x_1 \leq x_2$.

Theorem 9.35 (**Integral Test**). *Let* $m \in \mathbb{Z}$*, and suppose* f *is a function that is continuous, nonnegative and nonincreasing on* $[m, \infty)$*. Then* $\sum_{k=m}^{\infty} f(k)$ *converges if and only if* $\int_m^{\infty} f(x)dx$ *converges.*[8]

Proof. Suppose that $\int_m^{\infty} f(x)dx$ converges. Then there exists some $\alpha \in \mathbb{R}$ for which

$$\lim_{b\to\infty} \int_m^b f(x)dx = \alpha, \tag{9.6}$$

[8]It might be noticed that this version of the "Integral Test" is a wee bit more general than the version seen in the textbook; in particular f need only be nonnegative (not positive) and nonincreasing (not strictly decreasing).

where $\alpha \geq 0$ since $\int_m^b f(x)dx \geq 0$ for all $b \geq m$. If we define the sequence

$$a_n = \int_m^n f(x)dx, \quad n \geq m,$$

then (9.6) implies that $(a_n)_{n=m}^\infty$ is a sequence that converges to α and therefore by Theorem 9.18 it is also bounded. In fact $(a_n)_{n=m}^\infty$ is a nondecreasing sequence: for each $n \geq m$,

$$a_{n+1} = \int_m^{n+1} f(x)dx = \int_m^n f(x)dx + \int_n^{n+1} f(x)dx = a_n + \int_n^{n+1} f(x)dx \geq a_n,$$

where

$$\int_n^{n+1} f(x)dx \geq 0$$

since $f(x) \geq 0$ for all $x \in [n, n+1]$, and so it is not hard to see that α is an upper bound for $(a_n)_{n=m}^\infty$.

Now, certainly $f(m) \leq f(m) + \alpha$, and for each $n > m$,

$$\begin{aligned}\sum_{k=m}^n f(k) &= f(m) + \sum_{k=m+1}^n f(k) = f(m) + \sum_{k=m}^{n-1} f(k+1) = f(m) + \sum_{k=m}^{n-1} \int_k^{k+1} f(k+1)dx \\ &\leq f(m) + \sum_{k=m}^{n-1} \int_k^{k+1} f(x)dx = f(m) + \int_m^n f(x)dx = f(m) + a_n \leq f(m) + \alpha.\end{aligned}$$

This demonstrates that the nondecreasing (hence monotone) sequence

$$s_n = \sum_{k=m}^n f(k), \quad n \geq m$$

is bounded: $|s_n| \leq f(m) + \alpha$ for all $n \geq m$. By the Monotone Convergence Theorem $(s_n)_{k=m}^\infty$ converges, which is to say there is some $s \in \mathbb{R}$ such that $\lim_{n\to\infty} s_n = s$. Therefore the series $\sum_{k=m}^\infty f(k)$ converges.

To show that the convergence of $\sum_{k=m}^\infty f(k)$ implies the convergence of $\int_{k=m}^\infty f(x)dx$, it is easier to suppose that $\int_m^\infty f(x)dx$ diverges and then show that $\sum_{k=m}^\infty f(k)$ must diverge. So, suppose that $\int_m^\infty f(x)dx$ is divergent. Letting

$$F(b) = \int_m^b f(x)dx,$$

this means that $\lim_{b\to\infty} F(b)$ does not exist in $\mathbb{R}$. Because $f(x) \geq 0$ for all $x \in [m, \infty)$, we find that F is a nondecreasing function on $[m, \infty)$ and therefore by Corollary 2.2 $\lim_{b\to\infty} F(b) = \infty$.

Let $\alpha > 0$. Since

$$\lim_{b\to\infty} \int_m^\infty f(x)dx = \infty,$$

there is some $b_0 \geq m$ such that $\int_m^b f(x)dx > \alpha$ for all $b \geq b_0$. Choose $n_0 \in \mathbb{Z}$ such that $n_0 \geq b_0 + 1 \geq m + 1$. Suppose that $n \geq n_0$, so $n \geq m + 1$ and therefore $n - 1 \geq m$. Now,

$$\sum_{k=m}^n f(k) \geq \sum_{k=m}^{n-1} f(k) = \sum_{k=m}^{n-1} \int_k^{k+1} f(k)dx \geq \sum_{k=m}^{n-1} \int_k^{k+1} f(x)dx = \int_m^n f(x)dx > \alpha$$

where the second inequality holds since $f(x) \le f(k)$ for all $x \in [k, k+1]$, and the third inequality holds since $n > b_0$.

Since $\alpha > 0$ is arbitrary, it follows that for every $\alpha > 0$ there exists some integer $n_0 \ge m$ such that $\sum_{k=m}^{n} f(k) > \alpha$ for all $n \ge n_0$. Therefore

$$\lim_{n\to\infty} \sum_{k=m}^{n} f(k) = \infty,$$

and we conclude that $\sum_{k=m}^{\infty} f(k)$ diverges. ■

Example 9.36. Determine whether the series

$$\sum_{k=1}^{\infty} ke^{-3k^2}$$

converges or diverges.

Solution. Let $f(x) = xe^{-3x^2}$, which clearly is continuous and nonnegative on $[1, \infty)$. Now,

$$f'(x) = (1 - 6x^2)e^{-3x^2},$$

so we have $f'(x) < 0$ for $x > 1/\sqrt{6}$ or $x < -1/\sqrt{6}$, which certainly shows that f is nonincreasing on $[1, \infty)$ as well. The hypotheses of the Integral Test are therefore satisfied for $m = 1$.

Making the substitution $u = -3x^2$, we obtain

$$\begin{aligned}\int_1^{\infty} xe^{-3x^2} dx &= \lim_{b\to\infty} \int_1^{b} xe^{-3x^2} dx = \lim_{b\to\infty} \int_{-3}^{-3b^2} -\frac{1}{6}e^{u} du \\ &= \lim_{b\to\infty} \left[-\frac{1}{6}\left(e^{-3b^2} - e^{-3}\right)\right] = \frac{1}{6e^3},\end{aligned}$$

and thus the integral $\int_1^{\infty} xe^{-3x^2} dx$ converges. Therefore by the Integral Test $\sum_{k=1}^{\infty} ke^{-3k^2}$ converges as well. ■

A ***p*-series** is a series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p},$$

where p can be any real number. They arise often and will be handy when making use of the comparison tests in the next section.

Proposition 9.37. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for $p > 1$ and diverges for $p \le 1$.

Proof. Suppose that $p > 1$. The function $f(x) = 1/x^p$ is a continuous, nonnegative and nonincreasing function on $[1, \infty)$, so by the Integral Test $\sum_{k=1}^{\infty} 1/k^p$ converges if and only if $\int_1^{\infty} 1/x^p \, dx$ converges. We have

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p} \, dx &= \lim_{b\to\infty} \int_1^b x^{-p} \, dx = \lim_{b\to\infty} \left[\frac{1}{1-p} x^{1-p}\right]_1^b \\ &= \lim_{b\to\infty} \frac{1}{1-p} \left(b^{1-p} - 1\right) = \lim_{b\to\infty} \frac{1}{1-p} \left(\frac{1}{b^{p-1}} - 1\right) \\ &\overset{(1)}{=} \frac{1}{1-p}(0-1) = \frac{1}{p-1},\end{aligned}$$

where equality (1) holds since $p-1 > 0$ implies that $b^{p-1} \to \infty$ as $b \to \infty$, and thus the integral converges. Therefore $\sum_{k=1}^{\infty} 1/k^p$ converges.

Now suppose that $0 < p < 1$. Then $f(x) = 1/x^p$ is still continuous, nonnegative and nonincreasing on $[1, \infty)$; however, since $0 < 1 - p < 1$, we find that

$$\lim_{b\to\infty} \frac{1}{1-p} \left(b^{1-p} - 1\right) = \infty$$

since $b^{1-p} \to \infty$ as $b \to \infty$. Thus $\int_1^{\infty} 1/x^p \, dx$ diverges, and by the Integral Test $\sum_{k=1}^{\infty} 1/k^p$ diverges as well.

What about $p = 1$? The function $f(x) = 1/x$ still satisfies the hypotheses of the Integral Test, so, since

$$\int_1^{\infty} \frac{1}{x} \, dx = \lim_{b\to\infty} \int_1^{\infty} \frac{1}{x} \, dx = \lim_{b\to\infty} \left[\ln|x|\right]_1^b = \lim_{b\to\infty} \ln(b) = \infty,$$

we conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

(called the **harmonic series**) diverges.

The case when $p = 0$ is straightforward: since

$$\lim_{k\to\infty} \frac{1}{k^p} = \lim_{k\to\infty} \frac{1}{k^0} = \lim_{k\to\infty} (1) = 1 \neq 0,$$

the series diverges by the Divergence Test.

Finally we come to $p < 0$: since $-p > 0$, we find that

$$\lim_{k\to\infty} \frac{1}{k^p} = \lim_{k\to\infty} k^{-p} = \infty,$$

so again the series diverges by the Divergence Test. ■

9.5 – Comparison, Root and Ratio Tests

The Direct Comparison Test enables one to determine whether a series converges or diverges by comparing it to a series that is already known to converge or diverge.

Theorem 9.38 (Direct Comparison Test). *Suppose there exists some $k_0 \in \mathbb{Z}$ such that $0 \leq a_k \leq b_k$ for all $k \geq k_0$.*

1. *If $\sum_{k=k_0}^{\infty} b_k$ converges, then $\sum a_k$ converges.*
2. *If $\sum_{k=k_0}^{\infty} a_k$ diverges, then $\sum b_k$ diverges.*

Proof. Suppose there is some integer k_0 such that $0 \leq a_k \leq b_k$ for all $k \geq k_0$.

To prove part (1), suppose that $\sum_{k_0=k}^{\infty} b_k$ converges, so there exists some $t \in \mathbb{R}$ such that $\sum_{k=k_0}^{\infty} b_k = t$. Now, since $b_k \geq 0$ for each $k \geq k_0$, the sequence

$$\sum_{k=k_0}^{n} b_k, \quad n \geq k_0$$

is monotone increasing, so t must be the least upper bound for

$$\left(\sum_{k=k_0}^{n} b_k\right)_{n=k_0}^{\infty}$$

by Corollary 9.20. Thus, recalling that $a_k \leq b_k$ for each $k \geq k_0$, we obtain

$$0 \leq \sum_{k=k_0}^{n} a_k \leq \sum_{k=k_0}^{n} b_k \leq t$$

for each $n \geq k_0$. It's now seen that the monotone increasing sequence

$$\left(\sum_{k=k_0}^{n} a_k\right)_{n=k_0}^{\infty}$$

is bounded, and therefore by Theorem 9.19 this sequence converges. So, there is some $s \in \mathbb{R}$ such that

$$\sum_{k=k_0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=k_0}^{n} a_k = s,$$

which shows that the series $\sum_{k=k_0}^{\infty} a_k$ is convergent and hence, by Proposition 9.27, the series $\sum a_k$ converges. This proves (1).

Part (2) is equivalent to the statement "If $\sum b_k$ converges, then $\sum_{k=k_0}^{\infty} a_k$ converges," so suppose that $\sum b_k$ converges. The series $\sum_{k=k_0}^{\infty} b_k$ is obtained from $\sum b_k$ by adding (resp. subtracting) at most a finite number of terms to (resp. from) $\sum b_k$, so by Proposition 9.27 $\sum_{k=k_0}^{\infty} b_k$ also converges. Hence $\sum a_k$ converges by part (1), and Proposition 9.27 implies that $\sum_{k=k_0}^{\infty} a_k$ converges. ∎

Example 9.39. Use the Direct Comparison Test to determine whether the series

$$\sum_{k=1}^{\infty} \frac{2k+5}{5k^3+3k^2}$$

converges or diverges.

Solution. For any $k \geq 1$ we have $2k+5 \leq 2k+5k = 7k$, and from $5k^3+3k^2 \geq 5k^3$ we obtain

$$\frac{1}{5k^3+3k^2} \leq \frac{1}{5k^3}.$$

Now, combining this result with $2k+5 \leq 7k$ yields

$$0 \leq \frac{2k+5}{5k^3+3k^2} \leq \frac{7k}{5k^3+3k^2} \leq \frac{7k}{5k^3} = \frac{7}{5k^2} \leq \frac{7}{k^2}$$

for all $k \geq 1$. By Proposition 9.28(1), since $\sum_{k=1}^{\infty} 1/k^2$ is a convergent p-series we have

$$\sum_{k=1}^{\infty} \frac{7}{k^2} = 7\sum_{k=1}^{\infty} \frac{1}{k^2},$$

and so $\sum_{k=1}^{\infty} 7/k^2$ is also a convergent series. Therefore

$$\sum_{k=1}^{\infty} \frac{2k+5}{5k^3+3k^2}$$

converges by the Direct Comparison Test. ■

Theorem 9.40 (Limit Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series such that $a_k, b_k > 0$ for all k, and let $\lim_{k\to\infty} a_k/b_k = L$.*

1. *If $L \in (0,\infty)$, then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.*
2. *If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
3. *If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Proof. Suppose that $L \in (0,\infty)$. Since $L/2 > 0$ and $\lim_{k\to\infty} a_k/b_k = L$, there exists some $k_0 \in \mathbb{Z}$ such that $k \geq k_0$ implies that $|a_k/b_k - L| < L/2$. Thus for any $k \geq k_0$ we have

$$-\frac{L}{2} < \frac{a_k}{b_k} - L < \frac{L}{2},$$

which implies

$$0 < \frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2}$$

and finally

$$0 < \frac{L}{2}b_k < a_k < \frac{3L}{2}b_k$$

since $b_k > 0$. Now, if $\sum b_k$ diverges, then $\sum (L/2)b_k$ diverges also, and since $0 < (L/2)b_k < a_k$ for all $k \geq k_0$ the Direct Comparison Test concludes that $\sum a_k$ diverges. If $\sum b_k$ converges, then $\sum (3L/2)b_k$ converges also, and since $0 < a_k < (3L/2)b_k$ for all $k \geq k_0$ the Direct Comparison Test concludes that $\sum a_k$ converges. Therefore $\sum a_k$ and $\sum b_k$ either both diverge or both converges if $L \in (0,\infty)$.

Next, suppose that $L = 0$ and $\sum b_k$ converges. Since $\lim_{k\to\infty} a_k/b_k = 0$, there exists some $k_0 \in \mathbb{Z}$ such that $k \geq k_0$ implies that $|a_k/b_k| < 1$, where

$$\frac{|a_k|}{|b_k|} < 1 \quad \Rightarrow \quad |a_k| < |b_k| \quad \Rightarrow \quad 0 < a_k < b_k$$

since $a_k, b_k > 0$. Because $0 < a_k < b_k$ for all $k \geq k_0$ and $\sum b_k$ converges, by the Direct Comparison Test $\sum a_k$ also converges.

Finally, suppose that $L = \infty$ and $\sum b_k$ diverges. Since $\lim_{k\to\infty} a_k/b_k = \infty$ there exists some $k_0 \in \mathbb{Z}$ such that $k \geq k_0$ implies that $|a_k/b_k| > 1$, where

$$\frac{|a_k|}{|b_k|} > 1 \quad \Rightarrow \quad |a_k| > |b_k| \quad \Rightarrow \quad a_k > b_k > 0$$

since $a_k, b_k > 0$. Because $0 < b_k < a_k$ for all $k \geq k_0$ and $\sum b_k$ diverges, by the Direct Comparison Test $\sum a_k$ also diverges. ■

Example 9.41. Use the Limit Comparison Test to determine whether the series

$$\sum_{k=1}^{\infty} \frac{7k^3 - 3}{k^4 + 11}$$

converges or diverges.

Solution. As k grows very large the constant terms in the fraction become negligible, so that its value is very nearly equal to $7k^3/k^4 = 7/k$. Since $\sum 1/k$ is known to diverge by Proposition 9.37, we might guess that the given series diverges as well and attempt to use the Direct Comparison Test to prove it. However, constructing a winning string of inequalities to make this feasible is not necessarily easy to do. So, we use the Limit Comparison Test, with $\sum a_k$ being the given series and $\sum b_k$ being $\sum 1/k$. We evaluate L:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \left(\frac{7k^3 - 3}{k^4 + 11} \div \frac{1}{k} \right) = \lim_{k\to\infty} \frac{7k^4 - 3k}{k^4 + 11} = \lim_{k\to\infty} \frac{7 - 3/k^3}{1 + 11/k^4} = 7.$$

Hence $L = 7 \in (0, \infty)$, and so it follows from part (1) of the Limit Comparison Test that $\sum_{k=1}^{\infty} (7k^3 - 3)/(k^4 + 11)$ diverges. ■

Example 9.42. Use the Limit Comparison Test to determine whether the series

$$\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$$

converges or diverges.

Solution. Since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

by Lemma 3.18, by Proposition 2.29 we obtain

$$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = 1.$$

Then, letting $a_k = \sin(1/k)$ and $b_k = 1/k$ for integers $k \geq 1$, Theorem 9.9 implies that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = 1 \in (0, \infty).$$

We have $L = 1 \in (0, \infty)$, and since the series $\sum 1/k$ diverges it follows from part (1) of the Limit Comparison Test that $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ also diverges. ■

A series $\sum a_k$ is said to **converge absolutely** (or be **absolutely convergent**) if the series $\sum |a_k|$ converges. With the Direct Comparison Test we're in a position to prove the following useful result, which says that absolute convergence implies convergence.

Proposition 9.43. *If $\sum |a_k|$ converges, then $\sum a_k$ converges.*

Proof. Suppose that $\sum |a_k|$ converges, so $\sum |a_k| = s$ for some $s \in \mathbb{R}$. Then $\sum 2|a_k| = 2s$ by Proposition 9.28(1), so $\sum 2|a_k|$ converges also. Now, for each k we have $|a_k| = \pm a_k$. If $a_k \geq 0$, then $|a_k| = a_k$ and we obtain

$$0 \leq |a_k| + a_k = 2|a_k|.$$

If $a_k < 0$, then $|a_k| = -a_k$ and we obtain

$$0 = |a_k| + a_k \leq 2|a_k|.$$

Hence

$$0 \leq |a_k| + a_k \leq 2|a_k|$$

for all k, and since $\sum 2|a_k|$ converges, by the Direct Comparison Test $\sum(|a_k|+a_k)$ must converge; that is,

$$\sum(|a_k| + a_k) = t$$

for some $t \in \mathbb{R}$.

Finally, since $\sum |a_k|$ and $\sum(|a_k| + a_k)$ are convergent series with $\sum |a_k| = s$ and $\sum(|a_k| + a_k) = t$, by Proposition 9.28(2)

$$\sum a_k = \sum \left[(|a_k| + a_k) - |a_k|\right] = \sum(|a_k| + a_k) - \sum |a_k| = t - s \in \mathbb{R},$$

which shows that $\sum a_k$ converges. ■

It is a curious fact that most calculus textbooks give versions of the Root and Ratio Tests that are weaker than they have to be. So, here is where these notes come especially in handy, for here will be presented what are almost—but not quite—the strongest versions of these very important tests for infinite series convergence.

Theorem 9.44 (**Root Test**)**.** *Given the series $\sum a_k$, let $\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$.*

1. *If $\rho \in [0, 1)$, then $\sum |a_k|$ converges.*
2. *If $\rho \in (1, \infty]$, then $\sum a_k$ diverges.*

Proof. Suppose that

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \rho \tag{9.7}$$

for some $\rho \in [0,1)$. Let $\delta > 0$ be sufficiently small so that $r = \rho + \delta < 1$. Then there exists some integer $k_0 > 0$ such that

$$0 \le \sqrt[k]{|a_k|} < r < 1$$

for all $k \ge k_0$, whence $0 \le |a_k| < r^k$ for all $k \ge k_0$. Since $0 < r < 1$, $\sum_{k=0}^{\infty} r^k$ is a convergent geometric series by Proposition 9.31, so $\sum_{k=k_0}^{\infty} r^k$ converges by Proposition 9.27 and then the Direct Comparison Test implies that $\sum |a_k|$ converges.

Now suppose that (9.7) holds for some $\rho \in (1,\infty)$. Let $\delta > 0$ be sufficiently small so that $r = \rho - \delta > 1$. Then there exists some integer $k_0 > 0$ such that $\sqrt[k]{|a_k|} > r$ for all $k \ge k_0$, which implies that $|a_k| > r^k$ for all $k \ge k_0$. Now, since $r > 1$ we have $\lim_{k\to\infty} r^k = \infty$, whence $\lim_{k\to\infty} |a_k| = \infty$ and so $\lim_{k\to\infty} a_k \ne 0$. Therefore $\sum a_k$ diverges by the Divergence Test. ■

What is stronger about this version of the Root Test is that, unlike the book, the terms a_k of the series are not required to be nonnegative. That is, it is permissible to have $a_k < 0$, which means the test can be applied to quite a few more series! What has not changed is that if $\rho = 1$ then the test is inconclusive, but it is not necessary to spell this out in the statement of the theorem.

Example 9.45. Use the Root Test to determine whether the series

$$\sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right)^k$$

converges or diverges.

Solution. Here we have

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{\left|\sqrt{k} - \sqrt{k-1}\right|^k} &= \lim_{k\to\infty} \left(\sqrt{k} - \sqrt{k-1}\right) \\
&= \lim_{k\to\infty} \left(\frac{\sqrt{k} - \sqrt{k-1}}{1} \cdot \frac{\sqrt{k} + \sqrt{k-1}}{\sqrt{k} + \sqrt{k-1}}\right) \\
&= \lim_{k\to\infty} \frac{k - (k-1)}{\sqrt{k} + \sqrt{k-1}} = \lim_{k\to\infty} \frac{1}{\sqrt{k} + \sqrt{k-1}} = 0.
\end{aligned}$$

Since $\rho = 0 < 1$, the Root Test concludes that the series converges. ■

Theorem 9.46 (Ratio Test). *Given the series $\sum a_k$ for which $a_k = 0$ for at most a finite number of k values, let $\rho = \lim\limits_{k\to\infty} |a_{k+1}/a_k|$.*

1. *If $\rho \in [0,1)$, then $\sum |a_k|$ converges.*
2. *If $\rho \in (1,\infty]$, then $\sum a_k$ diverges.*

Proof. Suppose that $\lim_{k\to\infty}|a_{k+1}/a_k| = \rho$ for some $\rho \in [0,1)$. Let $\delta > 0$ be sufficiently small so that $r = \rho + \delta < 1$. Then there exists some integer $k_0 > 0$ such that $0 \leq |a_{k+1}/a_k| < r < 1$ for all $k \geq k_0$, so that

$$\begin{aligned}
&|a_{k_0+1}| < r|a_{k_0}|,\\
&|a_{k_0+2}| < r|a_{k_0+1}| \leq r^2|a_{k_0}|,\\
&|a_{k_0+3}| < r|a_{k_0+2}| \leq r^3|a_{k_0}|,
\end{aligned}$$

and in general,

$$|a_{k_0+k}| < r^k|a_{k_0}|$$

for all $k \geq 0$. From this we obtain

$$0 \leq |a_k| < |a_{k_0}|r^{k-k_0} = \frac{|a_{k_0}|}{r^{k_0}}r^k \tag{9.8}$$

for all $k \geq k_0$.

Since $|a_{k_0}|/r^{k_0} \neq 0$ and $0 < r < 1$, Proposition 9.31 implies that

$$\sum_{k=0}^{\infty}\frac{|a_{k_0}|}{r^{k_0}}r^k$$

is a convergent geometric series, and thus so too is $\sum_{k=k_0}^{\infty}(|a_{k_0}|/r^{k_0})r^k$ by Proposition 9.27. Hence, in light of (9.8), the Direct Comparison Test implies that $\sum|a_k|$ converges.

Now suppose $\lim_{k\to\infty}|a_{k+1}/a_k| = \rho$ for some $\rho \in (1,\infty]$. Let $\delta > 0$ be sufficiently small so that $r = \rho - \delta > 1$. Then there's some integer $k_0 > 0$ such that $1 < r \leq |a_{k+1}/a_k|$ for all $k \geq k_0$, so that

$$\begin{aligned}
&|a_{k_0+1}| > r|a_{k_0}|,\\
&|a_{k_0+2}| > r|a_{k_0+1}| \geq r^2|a_{k_0}|,\\
&|a_{k_0+3}| > r|a_{k_0+2}| \geq r^3|a_{k_0}|,
\end{aligned}$$

and in general,

$$|a_{k_0+k}| > r^k|a_{k_0}| > 0$$

for all $k \geq 0$. From $r > 1$ and $|a_{k_0}| > 0$ we have $\lim_{k\to\infty} r^k|a_{k_0}| = \infty$, so that $\lim_{k\to\infty}|a_{k_0+k}| = \infty$ and we obtain $\lim_{k\to\infty} a_k \neq 0$. Hence $\sum a_k$ diverges by the Divergence Test. ∎

Remark. Given Proposition 9.43, whenever part (1) of the Ratio Test or Root Test is used to conclude that $\sum|a_k|$ converges, it immediately follows that $\sum a_k$ converges as well.

Some books present a version of the Ratio Test which maintains that the terms of a series $\sum a_k$ must be positive, which is an unnecessarily restrictive hypothesis. All that is needed is to require that the terms a_k be nonzero for all but (at most) finitely many values of the index k, as reflected in the statement of the Ratio Test above. If there are at most finitely many terms a_k equalling 0, then there will exist some integer k_0 such that $a_k \neq 0$ for all $k \geq k_0$, and thus there need be no worries that a_k/a_{k+1} will be undefined due to division by 0.

Example 9.47. Use the Ratio Test to determine whether the series

$$\sum_{k=1}^{\infty} \frac{15^k}{(k+1)4^{2k+1}}$$

converges or diverges.

Solution. We evaluate $\lim_{k\to\infty} |a_{k+1}/a_k|$ for $a_k = 15^k/[(k+1)4^{2k+1}]$:

$$\begin{aligned}
\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|a_{k+1} \cdot \frac{1}{a_k}\right| \\
&= \lim_{k\to\infty} \left|\frac{15^{k+1}}{(k+2)4^{2k+3}} \cdot \frac{(k+1)4^{2k+1}}{15^k}\right| \\
&= \lim_{k\to\infty} \frac{15(k+1)}{(k+2)4^2} = \lim_{k\to\infty} \frac{15k+15}{16k+32} = \frac{15}{16}.
\end{aligned}$$

Since $\rho = 15/16 < 1$, the Ratio Test concludes that the series converges (absolutely). ■

Example 9.48. Use the Ratio Test to determine whether the series

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$$

converges or diverges.

Solution. Here we have

$$\begin{aligned}
\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|a_{k+1} \cdot \frac{1}{a_k}\right| \\
&= \lim_{k\to\infty} \left|\frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{(k!)^2}\right| = \lim_{k\to\infty} \frac{(k+1)!(k+1)!}{(2k+2)!} \cdot \frac{(2k)!}{k!k!} \\
&= \lim_{k\to\infty} \frac{k!(k+1) \cdot k!(k+1)}{(2k)!(2k+1)(2k+2)} \cdot \frac{(2k)!}{k!k!} = \lim_{k\to\infty} \frac{(k+1)^2}{(2k+1)(2k+2)} \\
&= \lim_{k\to\infty} \frac{k^2+2k+1}{4k^2+6k+2} = \frac{1}{4}.
\end{aligned}$$

Since $\rho = 1/4 < 1$, the Ratio Test concludes that the series converges (absolutely). ■

Example 9.49. Use the Ratio Test to determine whether the series

$$\sum_{k=1}^{\infty} e^{-k} k!$$

converges or diverges.

Solution. Here we have

$$\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{e^{-(k+1)}(k+1)!}{e^{-k}k!} = \lim_{k\to\infty} \frac{e^{-k}e^{-1}k!(k+1)}{e^{-k}k!} = \lim_{k\to\infty} \frac{k+1}{e} = \infty.$$

Since $\rho = \infty$, the Ratio Test concludes that the series diverges. ■

9.6 – Alternating Series

If (b_k) is a sequence such that $b_k > 0$ for all k, then a series of the form

$$\sum(-1)^k b_k \quad \text{or} \quad \sum(-1)^{k+1} b_k$$

is called **alternating**. Thus, the terms of an alternating series alternate between positive and negative values. An easy example is the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k},$$

which it known as the **alternating harmonic series**. Less obviously alternating is

$$\sum_{k=1}^{\infty}\frac{\cos \pi k}{\sqrt{k}}, \tag{9.9}$$

but notice that $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, and in general $\cos k\pi = (-1)^k$, so in fact (9.9) can be rewritten as

$$\sum_{k=1}^{\infty}(-1)^k\frac{1}{\sqrt{k}}.$$

The foremost tool for determining whether an alternating series converges or not is the following.

Theorem 9.50 (Alternating Series Test). *If (b_k) is such that $0 < b_{k+1} \le b_k$ for all k and $\lim_{k\to\infty} b_k = 0$, then the series*

$$\sum(-1)^{k+1} b_k$$

converges.

The proof is pending, but first notice that the test applies equally well to the series $\sum(-1)^k b_k$; that is, if the test concludes that $\sum(-1)^{k+1}b_k$ converges, then by Proposition 9.28(1) we have

$$(-1)^{-1}\sum(-1)^{k+1}b_k = \sum(-1)^{-1}(-1)^{k+1}b_k = \sum(-1)^k b_k,$$

and so it can be concluded that

$$\sum(-1)^k b_k$$

converges as well.

Example 9.51. Determine whether the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{e^{1/k}}{k}$$

converges or diverges.

Solution. Here we have $b_k = e^{1/k}/k$ for $k \geq 1$. Clearly $b_k > 0$ for all $k \geq 1$. To use the Alternating Series Test to establish convergence, we must show that $b_{k+1} \leq b_k$ for all $k \geq 1$ (i.e. $\{b_k\}$ is a nonincreasing sequence) and $\lim_{k\to\infty} b_k = 0$.

First the limit. From $\lim_{k\to\infty} 1/k = 0$ and $\lim_{k\to\infty} e^{1/k} = e^0 = 1$ we obtain

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{e^{1/k}}{k} = \lim_{k\to\infty} \frac{1}{k} \cdot \lim_{k\to\infty} e^{1/k} = 0 \cdot 1 = 0.$$

Next, observe that for each $k \geq 1$ we have $1/(k+1) < 1/k$, which implies that $e^{1/(k+1)} < e^{1/k}$ since the exponential function is strictly increasing on its domain. As a result we obtain

$$b_{k+1} = \frac{e^{1/(k+1)}}{k+1} = \frac{1}{k+1} \cdot e^{1/(k+1)} < \frac{1}{k} \cdot e^{1/k} = b_k,$$

and so $0 < b_{k+1} \leq b_k$ for all $k \geq 1$.

Therefore, by the Alternating Series Test, the series $\sum_{k=1}^{\infty} (-1)^{k+1} e^{1/k}/k$ converges. ■

Example 9.52. Determine whether the series

$$\sum_{k=3}^{\infty} (-1)^{k+1} \frac{k^2 - 4}{k^3 - 8}$$

converges or diverges.

Solution. To use the Alternating Series Test to establish convergence, we first need to establish that the sequence

$$b_k = (k^2 - 4)/(k^3 - 8), \quad k \geq 3,$$

is such that $0 < b_{k+1} \leq b_k$ for all $k \geq 3$. There are several ways to do this, but the direct approach will be taken here. To commence, observe that

$$b_k = \frac{k^2 - 4}{k^3 - 8} = \frac{(k-2)(k+2)}{(k-2)(k^2 + 2k + 4)} = \frac{k+2}{k^2 + 2k + 4},$$

which makes clear that $b_k > 0$ for all $k \geq 3$. Moreover, we have the chain of equivalencies

$$\begin{aligned} b_{k+1} \leq b_k \quad &\Leftrightarrow \quad \frac{(k+1)+2}{(k+1)^2 + 2(k+1) + 4} \leq \frac{k+2}{k^2 + 2k + 4} \quad \Leftrightarrow \quad \frac{k+3}{k^2 + 4k + 7} \leq \frac{k+2}{k^2 + 2k + 4} \\ &\Leftrightarrow \quad (k+3)(k^2 + 2k + 4) \leq (k+2)(k^2 + 4k + 7) \\ &\Leftrightarrow \quad 5k^2 + 10k + 12 \leq 6k^2 + 15k + 14 \\ &\Leftrightarrow \quad k^2 + 5k + 2 \geq 0, \end{aligned}$$

and so since $k^2 + 5k + 2 \geq 0$ is obviously true for $k \geq 3$, it follows that $b_{k+1} \leq b_k$ is true for $k \geq 3$.

The next thing to establish is that $\lim_{k\to\infty} b_k = 0$, but this is easy:

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{k^2 - 4}{k^3 - 8} = \lim_{k\to\infty} \frac{k^2 - 4}{k^3 - 8} \cdot \frac{1/k^3}{1/k^3} = \lim_{k\to\infty} \frac{1/k - 4/k^3}{1 - 8/k^3} = \frac{0 - 0}{1 - 0} = 0.$$

Therefore, by the Alternating Series Test, we conclude that

$$\sum_{k=3}^{\infty}(-1)^{k+1}\frac{k^2-4}{k^3-8}$$

converges. ■

To prove Theorem 9.50 we first need the following result.

Lemma 9.53. *Given a sequence $(a_n)_{n=1}^{\infty}$, let $s_n = \sum_{k=1}^{n} a_k$. If there is some $s \in \mathbb{R}$ such that $s_{2n} \to s$ and $s_{2n-1} \to s$ as $n \to \infty$, then the series $\sum_{k=1}^{\infty} a_k$ converges to s.*

Proof. Suppose that $s_{2n}, s_{2n-1} \to s$ as $n \to \infty$ for some $s \in \mathbb{R}$. By definition

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty}\sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n,$$

so the objective is to show that $\lim_{n\to\infty} s_n = s$.

Fix $\epsilon > 0$. Since $\lim_{n\to\infty} s_{2n} = s$, there exists some integer N_1 such that $|s_{2n}-s| < \epsilon$ whenever $n > N_1$. Also, since $\lim_{n\to\infty} s_{2n-1} = s$, there exists some integer N_2 such that $|s_{2n-1} - s| < \epsilon$ whenever $n > N_2$. Let $N = \max\{2N_1, 2N_2 - 1\}$, and suppose that $n > N$.

If n is even, then $n = 2m$ for some $m \in \mathbb{Z}$, so that

$$n > N \;\Rightarrow\; 2m > 2N_1 \;\Rightarrow\; m > N_1,$$

and we obtain

$$|s_n - s| = |s_{2m} - s| < \epsilon.$$

If n is odd, then $n = 2m - 1$ for some $m \in \mathbb{Z}$, so that

$$n > N \;\Rightarrow\; 2m-1 > 2N_2 - 1 \;\Rightarrow\; m > N_2,$$

and we obtain

$$|s_n - s| = |s_{2m-1} - s| < \epsilon.$$

Since $|s_n - s| < \epsilon$ for $n > N$, we conclude that $\lim_{n\to\infty} s_n = s$. Therefore $\sum_{k=1}^{\infty} a_k = s$. ■

In the following proof we'll assume that the series in Theorem 9.50 is specifically of the form

$$\sum_{k=1}^{\infty}(-1)^{k+1}b_k,$$

since the structure of the proof is unchanged if the series is given as

$$\sum_{k=m}^{\infty}(-1)^{k+1}b_k$$

for any integer $m \neq 1$.

Proof of the Alternating Series Test. Suppose $(b_k)_{k=1}^{\infty}$ is such that $0 < b_{k+1} \leq b_k$ for all k and $\lim_{k\to\infty} b_k = 0$. Setting $a_k = (-1)^{k+1} b_k$, we find that, for each $n \geq 1$,

$$\begin{aligned} s_{2n} := \sum_{k=1}^{2n} a_k &= a_1 + a_2 + a_3 + a_4 + \cdots + a_{2n-1} + a_{2n} \\ &= (b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{2n-1} - b_{2n}) \qquad (9.10) \\ &= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}. \qquad (9.11) \end{aligned}$$

From (9.10) it should be evident that $s_{2n} \geq 0$, since $b_k - b_{k+1} \geq 0$ for all k; and from (9.11) it can be seen that $s_{2n} < b_1$, since

$$b_2 - b_3, b_4 - b_5, \ldots, b_{2n-2} - b_{2n-1} \geq 0$$

and $b_{2n} > 0$. So the sequence $(s_{2n})_{n=1}^{\infty} = (s_2, s_4, s_6, \ldots)$ is bounded. Morever (s_{2n}) is monotone increasing:

$$s_{2n} = \sum_{k=1}^{2n} a_k = \sum_{k=1}^{2n-2} a_k + (a_{2n-1} + a_{2n}) = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2},$$

and so by the Monotone Convergence Theorem there is some $s \in \mathbb{R}$ such that

$$\lim_{n\to\infty} s_{2n} = s. \qquad (9.12)$$

As for the sequence $(s_{2n-1})_{n=1}^{\infty} = (s_1, s_3, s_5, \ldots)$, we have

$$\lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} (s_{2n} - a_{2n}) = \lim_{n\to\infty} (s_{2n} + b_{2n}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n} = s, \qquad (9.13)$$

where

$$\lim_{n\to\infty} b_{2n} = 0$$

follows from $\lim_{n\to\infty} b_n = 0$ by Proposition 9.23.

From (9.12) and (9.13) we conclude by Lemma 9.53 that $\lim_{n\to\infty} s_n = s$, and therefore

$$\sum_{k=1}^{\infty} (-1)^{k+1} b_k = \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k+1} b_k = \lim_{n\to\infty} s_n = s.$$

That is, the series converges. ■

Let $\sum_{k=m}^{\infty} (-1)^{k+1} b_k$ be a convergent alternating series, so that

$$\sum_{k=m}^{\infty} (-1)^{k+1} b_k = s$$

for some $s \in \mathbb{R}$. If

$$s_n = \sum_{k=m}^{n} (-1)^{k+1} b_k$$

for each $n \geq m$ (where m is usually 0 or 1), then the absolute error in approximating s by s_n is

$$R_n = |s - s_n|$$

and is called the ***n*th remainder** for the series.

Theorem 9.54 (**Alternating Series Estimation Theorem**). *If $\sum(-1)^{k+1}b_k$ is a convergent alternating series such that $0 \leq b_{k+1} \leq b_k$ for all k, then $R_n \leq b_{n+1}$ for all n.*

Proof. Suppose that
$$\sum_{k=1}^{\infty}(-1)^{k+1}b_k = s$$
for some $s \in \mathbb{R}$, and $0 \leq b_{k+1} \leq b_k$ for all $k \geq 1$.

Let $n \geq 1$ be an odd integer. Then
$$s_{n+1} = s_n - b_{n+1} \leq s_n$$
shows that
$$s_{n+1} \leq s_{n+m} \leq s_n \tag{9.14}$$
in the case when $m = 1$. Let $m \geq 1$ be arbitrary and suppose that (9.14) holds. If m is odd, then $n + m + 2$ is even so that
$$s_{n+m+1} = s_{n+m} + (-1)^{n+m+2}b_{n+m+1} = s_{n+m} + b_{n+m+1} \geq s_{n+m} \geq s_{n+1},$$
and $m + 1 \geq 2$ is even so that
$$\begin{aligned} s_{n+m+1} &= s_n - b_{n+1} + b_{n+2} - b_{n+3} + b_{n+4} + \cdots - b_{n+m} + b_{n+m+1} \\ &= s_n + \underbrace{(-b_{n+1} + b_{n+2})}_{\leq 0} + \underbrace{(-b_{n+3} + b_{n+4})}_{\leq 0} + \cdots + \underbrace{(-b_{n+m} + b_{n+m+1})}_{\leq 0} \leq s_n. \end{aligned}$$
Hence we have
$$s_{n+1} \leq s_{n+m+1} \leq s_n \tag{9.15}$$
if m is odd. If m is even, then $n + m + 2$ is odd so that
$$s_{n+m+1} = s_{n+m} + (-1)^{n+m+2}b_{n+m+1} = s_{n+m} - b_{n+m+1} \leq s_{n+m} \leq s_n,$$
and $m + 1 \geq 3$ is odd so that
$$s_{n+m+1} = s_{n+1} + \underbrace{(b_{n+2} - b_{n+3})}_{\geq 0} + \cdots + \underbrace{(b_{n+m} - b_{n+m+1})}_{\geq 0} \geq s_{n+1}.$$
Hence (9.15) holds if m is even, and we conclude by the principle of induction that (9.14) holds for all $m \geq 1$ in the case when n is odd. By the Squeeze Theorem
$$\lim_{m\to\infty} s_{n+1} \leq \lim_{m\to\infty} s_{n+m} \leq \lim_{m\to\infty} s_n,$$
and therefore
$$s_{n+1} \leq s \leq s_n \tag{9.16}$$
if n is odd.

Next, suppose that $n \geq 1$ is an even integer. Then
$$s_{n+1} = s_n + (-1)^{n+2}b_{n+1} = s_n + b_{n+1} \geq s_n$$

shows that

$$s_n \leq s_{n+m} \leq s_{n+1} \tag{9.17}$$

in the case when $m = 1$. Fix $m \geq 1$ and suppose (9.17) holds. If m is odd, then $n + m + 2$ is odd so that

$$s_{n+m+1} = s_{n+m} + (-1)^{n+m+2} b_{n+m+1} = s_{n+m} - b_{n+m+1} \leq s_{n+m} \leq s_{n+1},$$

and $m + 1 \geq 2$ is even so that

$$s_{n+m+1} = s_n + \underbrace{(b_{n+1} - b_{n+2})}_{\geq 0} + \cdots + \underbrace{(b_{n+m} - b_{n+m+1})}_{\geq 0} \geq s_n.$$

Hence we have

$$s_n \leq s_{n+m+1} \leq s_{n+1} \tag{9.18}$$

if m is odd. If m is even, then $n + m + 2$ is even so that

$$s_{n+m+1} = s_{n+m} + (-1)^{n+m+2} b_{n+m+1} = s_{n+m} + b_{n+m+1} \geq s_{n+m} \geq s_n,$$

and $m + 1 \geq 3$ is odd so that

$$s_{n+m+1} = s_{n+1} + \underbrace{(-b_{n+2} + b_{n+3})}_{\leq 0} + \cdots + \underbrace{(-b_{n+m} + b_{n+m+1})}_{\leq 0} \leq s_{n+1}.$$

Hence (9.18) holds if m is even, and we conclude by the principle of induction that (9.17) holds for all $m \geq 1$ in the case when n is even. By the Squeeze Theorem

$$\lim_{m\to\infty} s_n \leq \lim_{m\to\infty} s_{n+m} \leq \lim_{m\to\infty} s_{n+1},$$

and therefore

$$s_n \leq s \leq s_{n+1} \tag{9.19}$$

if n is even.

From (9.16) we obtain

$$s_{n+1} - s_n \leq s - s_n \leq 0,$$

and thus $|s - s_n| \leq |s_{n+1} - s_n|$ if n is odd. From (9.19) we obtain

$$0 \leq s - s_n \leq s_{n+1} - s_n,$$

and thus $|s - s_n| \leq |s_{n+1} - s_n|$ if n is even. That is, for all $n \geq 1$,

$$|s - s_n| \leq |s_{n+1} - s_n| = |(-1)^{n+2} b_{n+1}| = b_{n+1},$$

and the proof is done. ■

The proof for Theorem 9.54 is easily adapted to suit an alternating series with an index k that starts at some other integer other than 1, or one could simply reindex.

Example 9.55. Approximate the value of the convergent series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(3k+2)^4}$$

with an absolute error less than 10^{-5}.

Solution. We evaluate $b_k = (3k+2)^{-4}$ for successive values of k until we obtain a number less than 10^{-5}:

$$\begin{aligned}
b_1 &= [3(1)+2]^{-4} = 5^{-4} = 1.60 \times 10^{-3} \\
b_2 &= [3(2)+2]^{-4} = 8^{-4} \approx 2.44 \times 10^{-4} \\
b_3 &= [3(3)+2]^{-4} = 11^{-4} \approx 6.83 \times 10^{-5} \\
b_4 &= [3(4)+2]^{-4} = 14^{-4} \approx 2.60 \times 10^{-5} \\
b_5 &= [3(5)+2]^{-4} = 17^{-4} \approx 1.20 \times 10^{-5} \\
b_6 &= [3(6)+2]^{-4} = 20^{-4} = 6.25 \times 10^{-6}
\end{aligned}$$

Thus, by Theorem 9.54 we have

$$R_5 = |s - s_5| \le b_6 = 6.25 \times 10^{-6} < 10^{-5},$$

which is to say that the approximation

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(3k+2)^4} \approx s_5 = 5^{-4} - 8^{-4} + 11^{-4} - 14^{-4} + 17^{-4} \approx 0.00141$$

has an absolute error that is less than 10^{-5}. ∎

10
Series Functions

10.1 – Taylor Polynomials

Polynomial functions are well behaved. They are continuous everywhere, have continuous derivatives of all orders everywhere, and repeated differentiation leads always to the zero function. It also turns out that, given any function f that has continuous derivatives of all orders, and given any $x_0 \in \mathrm{Dom}(f)$, a polynomial function p can be found that approximates f to an arbitrary degree of accuracy in some neighborhood of x_0. The precise way of going about this is to construct what is called a Taylor polynomial.

Definition 10.1. *Let f be a function for which $f'(x_0), f''(x_0), \ldots, f^{(n)}(x_0) \in \mathbb{R}$. For $n \geq 0$ the* ***nth-order Taylor polynomial for f at x_0*** *is the polynomial function $p_n(\,\cdot\,; x_0)$ given by*

$$p_n(x; x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{10.1}$$

for all $x \in \mathbb{R}$, where we define $f^{(0)} = f$ and $(x - x_0)^0 = 1$.

For each $0 \leq k \leq n$ we call

$$\frac{f^{(k)}(x_0)}{k!}$$

in (10.1) the ***k*th coefficient** of $p_n(x; x_0)$. Expanding the sum in (10.1) gives

$$p_n(x; x_0) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

which makes apparent that

$$p_n(x_0; x_0) = f(x_0), \tag{10.2}$$

and also $p_0(x; x_0) = f(x_0)$ for all x.

The **remainder** $R_n(\,\cdot\,; x_0)$ associated with the nth-order Taylor polynomial for f at x_0 is given by

$$R_n(x; x_0) = f(x) - p_n(x; x_0),$$

and can be seen to be the resultant error when using p_n to approximate f for any $x \in \mathrm{Dom}(f)$. The **absolute error** in approximating $f(x)$ with $p_n(x; x_0)$ is

$$|R_n(x; x_0)| = |f(x) - p_n(x; x_0)|\,.$$

Theorem 10.2 (**Taylor's Theorem**). *Suppose f has derivatives of all orders up to $n+1$ on $[a, b]$, and let $x_0 \in [a, b]$. For each $x \in [a, b]$ with $x \neq x_0$, there exists c between x and x_0 such that*

$$f(x) = p_n(x; x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{10.3}$$

Proof. Let $x \in [a, b]$ be such that $x \neq x_0$ (so here x is a constant). Let M be the number for which

$$f(x) = p_n(x; x_0) + \frac{M}{(n+1)!}(x - x_0)^{n+1}. \tag{10.4}$$

To show is that there exists c between x and x_0 such that $M = f^{(n+1)}(c)$. Define the function $g : [a, b] \to \mathbb{R}$ by

$$g(t) = -f(x) + p_n(x; t) + \frac{M}{(n+1)!}(x - t)^{n+1}.$$

Because f has derivatives of all orders up to $n+1$ on $[a, b]$, all the terms in the polynomial

$$p_n(x; t) = f(t) + f'(t)(x - t) + \frac{f''(t)}{2!}(x - t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x - t)^n$$

are differentiable on $[a, b]$, and hence g itself is differentiable on $[a, b]$. Now, from (10.4) we see that $g(x) = 0$, while from (10.2) comes

$$g(x_0) = -f(x_0) + p_n(x_0; x_0) = -f(x_0) + f(x_0) = 0,$$

and so by the Mean Value Theorem (Theorem 4.14) there exists c between x and x_0 such that $g'(c) = 0$. Since

$$\begin{aligned}
g'(t) &= \frac{d}{dt}\left[-f(x) + f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x - t)^k + \frac{M}{(n+1)!}(x - t)^{n+1}\right] \\
&= f'(t) + \sum_{k=1}^{n}\left(\frac{f^{(k+1)}(t)}{k!}(x - t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x - t)^{k-1}\right) - \frac{M}{n!}(x - t)^n \\
&= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x - t)^k - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x - t)^k - \frac{M}{n!}(x - t)^n \\
&= \frac{f^{(n+1)}(t)}{n!}(x - t)^n - \frac{M}{n!}(x - t)^n,
\end{aligned}$$

it follows from $g'(c) = 0$ that

$$\frac{M}{n!}(x - c)^n = \frac{f^{(n+1)}(c)}{n!}(x - c)^n,$$

and therefore $M = f^{(n+1)}(c)$. ∎

Remark. Taylor's Theorem still holds, and the proof is the same, if we assume the very slightly weaker hypothesis that f has derivatives of all orders up to n on $[a, b]$, and $f^{(n)}$ is differentiable on (a, b) and continuous on $[a, b]$. If we do this, and we also set $n = 1$, then we find that Taylor's Theorem becomes precisely the statement of the Mean Value Theorem, with (10.3) becoming

$$f(x) = p_0(x; x_0) + f'(c)(x - x_0),$$

and hence

$$f'(c) = \frac{f(x) - f(x_0)}{x - x_0}$$

since $p_0(x; x_0) = f(x_0)$. Thus Taylor's Theorem can be said to be a generalization of the Mean Value Theorem.

Proposition 10.3. *Suppose that $f : [a, b] \to \mathbb{R}$, $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b) for some $n \geq 0$, $a \leq x_0, x \leq b$ with $x \neq x_0$, and I is the open interval with endpoints x and x_0. If p_n is the nth-order Taylor polynomial for f with center x_0 and there exists some $M \in \mathbb{R}$ such that $|f^{(n+1)}(t)| \leq M$ for all $t \in I$, then*

$$|R_n(x)| \leq M\frac{|x - x_0|^{n+1}}{(n + 1)!}.$$

Proof. By Taylor's Theorem there exists some c between x and x_0 such that

$$R_n(x; x_0) = f(x) - p_n(x; x_0) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1},$$

and since $c \in I$ we obtain

$$|R_n(x; x_0)| = \left|\frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1}\right| = \left|f^{(n+1)}(c)\right| \frac{|x - x_0|^{n+1}}{(n + 1)!} \leq M\frac{|x - x_0|^{n+1}}{(n + 1)!},$$

as was to be shown. ■

Example 10.4. Find the nth-order Taylor polynomial for $f(x) = (1 + x)^{-2}$ centered at 0 for $n = 1, 2, 3, 4, 5$. Then find a general expression for $p_n(x; 0)$.

Solution. First we obtain the needed derivatives for f, along with their values at 0.

$$\begin{aligned}
f'(x) &= -2(1 + x)^{-3} &&\Rightarrow\ f'(0) = -2\\
f''(x) &= 6(1 + x)^{-4} &&\Rightarrow\ f''(0) = 6\\
f'''(x) &= -24(1 + x)^{-5} &&\Rightarrow\ f'''(0) = -24\\
f^{(4)}(x) &= 120(1 + x)^{-6} &&\Rightarrow\ f^{(4)}(0) = 120\\
f^{(5)}(x) &= -720(1 + x)^{-7} &&\Rightarrow\ f^{(5)}(0) = -720
\end{aligned}$$

Thus we have

$$p_1(x; 0) = f(0) + f'(0)x = 1 - 2x,$$

$$p_2(x; 0) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 - 2x + 3x^2,$$

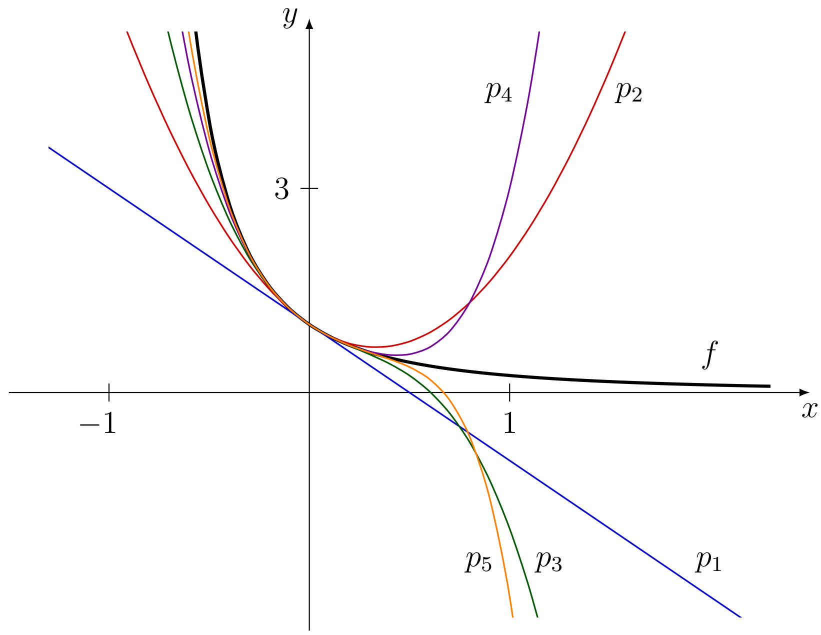


FIGURE 37.

$$p_3(x;0) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 - 2x + 3x^2 - 4x^3,$$

$$p_4(x;0) = p_3(x;0) + \frac{f^{(4)}(0)}{4!} = 1 - 2x + 3x^2 - 4x^3 + 5x^4,$$

$$p_5(x;0) = p_4(x;0) + \frac{f^{(5)}(0)}{5!} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5.$$

Figure 37 shows the graphs of these Taylor polynomials. It can be seen that p_n provides a better approximation for f in the neighborhood of 0 as n increases. It can also be seen that the terms in the Taylor polynomials fit a pattern, so that we may conjecture that

$$p_n(x;0) = \sum_{k=0}^{n} (-1)^k (k+1)x^k$$

for each $n \geq 0$. This can be proven formally by induction. ■

Example 10.5. Find the nth-order Taylor polynomial for $f(x) = \sin(x)$ centered at 0 for $n = 0, 1, 2, 3, 4, 5$. Then find a general expression for $p_n(x)$.

Solution. First we obtain the needed derivatives for f, along with their values at 0.

$$\begin{aligned}
f'(x) &= \cos(x) &&\Rightarrow\quad f'(0) = 1 \\
f''(x) &= -\sin(x) &&\Rightarrow\quad f''(0) = 0 \\
f'''(x) &= -\cos(x) &&\Rightarrow\quad f'''(0) = -1 \\
f^{(4)}(x) &= \sin(x) &&\Rightarrow\quad f^{(4)}(0) = 0 \\
f^{(5)}(x) &= \cos(x) &&\Rightarrow\quad f^{(5)}(0) = 1
\end{aligned}$$

As usual we define $f^{(0)}(x) = f(x)$, and so we have

$$\begin{aligned}
p_0(x;0) &= f(0) = 0,\\
p_1(x;0) &= f(0) + f'(0)x = x,\\
p_2(x;0) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = x,\\
p_3(x;0) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - \frac{x^3}{6},\\
p_4(x;0) &= p_3(x;0) + \frac{f^{(4)}(0)}{4!} = x - \frac{x^3}{6},\\
p_5(x;0) &= p_4(x;0) + \frac{f^{(5)}(0)}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}.
\end{aligned}$$

If we write

$$p_5(x;0) = (0)\frac{x^0}{0!} + (1)\frac{x^1}{1!} + (0)\frac{x^2}{2!} + (-1)\frac{x^3}{3!} + (0)\frac{x^4}{4!} + (1)\frac{x^5}{5!} + \cdots,$$

then generally we have

$$p_n(x;0) = \sum_{k=0}^{n} \frac{\sigma(k)}{k!}x^k,$$

where σ is the function given by

$$\sigma(k) = (-1)^{\llbracket k/2 \rrbracket}\left[\frac{(-1)^{k+1}+1}{2}\right]. \tag{10.5}$$

In particular, $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(2) = 0$, $\sigma(3) = -1$, $\sigma(4) = 0$, $\sigma(5) = 1$, and so on. ■

Example 10.6. Let $f(x) = (1+x)^{-2}$ as in Example 10.4. Find an upper bound on the absolute error that may be incurred by approximating $f(0.1)$ using the 4th-order Taylor polynomial for f with center 0.

Solution. The absolute error in question is $|R_4(0.1;0)| = |f(0.1) - p_4(0.1;0)|$. From Example 10.4 we found that $f^{(5)}(x) = -720(1+x)^{-7}$, which certainly is continuous on $[0, 0.1]$ and differentiable on $(0, 0.1)$. Now, for any $0 < t < 0.1$,

$$\left|f^{(5)}(t)\right| = \frac{720}{|1+t|^7} \le \frac{720}{|1+0|^7} = 720,$$

and so by Proposition 10.3 we obtain an upper bound on $|R_4(x;0)|$:

$$|R_4(x;0)| \le (720)\frac{|0.1-0|^5}{(4+1)!} = \frac{720(0.1)^5}{5!} = 6 \times 10^{-5}.$$

Thus, if we use $p_4(0.1;0)$ to estimate $f(0.1)$, the absolute error will be no greater than 6×10^{-5}.

Of course, nothing here prevents us from actually calculating the absolute error in this case. Since

$$f(0.1) = (1+0.1)^{-2} \approx 0.826446281$$

and
$$p_4(0.1) = 1 - 2(0.1) + 3(0.1)^2 - 4(0.1)^3 + 5(0.1)^4 = 0.8265,$$
it can be seen that the absolute error is about 5.3719×10^{-5}. This is indeed less than the upper bound 6×10^{-5}. ■

Example 10.7. Find an upper bound on the absolute error in approximating $f(x) = (1+x)^{-2}$ on the interval $[-0.3, 0.3]$ using the 4th-order Taylor polynomial for f with center 0.

Solution. The goal here is to find some number N such that $|R_4(x;0)| \le N$ for all $x \in [-0.3, 0.3]$. Example 10.4 gives an expression for $f^{(5)}$, which is seen to be continuous on $[-0.3, 0.3]$ and differentiable on $(-0.3, 0.3)$, and so Proposition 10.3 can be used with $x_0 = 0$ in order to determine a value for N.

Fix $x \in [-0.3, 0.3]$, and let I be the open interval with endpoints 0 and x. For any $t \in I$ we have
$$\left|f^{(5)}(t)\right| = \frac{720}{|1+t|^7} \le \frac{720}{|1+(-0.3)|^7} \approx 8742.7 < 8742.8,$$
since $t \in I$ implies that $-0.3 \le t \le 0.3$, and so by Proposition 10.3 and the fact that $|x| \le 0.3$ we obtain
$$|R_4(x;0)| \le (8742.8)\frac{|x|^5}{5!} \le (8742.8)\frac{0.3^5}{5!} \approx 0.1770 < 0.1771.$$
Now, since $x \in [-0.3, 0.3]$ is arbitrary, it follows that $|R_4(x;0)| \le 0.1771$ for all $-0.3 \le x \le 0.3$. That is, 0.1771 serves as an upper bound on the absolute error in approximating f on $[-0.3, 0.3]$ using p_4.

In the case when $x = -0.3$ we have $f(-0.3) = (1-0.3)^{-2} \approx 2.0408$ and $p_4(-0.3; 0) = 2.0185$, and so the absolute error is 0.0223—well less than 0.1771. ■

10.2 – Power Series

Definition 10.8. *An infinite series of the form*

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k \tag{10.6}$$

is a ***power series with center*** x_0*, and the* c_k *values are the* ***coefficients*** *of the power series.*

Just as an infinite series as defined in Section 9.3 need not have its index k start at 1, the index of a power series as defined here does not need to start at 0. However having the initial value of k be 0 is the most common scenario.

Power series may be used to define functions. That is, we can define a function f by

$$f(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k,$$

with the understanding that the domain of f consists of the set of all $x \in \mathbb{R}$ for which the series converges. Define

$$S = \left\{ x \in \mathbb{R} : \sum_{k=0}^{\infty} c_k(x-x_0)^k \in \mathbb{R} \right\}$$

to be the "set of convergence" for the series (10.6). As the next theorem makes clear, given any power series (10.6) the set S can only ever be $\{x_0\}$, $(-\infty, \infty)$, or an interval with endpoints $x_0 - R$ and $x_0 + R$ for some $R > 0$. Here R is called the **radius of convergence** of the power series. We define $R = 0$ if $S = \{x_0\}$, and $R = \infty$ if $S = (-\infty, \infty)$.

Theorem 10.9. *A power series* $\sum c_k(x-x_0)^k$ *exhibits one of three behaviors:*

1. *The series converges absolutely for all* $x \in \mathbb{R}$*, so that* $S = (-\infty, \infty)$ *and* $R = \infty$.
2. *The series converges only for* $x = x_0$*, so that* $S = \{x_0\}$ *and* $R = 0$.
3. *For some* $0 < R < \infty$ *the series converges absolutely for all* $x \in (x_0 - R, x_0 + R)$ *and diverges for all* $x \in (-\infty, x_0 - R) \cup (x_0 + R, \infty)$.

In part (3) of the theorem notice that nothing is said about whether the power series converges at $x = x_0 \pm R$, and that is because nothing can be said in general. If part (3) applies to a particular power series, then the set of convergence S of the series will be an **interval of convergence** that may be of the form $(x_0 - R, x + R)$, $[x_0 - R, x + R)$, $(x_0 - R, x + R]$, or $[x_0 - R, x + R]$. The endpoints $x_0 - R$ and $x_0 + R$ will have to be investigated individually to determine whether the series converges or diverges there.

To determine for what values of x a power series converges, we only need the tests for convergence that were developed in the previous chapter.

Example 10.10. Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k^3}, \tag{10.7}$$

and state the radius of convergence.

Solution. Clearly the series converges when $x = 0$. Assuming $x \neq 0$, we can employ the Ratio Test with $a_k = (-1)^{k-1}x^k/k^3$:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^k x^{k+1}}{(k+1)^3}\cdot\frac{k^3}{(-1)^{k-1}x^k}\right| = \lim_{k\to\infty}\left|\frac{-k^3x}{(k+1)^3}\right| = \lim_{k\to\infty}\frac{k^3}{(k+1)^3}|x| = |x|,$$

since

$$\lim_{k\to\infty}\frac{k^3}{(k+1)^3} = 1.$$

Thus the series converges if $|x| < 1$, or equivalently $-1 < x < 1$. The Ratio Test is inconclusive when $x = -1$ or $x = 1$, so we analyze these endpoints separately.

When $x = -1$ the series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(-1)^k}{k^3} = \sum_{k=1}^{\infty}\frac{(-1)^{2k-1}}{k^3} = \sum_{k=1}^{\infty}\frac{-1}{k^3}.$$

Recall that $\sum 1/k^3$ is a convergent p-series, and thus $\sum 1/k^3 = s$ for some $s \in \mathbb{R}$. Now Proposition 9.12 implies that

$$\sum_{k=1}^{\infty}\frac{-1}{k^3} = -\sum_{k=1}^{\infty}\frac{1}{k^3} = -s,$$

which shows that $\sum -1/k^3$ also converges.

When $x = 1$ the series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3},$$

which is an alternating series $\sum(-1)^{k-1}b_k$ with $b_k = 1/k^3$. Since $\lim_{k\to\infty} b_k = 0$ and

$$b_{k+1} = \frac{1}{(k+1)^3} < \frac{1}{k^3} = b_k$$

for all k, by the Alternating Series Test this series converges.

We conclude that the series (10.7) converges on the interval $[-1, 1]$, and the radius of convergence is $R = \frac{1}{2}|1-(-1)| = 1$. ■

Example 10.11. Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty}(-1)^k\frac{(x+2)^k}{k\cdot 2^k}, \tag{10.8}$$

and state the radius of convergence.

Solution. Clearly the series converges when $x = -2$. Assuming $x \neq -2$, we can employ the Ratio Test with

$$a_k = (-1)^k\frac{(x+2)^k}{k\cdot 2^k}$$

to obtain

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(x+2)^{k+1}}{(k+1)\cdot 2^{k+1}}\cdot\frac{k\cdot 2^k}{(-1)^k(x+2)^k}\right| \\ &= \lim_{k\to\infty}\left|\frac{(-1)(x+2)}{2(k+1)}\cdot\frac{k}{1}\right| = \lim_{k\to\infty}\frac{k}{2k+2}|x+2|\end{aligned}$$

$$= \frac{1}{2}|x+2|.$$

Thus the series converges if $\frac{1}{2}|x+2| < 1$, which implies $|x+2| < 2$ and thus $-4 < x < 0$. The Ratio Test is inconclusive when $x = -4$ or $x = 0$, so we analyze these endpoint separately.

When $x = -4$ the series becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k(-2)^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{2^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{1}{k},$$

which is the harmonic series and therefore diverges.

When $x = 0$ the series becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k},$$

which is an alternating series $\sum (-1)^k b_k$ with $b_k = 1/k$. Since $\lim_{k\to\infty} b_k = 0$ and

$$b_{k+1} = \frac{1}{k+1} < \frac{1}{k} = b_k$$

for all k, by the Alternating Series Test this series converges.

Therefore the series (10.8) converges on the interval $(-4, 0]$, and the radius of convergence is $R = \frac{1}{2}|0 - (-4)| = 2$. ■

Example 10.12. Find the interval of convergence of the power series

$$\sum_{k=2}^{\infty} \frac{x^k}{(\ln k)^k}, \tag{10.9}$$

and state the radius of convergence.

Solution. In this case it would be easier to use the Root Test with $a_k = x^k/(\ln k)^k$; so, for any $x \in \mathbb{R}$, we obtain

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left|\frac{x^k}{(\ln k)^k}\right|} = \lim_{k\to\infty} \sqrt[k]{\frac{|x|^k}{|\ln k|^k}} = \lim_{k\to\infty} \frac{|x|}{|\ln k|} = 0,$$

since $\ln k \to \infty$ as $k \to \infty$. Thus, the series (10.9) converges for all real x, which implies that the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$. ■

Theorem 10.13. *Suppose the series $\sum_{k=0}^{\infty} c_k(x-x_0)^k$ converges on an interval I, and define $f : I \to \mathbb{R}$ by $f(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k$.*

1. *f is continuous on I.*
2. *f is differentiable on* $\operatorname{Int}(I)$*, with*

$$f'(x) = \sum_{k=1}^{\infty} k c_k (x-x_0)^{k-1}.$$

for all $x \in \operatorname{Int}(I)$.

3. *f is integrable on* $\text{Int}(I)$, *with*

$$\int f = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-x_0)^{k+1} + c$$

for arbitrary constant c.

The first part of Theorem 10.13 states that, for any $x \in \text{Int}(I)$,

$$\lim_{t\to x} \sum_{k=0}^{\infty} c_k(t-x_0)^k = \lim_{t\to x} f(t) = f(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k = \sum_{k=0}^{\infty} \lim_{t\to x} c_k(t-x_0)^k, \tag{10.10}$$

which is to say the limit of a convergent series can be carried out "termwise" so long as the limit operates in the interior of the interval of convergence I of the series. If x is an endpoint of I then the appropriate one-sided limit is executed in (10.10) instead. The second and third parts of the theorem state that a convergent power series can be differentiated and integrated "termwise" on the interior of I, meaning the differentiation or integration operator can be brought inside the series:

$$\frac{d}{dx}\sum_{k=0}^{\infty} c_k(x-x_0)^k = \sum_{k=0}^{\infty} \frac{d}{dx}\left[c_k(x-x_0)^k\right] = \sum_{k=0}^{\infty} kc_k(x-x_0)^{k-1} = \sum_{k=1}^{\infty} kc_k(x-x_0)^{k-1},$$

and

$$\int \left[\sum_{k=0}^{\infty} c_k(x-x_0)^k\right] dx = \sum_{k=0}^{\infty}\left[\int c_k(x-x_0)^k\,dx\right] + c = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-x_0)^{k+1} + c.$$

Moreover the new series that results from differentiating or integrating the old series will be convergent on $\text{Int}(I)$ as mentioned in the theorem, although nothing can be said in general about the behavior of the new series at the endpoints of I.

Example 10.14. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\tfrac{1}{2}\right).$$

Solution. Recall that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for any $x \in (-1,1)$. Since $-\ln(1-x)$ is an antiderivative for $(1-x)^{-1}$ on $(-1,1)$, for $x \in (-1,1)$ the Fundamental Theorem of Calculus gives

$$\int_0^x \left(\sum_{n=0}^{\infty} t^n\right) dt = \int_0^x \frac{1}{1-t}\,dt = \big[-\ln(1-t)\big]_0^x = -\ln(1-x). \tag{10.11}$$

On the other hand Theorem 10.13(3) shows that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \tag{10.12}$$

is an antiderivative for $\sum_{n=0}^{\infty} x^n$ on $(-1,1)$, and so

$$\int_0^x \left(\sum_{n=0}^{\infty} t^n\right) dt = \left[\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}\right]_0^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \tag{10.13}$$

for $x \in (-1,1)$. Combining (10.11) and (10.13) gives

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x) \tag{10.14}$$

for $x \in (-1,1)$.

Using the Ratio Test and Alternating Series Test, it is straightforward to show that the series in (10.12) converges on $[-1,1)$. Thus by Theorem 10.13(1) the function f is continuous on $[-1,1)$, and so in particular

$$\lim_{x \to -1^+} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \lim_{x \to -1^+} f(x) = f(-1) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}. \tag{10.15}$$

But by (10.14) we also have

$$\lim_{x \to -1^+} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \lim_{x \to -1^+} \big[-\ln(1-x)\big] = -\ln(2) \tag{10.16}$$

since the natural logarithm function is continuous on its domain. Combining (10.15) and (10.16) gives

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = -\ln(2).$$

Now we simply observe that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

and $\ln\left(\frac{1}{2}\right) = -\ln(2)$ to obtain the desired result. ■

10.3 – Taylor Series

Definition 10.15. *Let f be a function that has derivatives of all orders on an open interval I containing x_0. Then the power series of the form*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

*is the **Taylor series for** f **centered at** x_0. A Taylor series centered at 0 is called a **Maclaurin series**.*

Recalling the definition for the nth-order Taylor polynomial $P_n(x)$ for f with center x_0 given in Section 10.1, it can be seen that

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \lim_{n\to\infty} P_n(x).$$

In what follows recall that $R_n(x) = f(x) - P_n(x)$.

Theorem 10.16. *Let $P_n(x)$ be the nth-order Taylor polynomial for f centered at x_0. If $\lim_{n\to\infty} R_n(x) = 0$ for all x on an open interval I containing x_0, then*

$$f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

for all $x \in I$.

Proof. Suppose that $\lim_{n\to\infty} R_n(x) = 0$ for all x on an open interval I. Then $P_n(x)$ exists for all $n \geq 0$ and $x \in I$, which implies that f has derivatives of all orders on I and so the Taylor series for f centered at x_0 exists. Let $x \in I$. Then

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k &= \lim_{n\to\infty}\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \lim_{n\to\infty} P_n(x) \\ &= \lim_{n\to\infty}[f(x) - R_n(x)] = \lim_{n\to\infty} f(x) - \lim_{n\to\infty} R_n(x) \\ &= f(x) - 0 = f(x).\end{aligned}$$

Therefore $f(x)$ equals the value at x of Taylor series for f centered at x_0. ∎

Example 10.17. Show that

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

for all $x \in (-\infty, \infty)$.

Solution. In Example 10.5 we determined that the nth-order Taylor polynomial centered at 0 for the sine function is

$$P_n(x;0) = \sum_{k=0}^{n} \frac{\sigma(k)}{k!} x^k,$$

with σ defined by equation (10.5). Fix $n \geq 0$. We have that $\sin^{(n)}$ is continuous and differentiable on $(-\infty, \infty)$. Let $x \neq 0$, and let I be the open interval with endpoints 0 and x. Since $|\sin^{(n+1)}(t)|$ is either $|\sin t|$ or $|\cos t|$, it is clear that $|\sin^{(n+1)}(t)| \leq 1$ for all $t \in I$. Therefore

$$|R_n(x;0)| \leq 1 \cdot \frac{|x-0|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!} \tag{10.17}$$

by Proposition 10.3, and since $n \geq 0$ is arbitrary we conclude that the inequality (10.17) holds for all $n \geq 0$. Observing that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0,$$

by the Squeeze Theorem we obtain

$$\lim_{n\to\infty} |R_n(x;0)| = 0$$

and hence $R_n(x;0) \to 0$ as $n \to \infty$ for all $x \neq 0$. However, because $P_n(0;0) = 0 = \sin 0$ for all $n \geq 0$, we have $R_n(0;0) = 0$ for all $n \geq 0$ and thus

$$\lim_{n\to\infty} R_n(x;0) = 0$$

holds for all $x \in (-\infty, \infty)$. So, by Theorem 10.16 we conclude that

$$\sin(x) = \lim_{n\to\infty} P_n(x;0) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\sigma(k)}{k!} x^k$$

for all $x \in (-\infty, \infty)$.

Finally, it's a straightforward enough matter to verify that, for any $n \geq 0$,

$$\sum_{k=0}^{2n+1} \frac{\sigma(k)}{k!} x^k = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

and therefore

$$\sin(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\sigma(k)}{k!} x^k = \lim_{n\to\infty} \sum_{k=0}^{2n+1} \frac{\sigma(k)}{k!} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

as desired. ∎

10.4 – Applications of Taylor Series

Example 10.18. Approximate the value of the definite integral

$$\int_0^{0.2} \sin(x^2)\,dx$$

with an absolute error less than 10^{-10}.

Solution. In Example 10.17 we found that

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

for all $x \in (-\infty, \infty)$, and thus we have

$$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} \tag{10.18}$$

for all $-\infty < x < \infty$. In particular the series at right in (10.18) converges on $(-\infty, \infty)$, and so by Theorem 10.13(3)

$$\int \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} \right] dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!} + c$$

for all $x \in (-\infty, \infty)$ and arbitrary constant c. Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_0^{0.2} \sin(x^2)\,dx &= \int_0^{0.2} \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} \right] dx = \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!} \right]_0^{0.2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (0.2)^{4k+3}}{(4k+3)(2k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k (0)^{4k+3}}{(4k+3)(2k+1)!} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{0.2^{4k+3}}{(4k+3)(2k+1)!}
\end{aligned}$$

We have arrived at an alternating series $\sum (-1)^k b_k$ with

$$b_k = \frac{0.2^{4k+3}}{(4k+3)(2k+1)!}$$

for $k \geq 0$. Evaluating the first few b_k values,

$$\begin{aligned}
b_0 &= 0.2^3/(3 \cdot 1!) \approx 2.6667 \times 10^{-3} \\
b_1 &= 0.2^7/(7 \cdot 3!) \approx 3.0476 \times 10^{-7} \\
b_2 &= 0.2^{11}/(11 \cdot 5!) \approx 1.5515 \times 10^{-11}
\end{aligned}$$

By the Alternating Series Estimation Theorem the approximation

$$\sum_{k=0}^{\infty} (-1)^k \frac{0.2^{4k+3}}{(4k+3)(2k+1)!} \approx b_0 - b_1 = \frac{0.2^3}{3} - \frac{0.2^7}{42}$$

will have an absolute error that is less than $b_2 \approx 1.5515 \times 10^{-11} < 10^{-10}$. Therefore the approximation

$$\int_0^{0.2} \sin(x^2)\,dx \approx \frac{0.2^3}{3} - \frac{0.2^7}{42}$$

has an absolute error less than 10^{-10}. In fact it will be less than 1.6×10^{-11}! ■

11

Parametric and Polar Curves

11.1 – Parametric Equations

Definition 11.1. ***Parametric equations*** *are a set of equations that define a list of quantitative variables $x_1, \ldots, x_n$ as functions of independent variables $t_1, \ldots, t_m$ called* ***parameters****.*

In the xy-plane there are two obvious quantitative variables: x and y, which together specify a point (x, y). If we define x and y as functions of a single parameter t, then the resultant equations $x = f(t)$ and $y = g(t)$ form a set of two parametric equations. Usually we also specify an interval I for the parameter t, and write

$$(x, y) = (f(t), g(t)), \quad t \in I \tag{11.1}$$

to indicate that $x = f(t)$ and $y = g(t)$ for all $t \in I$. The **graph** of the parametric equations in (11.1) is the set Γ of points in the xy-plane given by

$$\Gamma = \big\{(f(t), g(t)) : t \in I\big\}.$$

We call Γ a **curve** if both f and g are continuous on I.

Conversely, given a set of points C that is a curve in the xy-plane, if there exists an interval I and continuous functions $f, g : I \to \mathbb{R}$ such that the parametric equations (11.1) have graph equal to C, then (11.1) is a **parametrization** of the curve C.

Example 11.2. The graph of the parametric equations

$$(x, y) = (\sin 2t, 2\sin t), \quad t \in [0, 2\pi],$$

is the curve in the xy-plane shown in Figure 38. In general curves defined by parametric equations of the form

$$(x, y) = \big(A\sin(at + \delta), B\sin(bt)\big), \quad t \in [0, 2\pi],$$

for constants A, B, a, b, δ are called **Lissajous curves**. ■

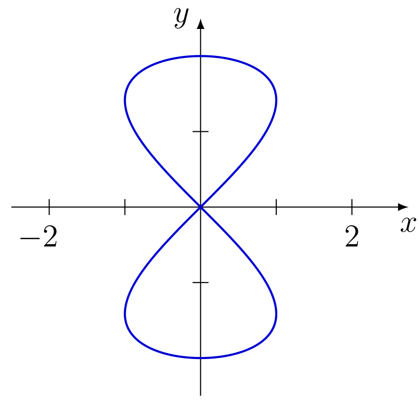


FIGURE 38. A Lissajous curve.

Any real-valued function φ defined on an interval I has graph

$$\Gamma = \{(x, \varphi(x)) : x \in I\},$$

and it is easy to see that the parametric equations

$$(x, y) = (t, \varphi(t)), \quad t \in I \tag{11.2}$$

have the same graph Γ. We call (11.2) a parametrization of the function φ, and in general any set of parametric equations whose graph equals the graph of φ is a **parametrization** of φ.

Recall that a function φ is **continuously differentiable** on an open interval I if φ is differentiable on I and φ' is continuous on I. Suppose we have parametric equations given by (11.1), with f and g both continuously differentiable on $\text{Int}(I)$. If $f'(\tau) \neq 0$ for some $\tau \in \text{Int}(I)$, then the continuity of f' implies there is some $\delta > 0$ such that either $f' > 0$ on $I_0 := (\tau - \delta, \tau + \delta)$ or $f' < 0$ on I_0. Thus f is either increasing or decreasing on I_0, and so is one-to-one there. Letting $f(I_0) = J_0$, it follows that $f : I_0 \to J_0$ has an inverse $f^{-1} : J_0 \to I_0$ such that

$$f(t) = x \quad \Leftrightarrow \quad f^{-1}(x) = t$$

for all $t \in I_0$ and $x \in J_0$, where J_0 is an open interval by Lemma 7.8. Then

$$y = g(t) = g(f^{-1}(x)) = (g \circ f^{-1})(x) \tag{11.3}$$

for all $t \in I_0$ and $x \in J_0$. We now see that y is a function of x, at least for $x \in J_0$, and write

$$y(x) = (g \circ f^{-1})(x)$$

for $x \in J_0$.

Now, since f is one-to-one and differentiable on I_0, with $f'(t) \neq 0$ for each $t \in I_0$, Theorem 7.10 implies f^{-1} is differentiable at each $x = f(t) \in J_0$ and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{11.4}$$

Thus, by the Chain Rule, equations (11.3) and (11.4) imply that

$$y'(x) = (g \circ f^{-1})'(x) = g'(f^{-1}(x)) \cdot (f^{-1})'(x) = \frac{g'(f^{-1}(x))}{f'(f^{-1}(x))} \tag{11.5}$$

for all $x \in J_0$. In particular we have

$$y'(f(\tau)) = \frac{g'(\tau)}{f'(\tau)},$$

and since τ is an arbitrary point in $\mathrm{Int}(I)$ for which $f'(\tau) \neq 0$, we have proven the following theorem.

Theorem 11.3. *Let I be an interval, let $(x, y) = (f(t), g(t))$ for all $t \in I$, and suppose f and g are continuously differentiable on $\mathrm{Int}(I)$. Then*

$$y'(f(t)) = \frac{g'(t)}{f'(t)} \tag{11.6}$$

for all $t \in \mathrm{Int}(I)$ such that $f'(t) \neq 0$.

So long as it's understood that $x = f(t)$, equation (11.6) may be written as

$$y'(x) = \frac{g'(t)}{f'(t)}. \tag{11.7}$$

It is common practice to denote parametric equations simply as $(x(t), y(t))$, so that $x = x(t)$ and $y = y(t)$. Then (11.7) becomes

$$y'(x) = \frac{y'(t)}{x'(t)} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

To have $y'(x)$ and $y'(t)$ in the same formula can be confusing, but bear in mind that the symbol $y'(x)$ in (11.7) is really $(g \circ f^{-1})'(x)$, as we see in (11.5).

Theorem 11.3 allows us to compute the usual slope of a curve C defined by parametric equations $(x, y) = (f(t), g(t))$ in the standard xy-plane, wherever the slope exists. By (11.6), the curve C has a horizontal tangent line at the point $(f(t), g(t))$ if and only if $y'(f(t)) = 0$, and the latter occurs if and only if $g'(t) = 0$ and $f'(t) \neq 0$. Also C has a vertical tangent line at $(f(t), g(t))$ if and only if $g'(t) \neq 0$ and $f'(t) = 0$. The key to seeing this begins with the observation that Theorem 11.3 has the following symmetrical result.

Corollary 11.4. *Let I be an interval, let $(x, y) = (f(t), g(t))$ for all $t \in I$, and suppose f and g are continuously differentiable on $\mathrm{Int}(I)$. Then*

$$x'(g(t)) = \frac{f'(t)}{g'(t)}$$

for all $t \in \mathrm{Int}(I)$ such that $g'(t) \neq 0$.

Example 11.5. Consider the curve C given by the parametric equations

$$(x, y) = (\sin 2t, 2 \sin t), \quad t \in [0, 2\pi],$$

first considered in Example 11.2.

(a) Find an expression for the slope of C at any point $(f(t), g(t))$, $t \in [0, 2\pi]$.
(b) Find the points on C where the tangent line is horizontal.
(c) Find the points on C where the tangent line is vertical.

Solution.
(a) Here $x = f(t) = \sin 2t$ and $y = g(t) = 2\sin t$. By Theorem 11.3,

$$y'(f(t)) = \frac{g'(t)}{f'(t)} = \frac{\cos t}{\cos 2t}$$

for all $t \in (0, 2\pi)$ such that $\cos 2t \neq 0$. What about the points on C corresponding to $t = 0$ and $t = 2\pi$? These two values for t in fact correspond to the same point $(0, 0)$:

$$(f(0), g(0)) = \big(\sin(2\cdot 0), 2\sin 0\big) = (0, 0) = \big(\sin(2\cdot 2\pi), 2\sin(2\pi)\big) = (f(2\pi), g(2\pi)).$$

In order to treat this point in a proper fashion using Theorem 11.3, we note that another parametrization for C is

$$(x, y) = (\sin 2t, 2\sin t), \quad t \in [-\pi, \pi].$$

Theorem 11.3 now implies that

$$y'(f(t)) = \frac{g'(t)}{f'(t)} = \frac{\cos t}{\cos 2t}$$

for all $t \in (-\pi, \pi)$ such that $\cos 2t \neq 0$, which includes $t = 0$ and hence the point $(0, 0) \in C$. Therefore

$$y'(f(t)) = \frac{\cos t}{\cos 2t} \tag{11.8}$$

for all $t \in [0, 2\pi]$ such that $\cos 2t \neq 0$.

(b) To find points on C where the tangent line is horizontal, by (11.8) we find $t \in [0, 2\pi]$ for which $\cos t = 0$ and $\cos 2t \neq 0$. The solutions are $t = \frac{\pi}{2}, \frac{3\pi}{2}$, which yield the points

$$\big(f(\tfrac{\pi}{2}), g(\tfrac{\pi}{2})\big) = (0, 2) \quad \text{and} \quad \big(f(\tfrac{3\pi}{2}), g(\tfrac{3\pi}{2})\big) = (0, -2).$$

(c) To find points on C where the tangent line is vertical, by (11.8) we find $t \in [0, 2\pi]$ for which $\cos t \neq 0$ and $\cos 2t = 0$. The solutions are $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, which yield the points

$$\big(f(\tfrac{\pi}{4}), g(\tfrac{\pi}{4})\big) = \big(\sin\tfrac{\pi}{2}, 2\sin\tfrac{\pi}{4}\big) = \big(1, \sqrt{2}\,\big), \quad \big(f(\tfrac{3\pi}{4}), g(\tfrac{3\pi}{4})\big) = \big(-1, \sqrt{2}\,\big),$$

$$\big(f(\tfrac{5\pi}{4}), g(\tfrac{5\pi}{4})\big) = \big(1, -\sqrt{2}\,\big), \quad \big(f(\tfrac{7\pi}{4}), g(\tfrac{7\pi}{4})\big) = \big(-1, -\sqrt{2}\,\big).$$

See again Figure 38. ■

11.2 – Polar Coordinates

In what follows we will say that two pairs of polar coordinates (r_1, θ_1) and (r_2, θ_2) are **equivalent**, and write $(r_1, \theta_1) \sim (r_2, \theta_2)$, if they have the same rectangular coordinates. Thus $(1, 0) \sim (1, 2\pi)$, since both lie at $(x, y) = (1, 0)$; and $(3, \pi/4) \sim (-3, 5\pi/4)$, since both lie at $(x, y) = (3/\sqrt{2}, 3/\sqrt{2})$.

Example 11.6. Find the smallest $\theta_0 > 0$ for which the set

$$S = \{(r, \theta) : 0 \leq \theta \leq \theta_0 \text{ and } r = 4\cos 3\theta\}$$

will equal the graph of the polar equation $r = 4\cos 3\theta$.

Solution. When $\theta = 0$ we find that $r = 4\cos 0 = 4$, so the point $(r, \theta) = (4, 0)$ lies on the graph of the equation. If we denote r by $f(\theta)$, so that $f(\theta) = 4\cos 3\theta$, then what must be done is to find the smallest $\theta_0 > 0$ such that $(f(\theta_0), \theta_0) \sim (4, 0)$. We start by examining the positive values of θ for which $f(\theta) = 4\cos 3\theta = \pm 4$ (recall that both $r = 4$ and $r = -4$ imply a distance of 4 from the origin):

$$\begin{aligned} f(\theta) = 4 \;&\Rightarrow\; 4\cos 3\theta = 4 \;\Rightarrow\; \cos 3\theta = 1 \;\Rightarrow\; 3\theta = 2\pi, 4\pi, 6\pi, \ldots \\ &\Rightarrow\; \theta = 2\pi/3, 4\pi/3, 2\pi, \ldots \end{aligned} \tag{11.9}$$

and

$$\begin{aligned} f(\theta) = -4 \;&\Rightarrow\; 4\cos 3\theta = -4 \;\Rightarrow\; \cos 3\theta = -1 \;\Rightarrow\; 3\theta = \pi, 3\pi, 5\pi, \ldots \\ &\Rightarrow\; \theta = \pi/3, \pi, 5\pi/3, \ldots \end{aligned} \tag{11.10}$$

The smallest value in (11.9) which when paired with $r = 4$ gives coordinates equivalent to $(4, 0)$ is $\theta = 2\pi$; that is, $(4, 2\pi) \sim (4, 0)$. The smallest value in (11.10) which when paired with $r = -4$ gives coordinates equivalent to $(4, 0)$ is $\theta = \pi$; that is, $(-4, \pi) \sim (4, 0)$. Thus we conclude

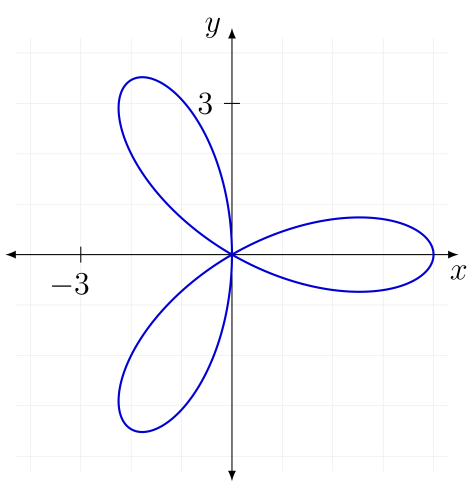


Figure 39. The graph of the polar curve $r = 4\cos 3\theta$, called a "rose."

that θ_0 should be π, the smaller of the two θ values obtained. That is, S equals the graph of $r = 4\cos 3\theta$ if $\theta_0 = \pi$. See Figure 39. ■

11.3 – Calculus in Polar Coordinates

A curve given by a polar equation $r = f(\theta)$ can always be defined by parametric equations using θ as the parameter and recalling that, in general, $x = r\cos\theta$ and $y = r\sin\theta$:

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Letting $g(\theta) = f(\theta)\cos\theta = x$ and $h(\theta) = f(\theta)\sin\theta = y$, and assuming that g and h are continuously differentiable functions with $g' \neq 0$ on some open interval I, then by Theorem 11.3 we obtain

$$y'(x) = \frac{h'(\theta)}{g'(\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \tag{11.11}$$

for all $\theta \in I$. From this the following proposition results.

Proposition 11.7. *Let f be a differentiable function at θ_0. If*

$$f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0 \neq 0,$$

then the slope of the tangent line to the curve with polar equation $r = f(\theta)$ at $(f(\theta_0), \theta_0)$ is

$$\frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0}.$$

Example 11.8. Find the points where the polar curve $r = 2+2\sin\theta$ has a horizontal or vertical tangent line.

Solution. First, when $\theta = 0$ we have $r = f(\theta) = 2$, so the graph of the equation contains the point $(r, \theta) = (2, 0)$. Setting $f(\theta) = 2$ gives $2 + 2\sin\theta = 2$, or $\sin\theta = 0$, which has solution set $\{0, \pi, 2\pi, \ldots\}$; and setting $f(\theta) = -2$ gives $\sin\theta = -2$, which has no solutions. From the set $\{0, \pi, 2\pi, \ldots\}$ the smallest nonzero value that returns us to $(2, 0)$ is 2π, and so the set

$$S = \{(r, \theta) : 0 \leq \theta \leq 2\pi \text{ and } r = 2 + 2\sin\theta\}$$

should produce the complete graph of $r = 2+2\sin\theta$ (which can be verified by actually producing the graph). Thus, we need only entertain θ values in the interval $[0, 2\pi]$ in our search for points where the curve may have a horizontal or vertical tangent line.

By Proposition 11.7 the curve given by $f(\theta) = 2 + 2\sin\theta$ has a horizontal tangent line wherever

$$\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{2\cos\theta\sin\theta + (2 + 2\sin\theta)\cos\theta}{2\cos^2\theta - (2 + 2\sin\theta)\sin\theta} = 0,$$

which implies

$$2\cos\theta\sin\theta + (2 + 2\sin\theta)\cos\theta = 0 \tag{11.12}$$

and

$$2\cos^2\theta - (2 + 2\sin\theta)\sin\theta \neq 0. \tag{11.13}$$

From (11.12) we obtain

$$(\cos\theta)(2\sin\theta + 1) = 0.$$

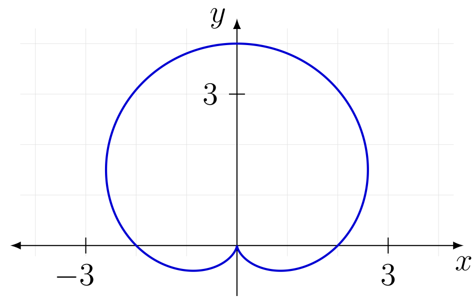


FIGURE 40. The graph of $r = 2 + 2\sin\theta$, called a "cardioid."

From $\cos\theta = 0$ we obtain solutions $\pi/2$ and $3\pi/2$ in $[0, 2\pi]$, and from $2\sin\theta + 1 = 0$ we obtain solutions $7\pi/6$ and $11\pi/6$ in $[0, 2\pi]$. The solution $3\pi/2$ violates the condition (11.13) so we discard it. Thus the curve has a horizontal tangent line at $(4, \pi/2)$, $(1, 7\pi/6)$, and $(1, 11\pi/6)$.

To find points where the curve has a vertical tangent line, we find solutions to

$$2\cos^2\theta - (2 + 2\sin\theta)\sin\theta = 0.$$

which gives $2\sin^2\theta + \sin\theta - 1 = 0$, and thus

$$(2\sin\theta - 1)(\sin\theta + 1) = 0.$$

From $2\sin\theta - 1 = 0$ we obtain solutions $\pi/6, 5\pi/6, 13\pi/6, \ldots$, and from $\sin\theta + 1 = 0$ we obtain solutions $3\pi/2, 7\pi/2, 11\pi/2, \ldots$. In the interval $[0, 2\pi]$, then, we arrive at the points $(3, \pi/6)$, $(3, 5\pi/6)$ and $(0, 3\pi/2) \sim (0, 0)$. See Figure 40 ■

Example 11.9. Find the points where the polar curve $r = \sin 2\theta$ has a horizontal tangent line.

Solution. The entire curve is generated for $0 \le \theta \le 2\pi$, so we only search for values of θ in the interval $[0, 2\pi]$.

Set $f(\theta) = \sin 2\theta$. To find where horizontal tangent lines reside, find θ for which

$$\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta}{2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta} = 0,$$

which entails solving

$$2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta = 0.$$

Using the identities

$$\sin 2t = 2\sin t\cos t \quad \text{and} \quad \cos 2t = \cos^2 t - \sin^2 t,$$

the equation becomes $(\sin\theta)(2 - 3\sin^2\theta) = 0$, so either $\sin\theta = 0$ or $\sin\theta = \pm\sqrt{2/3}$.

Solving $\sin\theta = \sqrt{2/3}$ gives two solutions: $\theta_1 = \tan^{-1}\sqrt{2}$ (an angle in Quadrant I) and $\theta_2 = \pi - \tan^{-1}\sqrt{2}$ (in Quadrant II).

Solving $\sin\theta = -\sqrt{2/3}$ gives $\theta_3 = \pi - \tan^{-1}\left(-\sqrt{2}\right)$ (in Quadrant III) and $\theta_4 = \tan^{-1}\left(-\sqrt{2}\right)$ (in Quadrant IV).

Putting the four angles θ_1, θ_2, θ_3 and θ_4 into $r = \sin 2\theta$ and noting that

$$\tan^{-1}\big(-\sqrt{2}\big) = -\tan^{-1}\big(\sqrt{2}\big),$$

we obtain four points:

$$\left(\pm\frac{2\sqrt{2}}{3}, \pm\tan^{-1}\sqrt{2}\right) \quad \text{and} \quad \left(\pm\frac{2\sqrt{2}}{3}, \pi \pm \tan^{-1}\sqrt{2}\right).$$

(These types of problems are seldom pleasant company.) Moving on to $\sin\theta = 0$, we obtain $\theta = 0, \pi$, which yields just one point, $(0, 0)$, although the curve passes through the point twice! ∎

12

Vectors and Coordinates

12.1 – Euclidean Space in Rectangular Coordinates

By "the plane" we have always meant the set of all ordered pairs of real numbers. Such a set we denote by $\mathbb{R}^2$, so that

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

As ever, an ordered pair (x, y) may be geometrically interpreted to be a point on the plane. More precisely (x, y) is taken to be the rectangular coordinates of a point, unless a different coordinate system has been specified such as the polar coordinate system introduced in §11.2. Note that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Cartesian product of the set of real numbers $\mathbb{R}$ with itself.

We now move up one dimension. By "space" we mean here the set of all ordered triples of real numbers. This set we denote by $\mathbb{R}^3$, which in fact equals $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ so that

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

An ordered triple (x, y, z) may naturally be regarded as a point in space. The coordinate system assumed here is the three-dimensional **rectangular coordinate system**, which is a natural extension of the two-dimensional rectangular coordinate system. The extension is accomplished by the addition to a z-axis that is perpendicular to both the x-axis and y-axis. Figure 41 illustrates two fundamentally different ways to do this, one way yielding a **left-handed coordinate system** (at left in the figure), and the other way yielding a **right-handed coordinate system** (at right in the figure). *We will always assume a right-handed coordinate*

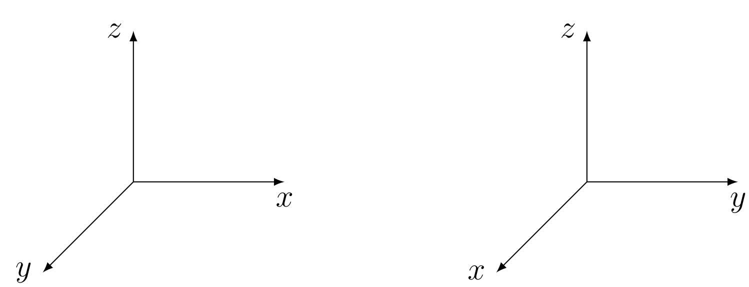


Figure 41. A left-handed system at left, and a right-handed system at right.

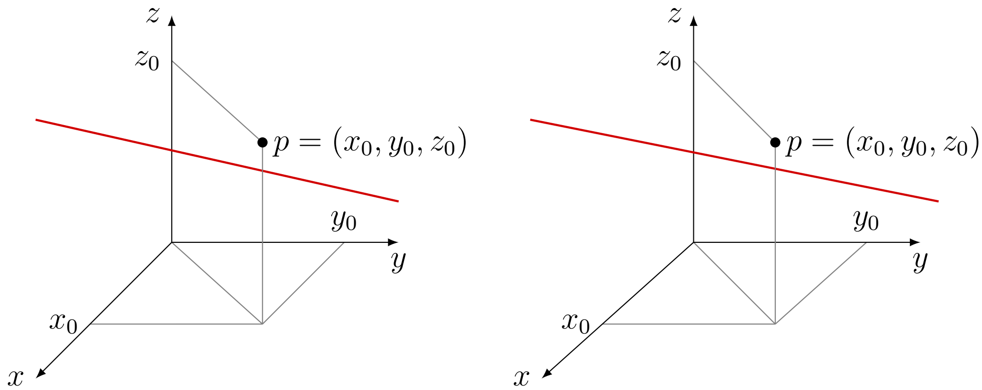


FIGURE 42. Stereoscopic image of three-space containing point and line.

system! This is the system for which the xy-plane would appear to have the conventional two-dimensional rectangular coordinate system from the perspective of someone looking "down" at it from a point on the positive z-axis.

If p is a point in space with coordinates (x_0, y_0, z_0), we write $p = (x_0, y_0, z_0)$. The location of such a point p in the case when $x_0 > 0$, $y_0 > 0$, and $z_0 > 0$ is illustrated in Figure 42, which is stereoscopic to give the effect of depth. Unless another coordinate system has been specified such as the cylindrical or spherical systems introduced in §12.5, we always assume the coordinates of a point in space are rectangular coordinates.

Space equipped with the three-dimensional rectangular coordinate system is sometimes called xyz-space, just as the plane equipped with the two-dimensional rectangular coordinate system is called the xy-plane. It is fair to say the xy-plane is a subset of xyz-space, although in xyz-space any point on the xy-plane must be expressed as an ordered triple with z-coordinate equal to zero: $(x, y, 0)$. Put another way, in xyz-space the xy-plane is the plane given by the equation $z = 0$. Similarly $x = 0$ is the yz-plane, and $y = 0$ is the xz-plane. More will be said about planes in space in §13.6.

In the two-dimensional rectangular coordinate system the xy-plane is partitioned into four quadrants: Q_1, Q_2, Q_3, Q_4. In xyz-space there are eight **octants** denoted $\mathsf{O}_1, \ldots, \mathsf{O}_8$. Octants O_1, O_2, O_3, O_4 lie above Q_1, Q_2, Q_3, Q_4, respectively, and octants O_5, O_6, O_7, O_8 lie below Q_1, Q_2, Q_3, Q_4. So the first and fifth octants, for example, are the sets

$$\mathsf{O}_1 = \{(x, y, z) : x > 0, y > 0, z > 0\} \quad \text{and} \quad \mathsf{O}_5 = \{(x, y, z) : x > 0, y > 0, z < 0\}.$$

Let $p_0 = (x_0, y_0, z_0)$ and $p = (x, y, z)$ be two points in $\mathbb{R}^3$. A simple algebraic argument that twice makes use of the Pythagorean Theorem shows that the distance between p_0 and p, denoted by $d(p_0, p)$, is given by the formula

$$d(p_0, p) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}. \tag{12.1}$$

This is the natural extension of the distance formula in $\mathbb{R}^2$ as it is known from algebra.

Recall that in algebra a **circle** in $\mathbb{R}^2$ with center $(x_0, y_0) \in \mathbb{R}^2$ and radius $r > 0$, which we shall denote by the symbol $C_r(x_0, y_0)$, is defined to be the set of all points $(x, y) \in \mathbb{R}^2$ that are a distance r from (x_0, y_0). That is,

$$C_r(x_0, y_0) = \left\{(x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} = r\right\}.$$

The natural extension of the notion of a circle in $\mathbb{R}^2$ is that of a sphere in $\mathbb{R}^3$. Specifically, a **sphere** in $\mathbb{R}^3$ with center $p_0 = (x_0, y_0, z_0)$ and radius $r > 0$, denoted by the symbol $S_r(p)$, is the set of all points $p \in \mathbb{R}^3$ that are a distance of r away from p_0. That is,

$$S_r(p_0) = \left\{p \in \mathbb{R}^3 : d(p_0, p) = r\right\},$$

where $d(p_0, p)$ is given by (12.1). To say a point (x, y, z) lies on the sphere $S_r(p_0)$ is equivalent to saying $d(p_0, p) = r$, which by (12.1) (squaring both sides) is equivalent to saying

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{12.2}$$

We refer to (12.2) as the **center-radius form** of the equation of the sphere $S_r(p_0)$. The following example illustrates the general technique for getting the equation of any sphere into center-radius form.

Example 12.1. Describe the set of points that satisfies the equation

$$2x^2 + 2y^2 + 2z^2 + 4x - 5y - 6z = \tfrac{1}{2}.$$

Solution. First divide by the coefficient of the squared terms, which in this case is 2, to get

$$x^2 + y^2 + z^2 + 2x - \tfrac{5}{2}y - 3z = \tfrac{1}{4}.$$

Now,

$$\begin{aligned}
(x^2 + 2x) + (y^2 - \tfrac{5}{2}y) + (z^2 - 3z) &= \tfrac{1}{4} && \text{(group by variable)}\\
(x^2 + 2x + 1) + (y^2 - \tfrac{5}{2}y + \tfrac{25}{16}) + (z^2 - 3z + \tfrac{9}{4}) &= \tfrac{1}{4} + 1 + \tfrac{25}{16} + \tfrac{9}{4} && \text{(complete the square)}\\
(x + 1)^2 + (y - \tfrac{5}{4})^2 + (z - \tfrac{3}{2})^2 &= \tfrac{81}{16} && \text{(factor each trinomial)}
\end{aligned}$$

The last equation is in center-radius form. Comparing with (1), we see that the equation is that for a sphere with center at $(-1, 5/4, 3/2)$ and radius $\sqrt{81/16} = 9/4$. ■

At least as important as spheres is the notion of an **open ball** in $\mathbb{R}^3$ centered at $p_0 \in \mathbb{R}^3$ with radius $r > 0$, denoted by $B_r(p_0)$ and defined to be the set of all points in $\mathbb{R}^3$ that are *less than* r units from p_0:

$$B_r(p_0) = \left\{p \in \mathbb{R}^3 : d(p_0, p) < r\right\}.$$

The **closed ball** in $\mathbb{R}^3$ centered at $p_0 \in \mathbb{R}^3$ with radius $r > 0$ is the set

$$\overline{B}_r(p_0) = \left\{p \in \mathbb{R}^3 : d(p_0, p) \le r\right\}.$$

Throughout the remainder of these notes we shall be working primarily in either $\mathbb{R}^2$ or $\mathbb{R}^3$, though $\mathbb{R}^n$ will continue to be useful for the purposes of presenting certain results in a more general form. Even in scientific applications there is frequent need to work in $\mathbb{R}^n$ for some $n > 3$.

For example, we have only to consider relativity theory to see an instance when it's necessary to work in

$$\mathbb{R}^4 = \{(x, y, z, t) : x, y, z, t \in \mathbb{R}\},$$

where x, y, and z represent spatial coordinates and t is time.

12.2 – Vectors in Euclidean Space

Like a point in $\mathbb{R}^n$, a **vector** in $\mathbb{R}^n$ is an ordered n-tuple of real numbers. However, to begin with there is a difference in notation: vectors in $\mathbb{R}^n$ are presented as ordered lists of n real numbers placed between angle brackets $\langle$ and $\rangle$ instead of parentheses:

$$\langle x_1, x_2, \dots, x_n \rangle.$$

As with points, we may more compactly denote a vector by a single letter, but the difference is we shall always use a **bold-faced** letter for the purpose. Thus the vector $\langle x_1, x_2, \dots, x_n \rangle$ may be denoted by $\mathbf{x}$, and we write[9]

$$\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle.$$

We call $x_1, x_2, \dots, x_n$ the **components**, or **coordinates**, of the vector $\mathbf{x}$. Vectors in $\mathbb{R}^n$ are sometimes called **coordinate vectors** or **euclidean vectors** to distinguish them from other kinds of vectors encountered in other realms of mathematics.

Two vectors $\mathbf{x} = \langle x_1, \dots, x_n \rangle$ and $\mathbf{y} = \langle y_1, \dots, y_n \rangle$ are said to be **equal** if $x_k = y_k$ for all $1 \le k \le n$, in which case we write $\mathbf{x} = \mathbf{y}$. The **zero vector $\mathbf{0}$** is the vector whose coordinates are all equal to 0: $\mathbf{0} = \langle 0, \dots, 0 \rangle$.

For our purposes a **scalar** is always taken to be a real number, though in other contexts a scalar could be a complex number.

Definition 12.2. *Let $\mathbf{x} = \langle x_1, \dots, x_n \rangle$ and $\mathbf{y} = \langle y_1, \dots, y_n \rangle$ be vectors in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Then we define the **sum** of $\mathbf{x}$ and $\mathbf{y}$ to be the vector*

$$\mathbf{x} + \mathbf{y} = \langle x_1 + y_1, \dots, x_n + y_n \rangle,$$

*and the **scalar multiple** of $\mathbf{x}$ by c to be the vector*

$$c\mathbf{x} = \langle cx_1, \dots, cx_n \rangle.$$

*We also define the **negative** of $\mathbf{x}$ to be the vector $-\mathbf{x} = (-1)\mathbf{x}$, and the **difference** of $\mathbf{x}$ and $\mathbf{y}$ to be the vector*

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}).$$

All the operations in Definition 12.2 are known as **vector operations**. In explicit terms we have

$$-\mathbf{x} = (-1)\mathbf{x} = (-1)\langle x_1, \dots, x_n \rangle = \big\langle (-1)x_1, \dots, (-1)x_n \big\rangle = \langle -x_1, \dots, -x_n \rangle,$$

and

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-1)\mathbf{y} = \langle x_1, \dots, x_n \rangle + \langle -y_1, \dots, -y_n \rangle = \langle x_1 - y_1, \dots, x_n - y_n \rangle.$$

With Definition 12.2 it is straightforward to prove the following properties of vectors in $\mathbb{R}^n$.

Proposition 12.3. *Let $\mathbf{x} = \langle x_1, \dots, x_n \rangle$ and $\mathbf{y} = \langle y_1, \dots, y_n \rangle$ be coordinate vectors in $\mathbb{R}^n$, and $a, b \in \mathbb{R}$. Then*

1. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$
2. $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$
3. $(ab)\mathbf{x} = a(b\mathbf{x})$

[9] Vectors written by hand, when bold-facing is not necessarily practical, may be denoted with an arrow: $\vec{x}$.

Definition 12.4. *Two nonzero vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *are* ***parallel****, written* $\mathbf{x} \parallel \mathbf{y}$*, if there exists* $c \in \mathbb{R}$ *such that* $\mathbf{y} = c\mathbf{x}$*.*

Aside from notation, how is a euclidean vector in $\mathbb{R}^n$ really different from a point in $\mathbb{R}^n$? From a mathematical standpoint there is no difference: both are ordered n-tuples of real numbers. The sole difference is merely a matter of geometrical interpretation. The geometric interpretation is simple: a vector $\langle x_1, \ldots, x_n \rangle$ is thought of as an arrow in $\mathbb{R}^n$ with its **tail** located at any arbitrary point $(p_1, \ldots, p_n)$ and its **head** located at $(p_1 + x_1, \ldots, p_n + x_n)$. Thus, so long as $\langle x_1, \ldots, x_n \rangle \neq \mathbf{0}$, it is natural to think of $\langle x_1, \ldots, x_n \rangle$ as an arrow with a fixed direction and fixed length, but no fixed location in n-space. Put another way: many different arrows can represent the same vector $\langle x_1, \ldots, x_n \rangle$, as shown in Figure 43 in the $\mathbb{R}^2$ case. A point in $\mathbb{R}^n$, in contrast, has a fixed location but no direction or length. In physics, for instance, there are many concepts (such as location) that are regarded as "point quantities," and other concepts (such as velocity and force) that are more naturally thought of as having direction and length.

In short, the difference between $(x_1, \ldots, x_n)$ and $\langle x_1, \ldots, x_n \rangle$ is an artifice demanded by the sciences and other fields in inquiry that use mathematics as a tool. Mathematics itself, however, realizes many benefits by treating all n-tuples notationally as vectors, and letting context alone tell us which vectors in an analysis are better pictured in the mind's eye as points instead. The practice of "treating all n-tuples as vectors" is carried out quite easily: any point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ is represented by (or converted to) the corresponding vector $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$. We call the vector $\mathbf{x}$ the **position vector** of the point x, since if the tail of the arrow for $\mathbf{x}$ is placed at the origin $o = (0, \ldots, 0)$, then the arrow's head will be located precisely at x. In this way the arrow for $\mathbf{x}$ "points" to x.

Many times throughout these notes we shall indeed notationally present all n-tuples as vectors. The convenience of doing so is too great to pass up, and the mathematics is never compromised by the practice. To aid the intuition, however, we may still refer to some vectors as "points" wherever it seems good to do so.

Given two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we can draw an arrow with tail at x and head at y, and this arrow represents a vector denoted by $\overrightarrow{xy}$. Since

$$y = \big(x_1 + (y_1 - x_1), \ldots, x_n + (y_n - x_n)\big),$$

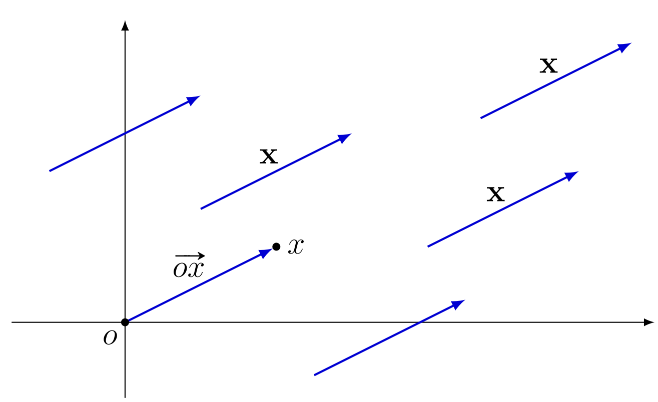


FIGURE 43. Different arrows, all representing vector $\mathbf{x}$.

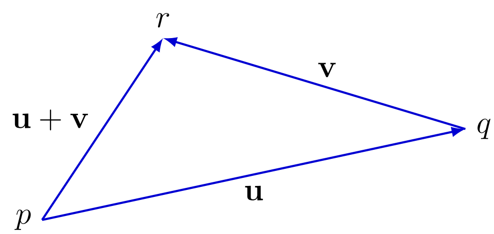


FIGURE 44.

it follows that

$$\overrightarrow{xy} = \langle y_1 - x_1, \ldots, y_n - x_n \rangle = \mathbf{y} - \mathbf{x}.$$

In particular we have

$$\overrightarrow{ox} = \langle x_1 - 0, \ldots, x_n - 0 \rangle = \langle x_1, \ldots, x_n \rangle = \mathbf{x},$$

which is the position vector for x.

If p, q, and r are three points in $\mathbb{R}^n$, and $\mathbf{u} = \overrightarrow{pq}$ while $\mathbf{v} = \overrightarrow{qr}$, then it is straightforward to verify that $\mathbf{u} + \mathbf{v} = \overrightarrow{pr}$. That is, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are vectors whose representative arrows form a triangle if their tails are placed at p, q, and p, respectively. See Figure 44.

There will arise occasions when many points, say m of them, must be referenced in an analysis, and these will be denoted by employing subscripts: $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$. For each $1 \le k \le m$ the vector $\mathbf{x}_k$ in $\mathbb{R}^n$ will be taken to have components given by

$$\mathbf{x}_k = \langle x_{k1}, x_{k2}, \ldots, x_{kn} \rangle.$$

Another notational convention: In $\mathbb{R}^2$ we will usually write $\mathbf{x} = \langle x, y \rangle$ instead of $\mathbf{x} = \langle x_1, x_2 \rangle$, while in $\mathbb{R}^3$ we will usually write $\mathbf{x} = \langle x, y, z \rangle$ rather than $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$.

The **magnitude** or **norm** of a vector $\mathbf{x} \in \mathbb{R}^n$, denoted by $\|\mathbf{x}\|$, is defined to be

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Geometrically, $\|\mathbf{x}\|$ is the length of any arrow in $\mathbb{R}^n$ that represents $\mathbf{x}$, as is easily confirmed by use of the n-space version of the distance formula. In similar fashion, the **direction** of a vector is just the direction of any one of its arrows.

Often it is desirable to obtain a vector that has the same direction as some given vector $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$, but with a magnitude equal to 1 (which is called a **unit vector**). This is easily accomplished by dividing $\mathbf{x}$ by its own magnitude to get a new vector

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

(Note: the "hat" accent $\hat{}$ will often be used to denote a unit vector.) It's easy to verify that $\hat{\mathbf{x}}$ is a unit vector:

$$\begin{aligned} \|\hat{\mathbf{x}}\| &= \left\| \frac{\langle x_1, \ldots, x_n \rangle}{\|\mathbf{x}\|} \right\| = \left\| \left\langle \frac{x_1}{\|\mathbf{x}\|}, \ldots, \frac{x_n}{\|\mathbf{x}\|} \right\rangle \right\| = \sqrt{\left(\frac{x_1}{\|\mathbf{x}\|} \right)^2 + \cdots + \left(\frac{x_n}{\|\mathbf{x}\|} \right)^2} \\ &= \frac{1}{\|\mathbf{x}\|} \sqrt{x_1^2 + \cdots + x_n^2} = \frac{1}{\|\mathbf{x}\|} \cdot \|\mathbf{x}\| = 1. \end{aligned}$$

Notice we can write $\mathbf{x} = \|\mathbf{x}\|\hat{\mathbf{x}}$, so that any vector can be broken down into a product of its magnitude and direction.

The **standard unit vectors** in $\mathbb{R}^2$ are $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, while in $\mathbb{R}^3$ they are $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$.

If $\mathbf{x}$ and $\mathbf{y}$ are two points in $\mathbb{R}^n$, then the **distance** (specifically the **euclidean distance**) between them is defined to be

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

which is just a natural extension of the definitions of distance between points in $\mathbb{R}^2$ or $\mathbb{R}^3$ that was given earlier.

Aside from giving vectors magnitudes and quantifying distances between vectors, the norm operation $\|\cdot\|$ induces what's called a "topology" on $\mathbb{R}^n$, which for us starts with the idea of an open ball. The **open ball** $B_\epsilon(\mathbf{a})$ centered at the point $\mathbf{a} = \langle a_1, \ldots, a_n\rangle$ with radius $\epsilon > 0$ is defined by

$$B_\epsilon(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{R}^n \;\middle|\; \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} < \epsilon\right\} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < \epsilon\}, \tag{12.3}$$

while the corresponding **closed ball** is

$$\overline{B}_\epsilon(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| \le \epsilon\}.$$

These definitions are a generalization of the notions of an open and closed ball in $\mathbb{R}^3$ given in the previous section, only now we make use of vector notation. We are now ready to set down a solid definition for what exactly it means to be an "open set" in $\mathbb{R}^n$.

Definition 12.5. *A set $U \subseteq \mathbb{R}^n$ is **open** if, for each $\mathbf{x} \in U$, there exists some $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq U$.*

Any point $\mathbf{x}$ in a set S for which there can be found some $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq S$ is called an **interior point** of S, so another way of defining an open set is to say it is a set that consists entirely of interior points. The collection $\mathcal{T}$ of all the subsets of $\mathbb{R}^n$ that are open according to Definition 12.5 forms what is called the **standard topology** on $\mathbb{R}^n$. (In general, a **topology** on any set of objects X is a collection of open subsets of X that satisfy certain properties.) In particular $\mathbb{R}^n$ and the empty set $\varnothing$ are open sets. It should be easy to see that the open balls defined by (12.3) are themselves open sets. The set of interior points of a set S is denoted by S° or $\mathrm{Int}(S)$.

A **neighborhood** of a point $\mathbf{x} \in \mathbb{R}^n$ is any open set that contains $\mathbf{x}$. For instance $B_\epsilon(\mathbf{x})$ is a neighborhood of $\mathbf{x}$ for any $\epsilon > 0$. We can now say that a set $U \subseteq \mathbb{R}^n$ is open if and only if for every $\mathbf{x} \in U$ there exists some neighborhood N of $\mathbf{x}$ such that $N \subseteq U$.

Another important concept is that of a boundary point: a point $\mathbf{x}$ is a **boundary point** of a set S if every open set that contains $\mathbf{x}$ contains at least one point in S and at least one point not in S. Note that it's possible for a boundary point of a set to not be an element of that set. The set of boundary points of a set S is denoted by ∂S.

Definition 12.6. *A set is **closed** if it contains all its boundary points.*

A hard look at Definition 12.5 should convince you that an open set U must not contain *any* of its boundary points, for if $\mathbf{x} \in U$ were a boundary point, then no matter what $\epsilon > 0$ we chose we would not have $B_\epsilon(\mathbf{x}) \subseteq U$ since $B_\epsilon(\mathbf{x})$—being itself an open set that contains $\mathbf{x}$—would contain a point $\mathbf{y} \notin U$.

According to Definition 12.6, for any set S it must be that $S \cup \partial S$ is closed. Boundary points and interior points taken together are called **closure points**, and it is common to define $\overline{S} = S \cup \partial S$ and call $\overline{S}$ the **closure** of S (i.e. the set of closure points of S).

We call a point $\mathbf{x}$ a **limit point** of a set S if every open set that contains $\mathbf{x}$ contains a point $\mathbf{y} \in S$ such that $\mathbf{y} \neq \mathbf{x}$. All interior points are limit points, but not all boundary points qualify as limit points. It's actually a fact that a set is closed if and only if it contains all its limit points.

Example 12.7. Consider the set $S = (1, 2) \cup \{3\}$ in $\mathbb{R}$. Only those points in the interval $(1, 2)$ are interior points. The points 1, 2, and 3 are boundary points. The closure points of S are the points in $[1, 2] \cup \{3\}$ (all the interior points and boundary points put together). Only those points in the interval $[1, 2]$ are limit points. A boundary point that is not a limit point is called an **isolated point**, so here the point 3 is an isolated point. Now notice that S is not an open set because not all of its points are interior points, and S is also not a closed set because it does not contain all its boundary points. A set that is not open is not necessarily closed! ■

The Cartesian product $(a, b) \times (c, d)$, called an **open box**, includes all points in the interior of the rectangle pictured in Figure 1, but *not* the boundary points. In $\mathbb{R}^2$, just as any point $\mathbf{x}$ in an open set U can be contained within an open ball $B_\epsilon(\mathbf{x})$ such that $B_\epsilon(\mathbf{x}) \subseteq U$, so too can an open box $(a, b) \times (c, d)$ be found such that $\mathbf{x} \in (a, b) \times (c, d) \subseteq U$.

In applications vectors may represent velocities, forces, or some other physical quantity, and it's often convenient to analyze a problem by moving vectors arrows around so that the initial point of one arrow is located at the terminal point of another arrow. Other times it may be expedient to place the initial points of all arrows at the origin of a chosen coordinate system. The very definition of vector addition is motivated by physical considerations: when two or more forces act on an object it turns out to be the case that, when the forces are expressed in component form, the *net* force that acts on the object is equal to the sum of the vectors as given by the rule $\langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$ from above.

Before considering some specific examples, there is an important concept that needs to be made clear. Imagine two objects, U and V, situated on a plane at the origin $(0, 0)$. Now, at time $t = 0$, U starts moving at a constant velocity $\mathbf{u} = \langle 3, 2\rangle$ (3 units to the right and 2 units up, per second), and V starts moving at a constant velocity $\mathbf{v} = \langle -2, -1\rangle$. Letting U_n and V_n denote the positions of U and V at time $t = n$, the situation is depicted in Figure 45. Suppose someone is riding on object U and observing object V over time. What does the person see? At time $t = 1$, V is 5 units to the left and 3 units down from the location of U; at $t = 2$, V is 10 to the left and 6 down; at $t = 3$, V is 15 left and 9 down. Thus, it appears to this person that V is traveling in such a way that it goes 5 units leftward and 3 units downward *per second*. (It may even be that this observer is not even aware that U is moving!) We say that V has a velocity of $\langle -5, -3\rangle$ **relative to** U. By the same reasoning U has a velocity of $\langle 5, 3\rangle$ **relative to** V. As for the "actual" velocity $\mathbf{u}$ of U, we sometimes say it is the velocity of U relative to the plane. In many physical settings the plane is called the "ground".

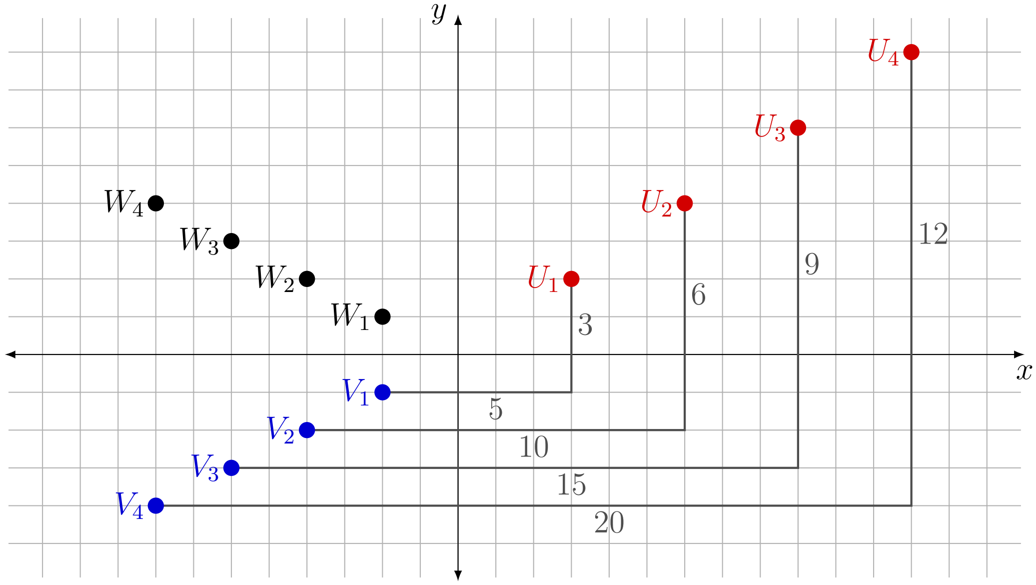


FIGURE 45. Objects in motion.

Suppose now we are told that an object W which also was set into motion from the origin $(0, 0)$ at time $t = 0$ has a velocity of $\mathbf{w}' = \langle -5, -1\rangle$ relative to U. Can we find $\mathbf{w}$, the velocity of W relative to the plane? At time $t = 1$, W will be 5 left and 1 down from U (located at the point U_1); at time $t = 2$, W will be 10 left and 2 down from U (now located at U_2), and so on. If we plot the positions of W with this information, we can figure out that $\mathbf{w} = \langle -2, 1\rangle$ (see Figure 45 again). But this is simply the sum of $\mathbf{w}'$ and $\mathbf{u}$:

$$\mathbf{w}' + \mathbf{u} = \langle -5, -1\rangle + \langle 3, 2\rangle = \langle -2, 1\rangle = \mathbf{w}.$$

Thus, in general, if we are told that an object U has velocity $\mathbf{u}$ relative to the plane and an object W has velocity $\mathbf{w}'$ relative to U, then to find the velocity $\mathbf{w}$ of W relative to the plane we simply compute $\mathbf{w}' + \mathbf{u}$. Keep this in mind in the examples to come.

Example 12.8. In still air, a parachute with a payload would fall vertically at a terminal speed of 40 m/s. Find the direction and magnitude of its terminal velocity relative to the ground if it falls in a steady wind blowing horizontally from west to east at 10 m/s.

Solution. Set up a rectangular coordinate system with the positive x-axis pointing east and the positive y-axis pointing up into the sky. The velocity vector of the parachute, once it attains terminal velocity in still air, is $\langle 0, -40\rangle$, while the velocity vector of the wind is $\langle 10, 0\rangle$. So, the terminal velocity vector of the parachute *in the wind* will just be the sum of these vectors: $\langle 0, -40\rangle + \langle 10, 0\rangle = \langle 10, -40\rangle$. This means for every 10 m the parachute goes east, it falls by 40 m. Calculating $\tan\theta = \frac{40}{10}$ gives $\theta \approx 75.96°$, so the parachute is drifting eastward along a straight-line trajectory that makes a 75.96° angle with the ground. The magnitude of the terminal velocity is

$$\|\langle 10, -40\rangle\| = \sqrt{10^2 + (-40)^2} = \sqrt{1700} \text{ m/s},$$

or approximately 41.23 m/s. ■

Example 12.9. A boat is towed with a force of 150 lb with a rope that makes an angle of 30° to the horizontal. Find the horizontal and vertical components of the force.

Solution. Let $\mathbf{F}$ represent the force exerted on the boat, where we're given that $\|\mathbf{F}\| = 150$ lb. Then $\mathbf{F} = \langle F_x, F_y \rangle$, where the horizontal and vertical components of $\mathbf{F}$, which are F_x and F_y, are given by

$$F_x = \|\mathbf{F}\| \cos\theta = 150\cos 30^\circ = 150 \cdot \frac{\sqrt{3}}{2} = 75\sqrt{3} \text{ lb}$$

and

$$F_y = \|\mathbf{F}\| \sin\theta = 150\sin 30^\circ = 150 \cdot \frac{1}{2} = 75 \text{ lb}.$$

We can write $\mathbf{F} = \langle 75\sqrt{3} \text{ lb}, 75 \text{ lb}\rangle$, or simply $\mathbf{F} = \langle 75\sqrt{3}, 75\rangle$ if the use of pounds as a unit of force is understood (the book is a little cavalier about this). ■

Example 12.10. An ant is walking due east at a constant speed of 2 mi/hr on a sheet of paper that rests on a table. Suddenly the sheet of paper starts moving southeast at $\sqrt{2}$ mi/hr. Describe the motion of the ant relative to the table.

Solution. The motion of the ant relative to the table is simply the motion of the ant as it would be perceived by an observer who is at rest with respect to the table. As usual, coordinatize the scene by making the tabletop the xy-plane, with the positive x-axis pointing east and the positive y-axis pointing north. This coordinate system will remain static in relation to the table, and in such a system the velocity vector of the ant is $\mathbf{a} = \langle 2, 0\rangle$ while the velocity vector of the paper is $\mathbf{p} = \langle 1, -1\rangle$. The latter vector is justified by noting that southeast means 45° south of east, so the velocity vector of the paper points 45° clockwise from our positive x-axis and has length $\sqrt{2}$ — meaning it is the hypotenuse of a classic 45°-45°-90° right triangle, which has legs of length 1. Now, both $\mathbf{a}$ and $\mathbf{p}$ are vectors that affect the position of the ant, and the net effect will be given by the *sum* of these vectors: $\mathbf{a} + \mathbf{p} = \langle 3, -1\rangle$. That is, for every 3 miles the ant travels east, it will travel 1 mile south. To get an angle, calculate $\tan\theta = \frac{3}{1}$ to get $\theta \approx 18.43^\circ$; that is, the direction the ant is traveling is about 18.43° south of east.

Finally, we have

$$\|\langle 3, -1\rangle\| = \sqrt{3^2 + (-1)^2} = \sqrt{10} \approx 3.16 \text{ mi/hr}$$

for the speed of the ant. ■

Example 12.11. A solid ball of osmium weighing 200 N and located at the point $p = (0, 0, -10)$ is suspended from three cables that are affixed to hooks located at points $q_1 = (3, -8, 0)$, $q_2 = (3, 6, 0)$ and $q_3 = (-2, 0, 0)$, as shown in Figure 46. Find the tension in each of the supporting cables pq_1, pq_2 and pq_3.

Solution. The osmium ball is stationary, and therefore the forces acting upon it must cancel out. If $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are the forces exerted by the cables pq_1, pq_2 and pq_3, respectively, and $\mathbf{F}_g$ is the force of gravity, then we obtain the equation

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_g = \langle 0, 0, 0\rangle. \tag{12.4}$$

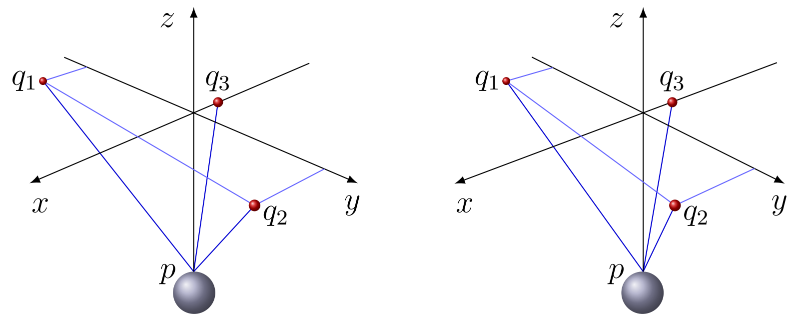


Figure 46. Solid ball of osmium, worth more than its weight in gold.

Thus, the forces $\mathbf{F}_i$ taken together constitute the "equal and opposite" force of the cables to the pull of gravity. Let $F_1 = \|\mathbf{F}_1\|$, $F_2 = \|\mathbf{F}_2\|$ and $F_3 = \|\mathbf{F}_3\|$, which are the tension values that we are to determine. It can be seen from Figure 46 that the direction of the force $\mathbf{F}_1$ must be the same as the direction of the arrow $\overrightarrow{pq}_1$, $\mathbf{F}_2$ must be in the direction of $\overrightarrow{pq}_2$, and $\mathbf{F}_3$ must be in the direction of $\overrightarrow{pq}_3$. Recalling that

$$\mathbf{F}_i = \|\mathbf{F}_i\|\hat{\mathbf{F}}_i = F_i\hat{\mathbf{F}}_i$$

for each i, where $\hat{\mathbf{F}}_i = \overrightarrow{pq}_i/\|\overrightarrow{pq}_i\|$ is the unit vector which points in the direction of $\mathbf{F}_i$, we obtain

$$\begin{aligned}
\mathbf{F}_1 &= F_1\left(\frac{\overrightarrow{pq}_1}{\|\overrightarrow{pq}_1\|}\right) = F_1\left(\frac{\langle 3,-8,10\rangle}{\sqrt{173}}\right) \approx \langle 0.228F_1, -0.608F_1, 0.760F_1\rangle \\
\mathbf{F}_2 &= F_2\left(\frac{\overrightarrow{pq}_2}{\|\overrightarrow{pq}_2\|}\right) = F_2\left(\frac{\langle 3,6,10\rangle}{\sqrt{145}}\right) \approx \langle 0.249F_2, 0.498F_2, 0.830F_2\rangle \\
\mathbf{F}_3 &= F_3\left(\frac{\overrightarrow{pq}_3}{\|\overrightarrow{pq}_3\|}\right) = F_3\left(\frac{\langle -2,0,10\rangle}{\sqrt{104}}\right) \approx \langle -0.196F_3, 0, 0.981F_3\rangle \\
\mathbf{F}_g &= \langle 0,0,-200\rangle
\end{aligned}$$

Putting all this into (12.4) gives a tidy little system of equations,

$$\begin{array}{rclclclcl}
0 &=& 0.228F_1 &+& 0.249F_2 &-& 0.196F_3 && \\
0 &=& -0.608F_1 &+& 0.498F_2 &&&& \\
0 &=& 0.760F_1 &+& 0.830F_2 &+& 0.981F_3 &-& 200
\end{array}$$

Solving this system yields the tension values $F_1 = 45.1$ N, $F_2 = 55.0$ N, and $F_3 = 122.4$ N, rounded to the tenths place. ■

12.3 – The Dot Product

There are two common ways in which vectors can be "multiplied," each way contrived to yield practical results in the sciences. The easier kind of vector multiplication is called the "dot product," which is also known as the "scalar product" since the outcome of the product is always a scalar quantity.

Definition 12.12. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two vectors in* $\mathbb{R}^n$*. Then the* ***dot product*** *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is given by*

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Thus, if $\mathbf{x}$ and $\mathbf{y}$ are vectors in a plane, then

$$\mathbf{x} \cdot \mathbf{y} = \langle x_1, x_2 \rangle \cdot \langle y_1, y_2 \rangle = x_1 y_1 + x_2 y_2.$$

Theorem 12.13. *For any vectors* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$*,*

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
2. $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$
3. $(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (c\mathbf{y})$
4. $\mathbf{x} \cdot \mathbf{x} > 0$ *if* $\mathbf{x} \neq \mathbf{0}$

Proof.
Proof of Part (2). We have

$$\begin{aligned}
\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= \langle x_1, \ldots, x_n \rangle \cdot \big(\langle y_1, \ldots, y_n \rangle + \langle z_1, \ldots, z_n \rangle\big) \\
&= \langle x_1, \ldots, x_n \rangle \cdot \langle y_1 + z_1, \ldots, y_n + z_n \rangle \\
&= \sum_{i=1}^{n} x_i (y_i + z_i) = \sum_{i=1}^{n} (x_i y_i + x_i z_i) \\
&= \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i z_i = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z},
\end{aligned}$$

using the well-known summation property

$$\sum (a_i + b_i) = \sum a_i + \sum b_i.$$

Proofs for the other dot product properties are left to the exercises. ■

There is a nice geometrical aspect of the dot product that is made explicit in the following theorem.

Theorem 12.14. *Let* $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ *be vectors in* $\mathbb{R}^n$*, and let* $0 \leq \theta \leq \pi$ *be the angle between* $\mathbf{x}$ *and* $\mathbf{y}$ *when they are represented as arrows with a common initial point. Then*

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta.$$

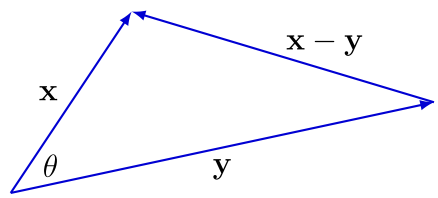


FIGURE 47.

Proof. It's sufficient to present the proof assuming that $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^2$, since in higher-dimensioned spaces the proof's structure remains the same. Let $0 < \theta < \pi$. The representative arrows for the vectors may be positioned so as to form a triangle as in Figure 47, where θ is depicted as an acute angle.

By the Law of Cosines we obtain

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta,$$

and since we're assuming that $\mathbf{x} = \langle x_1, x_2\rangle$ and $\mathbf{y} = \langle y_1, y_2\rangle$, we obtain $\mathbf{x}-\mathbf{y} = \langle x_1 - y_1, x_2 - y_2\rangle$ so that

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 = (x_1^2 + x_2^2) + (y_1^2 + y_2^2) - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta,$$

and hence

$$\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta = x_1y_1 + x_2y_2 = \mathbf{x}\cdot\mathbf{y}.$$

In the cases when $\theta = 0$ or $\theta = \pi$ we find that $\mathbf{y} = k\mathbf{x} = \langle kx_1, kx_2\rangle$ for some nonzero scalar k; that is, $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors, and we have

$$\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta = \|\mathbf{x}\|\|k\mathbf{x}\|\cos\theta = |k|(x_1^2 + x_2^2)\cos\theta. \tag{12.5}$$

If $\theta = 0$, then $k > 0$ so that $|k| = k$ and $\cos\theta = 1$; and if $\theta = \pi$, then $k < 0$ so that $|k| = -k$ and $\cos\theta = -1$. In either case, from (12.5) we obtain

$$\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta = k(x_1^2 + x_2^2) = \langle x_1, x_2\rangle \cdot \langle kx_1, kx_2\rangle = \mathbf{x}\cdot\mathbf{y}$$

as desired. ∎

Example 12.15. Find the measure of the angle between $\mathbf{x} = \langle -3, 7\rangle$ and $\mathbf{y} = \langle 5, -1\rangle$.

Solution. We have

$$\mathbf{x}\cdot\mathbf{y} = (-3)(5) + (7)(-1) = -22,$$

$$\|\mathbf{x}\| = \sqrt{(-3)^2 + 7^2} = \sqrt{58},$$

$$\|\mathbf{y}\| = \sqrt{5^2 + (-1)^2} = \sqrt{26}.$$

Thus

$$-22 = \mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta = \sqrt{58}\sqrt{26}\cos\theta = 2\sqrt{377}\cos\theta,$$

which gives $\cos\theta = -22/2\sqrt{377} \approx -0.5665$ and therefore

$$\theta \approx \arccos(-0.5665) \approx 124.51^\circ$$

is the answer. ∎

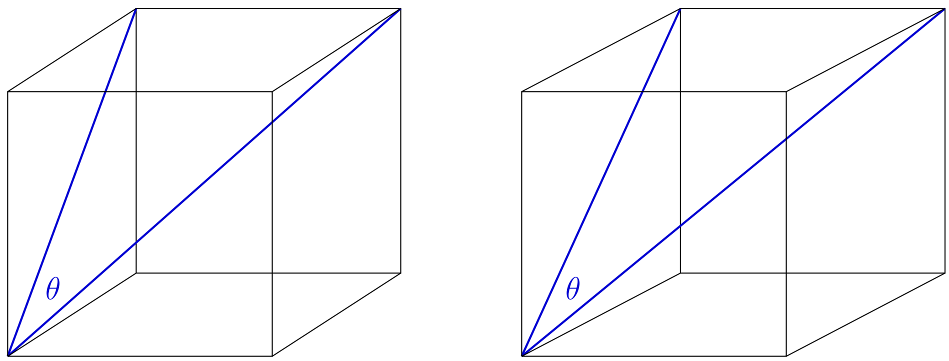


FIGURE 48.

Example 12.16. Find the measure of the angle between the diagonal of a cube and the diagonal of one of its faces.

Solution. The diagonals in question are shown in Figure 48, along with the angle θ between them. It will be convenient to regard the cube as existing in $\mathbb{R}^3$, with edges of length 1, and the vertex where the two diagonals meet situated at the origin $(0,0,0)$. We can then set up coordinate axes such that the cube diagonal has endpoints $(0,0,0)$ and $(1,1,1)$, and the face diagonal has endpoints $(0,0,0)$ and $(0,1,1)$. Thus the diagonals can be characterized as positions vectors $\mathbf{x} = \langle 1,1,1\rangle$ and $\mathbf{y} = \langle 0,1,1\rangle$. Now,

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{\langle 1,1,1\rangle \cdot \langle 0,1,1\rangle}{\sqrt{1^2+1^2+1^2}\,\sqrt{0^2+1^2+1^2}} = \frac{2}{\sqrt{6}},$$

and so

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35.264^\circ$$

is the angle's measure. ∎

Definition 12.17. *Two vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *are* ***orthogonal*** *if* $\mathbf{x}\cdot\mathbf{y} = 0$*, and we write* $\mathbf{x} \perp \mathbf{y}$*.*

Proposition 12.18. *If* $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ *are orthogonal vectors, then the angle between them is* 90°*.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero orthogonal vectors. Since the vectors are nonzero they can be represented by arrows with a common initial point, and so there is an angle $0^\circ \leq \theta \leq 180^\circ$ between them. By Theorem 12.14 and the orthogonality of the vectors we obtain

$$\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta = \mathbf{x}\cdot\mathbf{y} = 0. \tag{12.6}$$

Now, $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ implies that $\|\mathbf{x}\|, \|\mathbf{y}\| \neq 0$, so we can divide (12.6) by $\|\mathbf{x}\|\|\mathbf{y}\|$ to obtain $\cos\theta = 0$, and therefore $\theta = 90^\circ$. ∎

Definition 12.19. *Let* $\mathbf{y} \neq \mathbf{0}$*. The* ***orthogonal projection of*** $\mathbf{x}$ ***onto*** $\mathbf{y}$, $\operatorname{proj}_{\mathbf{y}}\mathbf{x}$*, is given by* $\operatorname{proj}_{\mathbf{y}}\mathbf{x} = (\|\mathbf{x}\|\cos\theta)\hat{\mathbf{y}}$*.*

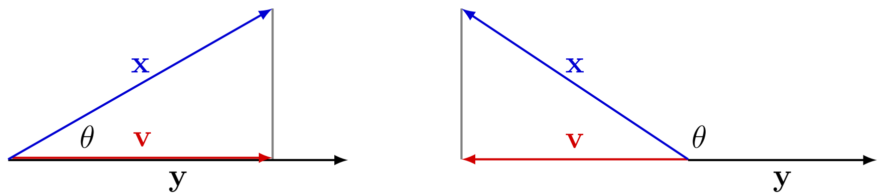


Figure 49. $\mathbf{v}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{y}$.

There is a rational explanation for this definition. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors that make an angle $0 < \theta < 90°$, as on the left-hand side of Figure 49. Consider the vector $\mathbf{v}$ in the figure. We have $\cos\theta = \|\mathbf{v}\|/\|\mathbf{x}\|$, so that $\|\mathbf{v}\| = \|\mathbf{x}\|\cos\theta$ is the magnitude of $\mathbf{v}$, while the direction of $\mathbf{v}$ is seen to be $\hat{\mathbf{y}}$. Hence

$$\mathbf{v} = (\|\mathbf{x}\|\cos\theta)\hat{\mathbf{y}} = \operatorname{proj}_{\mathbf{y}}\mathbf{x}.$$

The right-hand side of Figure 49 illustrates the situation when $90° < \theta < 180°$. We now have $\|\mathbf{v}\| = \|\mathbf{x}\|\cos(180° - \theta) = -\|\mathbf{x}\|\cos\theta$, and the direction of $\mathbf{v}$ is $-\hat{\mathbf{y}}$ so that

$$\mathbf{v} = (-\|\mathbf{x}\|\cos\theta)(-\hat{\mathbf{y}}) = (\|\mathbf{x}\|\cos\theta)\hat{\mathbf{y}} = \operatorname{proj}_{\mathbf{y}}\mathbf{x}$$

once again.

If $\theta = 90°$ it's easy to verify that $\mathbf{v} = \operatorname{proj}_{\mathbf{y}}\mathbf{x} = \mathbf{0}$, and the cases when θ is $0°$ or $180°$ are left to the reader to check. We see, then, that $\operatorname{proj}_{\mathbf{y}}\mathbf{x}$ literally represents the projection of $\mathbf{x}$ perpendicularly down onto $\mathbf{y}$, much like a shadow that is cast by $\mathbf{x}$ when there is a light source directly overhead.

Another notion that is hardly worth enshrining as a definition is that of the **scalar component of $\mathbf{x}$ in the direction of $\mathbf{y}$**, $\operatorname{scal}_{\mathbf{y}}\mathbf{x} = \|\mathbf{x}\|\cos\theta$. There is scarcely any point to remembering this formulation since it comes packaged in $\operatorname{proj}_{\mathbf{y}}\mathbf{x}$,

$$\operatorname{proj}_{\mathbf{y}}\mathbf{x} = (\operatorname{scal}_{\mathbf{y}}\mathbf{x})\hat{\mathbf{y}};$$

nevertheless it is the sort of thing many mainstream textbooks get excited about so it needed a mention.

Using $\hat{\mathbf{y}} = \mathbf{y}/\|\mathbf{y}\|$ and $\|\mathbf{x}\|\cos\theta = (\mathbf{x}\cdot\mathbf{y})/\|\mathbf{y}\|$, other formulations of $\operatorname{proj}_{\mathbf{y}}\mathbf{x}$ are as follows:

$$\operatorname{proj}_{\mathbf{y}}\mathbf{x} = \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|}\right)\left(\frac{\mathbf{y}}{\|\mathbf{y}\|}\right) = \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\right)\mathbf{y} = \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\right)\mathbf{y} \tag{12.7}$$

Proposition 12.20. *The vector* $\mathbf{x} - \operatorname{proj}_{\mathbf{y}}(\mathbf{x})$ *is orthogonal to* $\mathbf{y}$.

Proof. By Theorem 12.13(2) we have

$$\big(\mathbf{x} - \operatorname{proj}_{\mathbf{y}}(\mathbf{x})\big)\cdot\mathbf{y} = \mathbf{x}\cdot\mathbf{y} - \operatorname{proj}_{\mathbf{y}}(\mathbf{x})\cdot\mathbf{y},$$

and then by equation (12.7),

$$\big(\mathbf{x} - \operatorname{proj}_{\mathbf{y}}(\mathbf{x})\big)\cdot\mathbf{y} = \mathbf{x}\cdot\mathbf{y} - \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\right)\mathbf{y}\cdot\mathbf{y} = \mathbf{x}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{y} = 0.$$

Therefore $\mathbf{x} - \operatorname{proj}_{\mathbf{y}}(\mathbf{x}) \perp \mathbf{y}$. ∎

Example 12.21. Given $\mathbf{x} = \langle 4, 3, 0\rangle$ and $\mathbf{y} = \langle 1, 1, 1\rangle$, express $\mathbf{x}$ as the sum $\mathbf{x} = \mathbf{p} + \mathbf{n}$, where $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{n}$ is orthogonal to $\mathbf{y}$.

Solution. We start by obtaining $\mathbf{p}$, which will be the orthogonal projection of $\mathbf{x}$ onto $\mathbf{y}$:

$$\mathbf{p} = \operatorname{proj}_{\mathbf{y}} \mathbf{x} = \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\right)\mathbf{y} = \frac{7}{3}\langle 1, 1, 1\rangle = \left\langle \frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right\rangle.$$

Now, $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{y}$ by Proposition 12.20, and since $\mathbf{x} = \mathbf{p} + (\mathbf{x} - \mathbf{p})$ we conclude that

$$\mathbf{n} = \mathbf{x} - \mathbf{p} = \langle 4, 3, 0\rangle - \left\langle \frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right\rangle = \left\langle \frac{5}{3}, \frac{2}{3}, -\frac{7}{3}\right\rangle.$$

That is,

$$\mathbf{x} = \mathbf{p} + \mathbf{n} = \left\langle \frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right\rangle + \left\langle \frac{5}{3}, \frac{2}{3}, -\frac{7}{3}\right\rangle$$

is the desired decomposition of $\mathbf{x}$. ∎

Example 12.22. In physics the **work** W done by a constant force $\mathbf{F}$ when it displaces an object by $\mathbf{d}$ is given as

$$W = \|\mathbf{F}\|\|\mathbf{d}\|\cos\theta = \mathbf{F}\cdot\mathbf{d}.$$

The metric unit of work (and energy in general) is the joule, symbol J, written in terms of other SI units as $\mathrm{J} = \mathrm{N}\cdot\mathrm{m} = \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2$. Depicted in Figure 50 is a situation in which $\mathbf{F} = \langle 7.5, 9.5, 8.5\rangle$ moves an object from point $p = (3, 2, 0)$ to $q = (9, 12, 0)$. Letting $\mathbf{p} = \langle 3, 2, 0\rangle$ and $\mathbf{q} = \langle 9, 12, 0\rangle$, then we find that $\mathbf{d} = \mathbf{q} - \mathbf{p} = \langle 6, 10, 0\rangle$, and so

$$W = \mathbf{F}\cdot\mathbf{d} = \langle 7.5, 9.5, 8.5\rangle \cdot \langle 6, 10, 0\rangle = (7.5)(6) + (9.5)(10) + (8.5)(0) = 140 \text{ J}$$

is the work performed. ∎

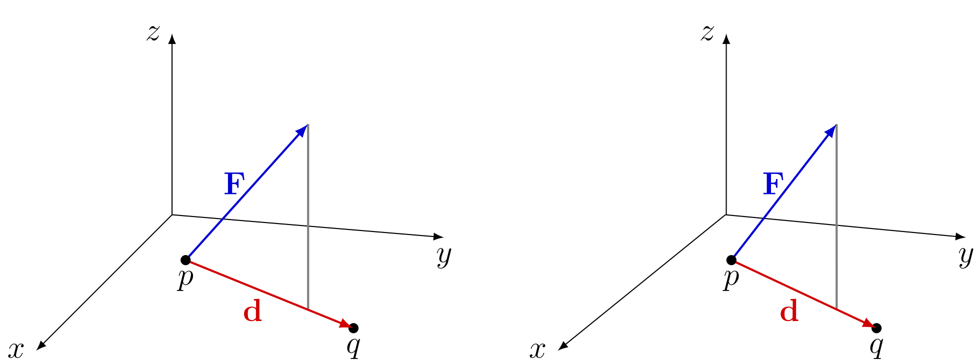


FIGURE 50. The wonderments of work, in 3D no less.

12.4 – The Cross Product

Definition 12.23. *The* ***cross product*** *of* $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ *and* $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ *is defined as*

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Since the cross product of two vectors produces another vector, it is sometimes called the "vector product". Unlike the dot product which works in any $\mathbb{R}^n$, the cross product only makes sense in $\mathbb{R}^3$. That is, $\mathbf{u}\times\mathbf{v}$ can only be calculated if $\mathbf{u}$ and $\mathbf{v}$ are ordered triples, and it turns out that if $\mathbf{w} = \mathbf{u}\times\mathbf{v}$ then $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$! Thus, the cross product is most often used to find a vector that is perpendicular to any two given vectors.

Theorem 12.24. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^3$*, and let* $c \in \mathbb{R}$*. Then*

1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$
3. $c(\mathbf{u}\times\mathbf{v}) = (c\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(c\mathbf{v})$
4. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$
5. $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
6. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

The proofs for the various properties of the cross product given in Theorem 12.24 are fairly straightforward and are left to the reader.

Proposition 12.25. *If* $\mathbf{w} = \mathbf{u}\times\mathbf{v}$*, then* $\mathbf{w}\perp\mathbf{u}$ *and* $\mathbf{w}\perp\mathbf{v}$*.*

Proof. Suppose $\mathbf{w} = \mathbf{u}\times\mathbf{v}$. Then

$$w = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle$$

by definition of the cross product. By definition of the dot product,

$$\begin{aligned}\mathbf{w}\cdot\mathbf{u} &= (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 \\ &= u_1u_2v_3 - u_1u_3v_2 + u_2u_3v_1 - u_1u_2v_3 + u_1u_3v_2 - u_2u_3v_1 \\ &= (u_1u_2v_3 - u_1u_2v_3) + (u_2u_3v_1 - u_2u_3v_1) + (u_1u_3v_2 - u_1u_3v_2) = 0.\end{aligned}$$

A similar argument will show that $\mathbf{w}\cdot\mathbf{v} = 0$. Therefore $\mathbf{w}\perp\mathbf{u}$ and $\mathbf{w}\perp\mathbf{v}$. ■

It's easy to verify that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, as you might expect in light of Proposition 12.25. But be careful, because as Proposition 12.25(1) tells us the cross product is not commutative: $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$. Its quirks notwithstanding, the cross product was invented precisely because it models many physical phenomena. The next theorem gives a couple more properties of the cross product that require a little more work to verify.

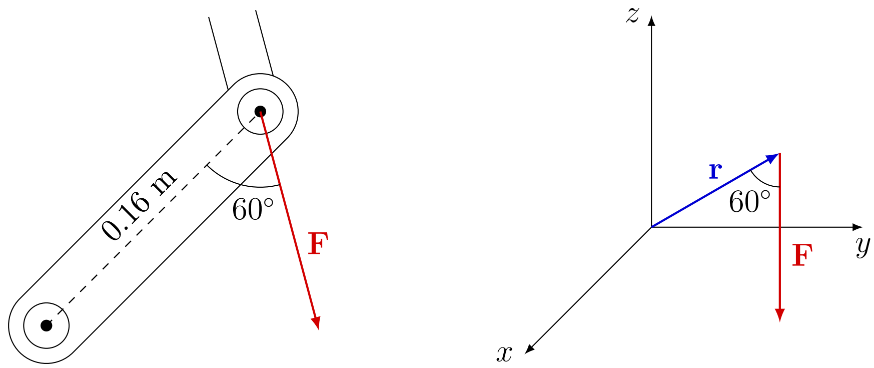


FIGURE 51.

Theorem 12.26. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in* $\mathbb{R}^3$*, and let* θ *be the angle between them. Then*

1. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$
2. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ *iff* $\mathbf{u} = c\mathbf{v}$ *for some* $c \in \mathbb{R}$.

Example 12.27. A force $\mathbf{F}$ with magnitude 2000 N acts on a crankshaft that is 0.16 meters long as shown at left in Figure 51. Find the torque on the crankshaft.

Solution. From physics, the torque $\boldsymbol{\tau}$ is a vector quantity given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. The right side of Figure 51 illustrates the situation in a conveniently placed coordinate system, so that $\mathbf{F} = \langle 0, 0, -2000\rangle$ and the position vector is

$$\mathbf{r} = \|\mathbf{r}\|\hat{\mathbf{r}} = 0.16\langle 0, \cos 30°, \sin 60°\rangle = 0.16\left\langle 0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle = \langle 0, 0.08\sqrt{3}, 0.08\rangle.$$

Thus we obtain...

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.08\sqrt{3} & 0.08 \\ 0 & 0 & -2000 \end{vmatrix} = (0.08\sqrt{3})(-2000)\mathbf{i} = -160\sqrt{3}\,\mathbf{i}$$

That is, $\boldsymbol{\tau} = \langle -160\sqrt{3}, 0, 0\rangle$, and in particular $\|\boldsymbol{\tau}\| = 277.1$ N·m. ■

12.5 – Cylindrical and Spherical Coordinates

13

Curves and Surfaces

13.1 – Vector-Valued Functions

Definition 13.1. *Let $S \subseteq \mathbb{R}$ and $n \geq 2$, and suppose $x_k : S \to \mathbb{R}$ for each $1 \leq k \leq n$. A **vector-valued function of one variable** is a function $\mathbf{r} : S \to \mathbb{R}^n$ given by*

$$\mathbf{r}(t) = \big\langle x_1(t), \ldots, x_n(t) \big\rangle$$

for all $t \in S$.

*We say $\mathbf{r}$ is **continuous** at $t_0 \in S$ if for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $t \in S$,*

$$|t - t_0| < \delta \quad \Rightarrow \quad \|\mathbf{r}(t) - \mathbf{r}(t_0)\| < \epsilon.$$

Finally, we write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{a}$$

if

$$\lim_{t \to t_0} \|\mathbf{r}(t) - \mathbf{a}\| = 0.$$

Proposition 13.2. *Let $\mathbf{r}(t) = \langle x_1(t), \ldots, x_n(t) \rangle$. If*

$$\lim_{t \to t_0} x_1(t) = a_1, \ldots, \lim_{t \to t_0} x_n(t) = a_n$$

for $a_1, \ldots, a_n \in \mathbb{R}$, then

$$\lim_{t \to t_0} \mathbf{r}(t) = \langle a_1, \ldots, a_n \rangle.$$

Proof. Let $\epsilon > 0$. For each $1 \leq i \leq n$, since $x_i(t) \to a_i$ as $t \to t_0$, there exists some $\delta_i > 0$ such that

$$0 < |t - t_0| < \delta_i \quad \Rightarrow \quad |x_i(t) - a_i| < \epsilon/\sqrt{n}.$$

Choose

$$\delta = \min\{\delta_1, \ldots, \delta_n\}.$$

Suppose that $0 < |t - t_0| < \delta$. Then

$$[x_i(t) - a_i]^2 < \epsilon^2/n$$

for each $1 \leq i \leq n$, and so

$$\begin{aligned}\big\|\mathbf{r}(t) - \langle a_1, \ldots, a_n\rangle\big\| &= \big\|\big\langle x_1(t) - a_1, \ldots, x_n(t) - a_n\big\rangle\big\| \\ &= \sqrt{[x_1(t) - a_1]^2 + \cdots + [x_n(t) - a_n]^2} \\ &< \left(\sum_{i=1}^{n} \frac{\epsilon^2}{n}\right)^{1/2} = (\epsilon^2)^{1/2} = \epsilon,\end{aligned}$$

which completes the proof. ■

The conclusion of Proposition 13.2 is most succinctly written as follows:

$$\lim_{t \to t_0} \big\langle x_1(t), \ldots, x_n(t)\big\rangle = \Big\langle \lim_{t \to t_0} x_1(t), \ldots, \lim_{t \to t_0} x_n(t)\Big\rangle .$$

Theorem 13.3. *A vector function $\mathbf{r}(t) = \langle x_1(t), \ldots, x_n(t)\rangle$ is continuous at t_0 if and only if the functions $x_1, \ldots, x_n$ are continuous at t_0.*

A **curve** in $\mathbb{R}^n$ is a continuous vector-valued function $\mathbf{r} : I \to \mathbb{R}^n$ for which the domain $I \subseteq \mathbb{R}$ is an interval. For $S \subseteq \mathbb{R}^n$, a **curve in** S is a curve $\mathbf{r} : I \to \mathbb{R}^n$ such that $\mathrm{Ran}(\mathbf{r}) \subseteq S$. A **planar curve** is a curve in a plane; that is, there is a plane $P \subseteq \mathbb{R}^n$ such that $\mathrm{Ran}(\mathbf{r}) \subseteq P$. Clearly any curve in $\mathbb{R}^2$ is necessarily a planar curve. A curve that is not planar is called **nonplanar**.

Any set $C \subseteq \mathbb{R}^n$ for which there exists a curve $\mathbf{r} : I \to \mathbb{R}^n$ such that $\mathrm{Ran}(\mathbf{r}) = C$ is also called a **curve**, in which case we say that

$$\mathbf{r}(t) = \big\langle x_1(t), \ldots, x_n(t)\big\rangle, \quad t \in I,$$

is a **parametrization** of C. The independent variable t is the **parameter**. Thus for each $t \in I$ the vector $\mathbf{r}(t)$ is the position vector of a point in the set C, which we naturally identify with the point itself. In explicit terms, if $(a_1, \ldots, a_n)$ is a point on C, then there exists some $t_0 \in I$ such that

$$\mathbf{r}(t_0) = \langle a_1, \ldots, a_n\rangle.$$

A **parametrized curve** is a curve C for which a parametrization $\mathbf{r}(t)$, $t \in I$, has been given.

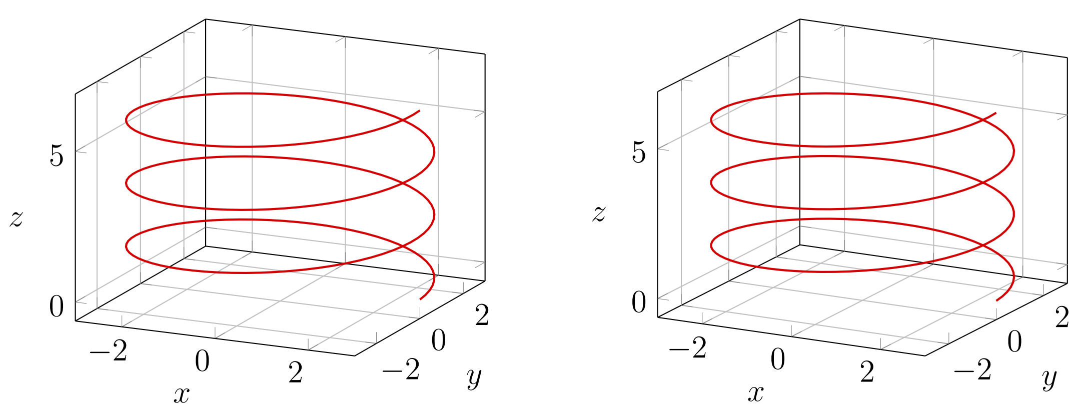


FIGURE 52. Stereoscopic image of a helical curve.

Common kinds of curves are lines, line segments, rays, circles, ellipses, parabolas, triangles, rectangles, and helixes.

Example 13.4. A helix is a nonplanar curve in $\mathbb{R}^3$ that coils around an axis rather like a spring. An example of a helix is shown in Figure 52, and an example of a parametrization for the helix is

$$\mathbf{r}(t) = \big\langle 3\cos(3t), 3\sin(3t), t\big\rangle, \quad t \in [0, 2\pi].$$

This is not the only parametrization possible! ■

Example 13.5. The circle in $\mathbb{R}^3$ lying in the xy-plane with center at $\mathbf{0} = \langle 0,0,0\rangle$ and radius 1, which we will denote by $C_1(\mathbf{0})$, is simply the set of points

$$C_1(\mathbf{0}) = \{(x, y, 0) : x^2 + y^2 = 1\}.$$

The set $C_1(\mathbf{0})$ is in fact a curve, and it will become a parametrized curve once we give it a parametrization $\mathbf{r}(t)$, $t \in I$. Consider the following functions:

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, 0\rangle, \quad t \in [0, 2\pi] \tag{13.1}$$

$$\mathbf{r}_2(t) = \langle \cos(-t), \sin(-t), 0\rangle, \quad t \in [0, 2\pi] \tag{13.2}$$

$$\mathbf{r}_3(t) = \langle \cos 2t, \sin 2t, 0\rangle, \quad t \in [0, 2\pi] \tag{13.3}$$

$$\mathbf{r}_4(t) = \langle \cos t, \sin t, 0\rangle, \quad t \in [0, 4\pi] \tag{13.4}$$

All four functions are continuous vector-valued functions on an interval domain, and therefore all are curves. Moreover, since all the functions have range equal to $C_1(\mathbf{0})$, it follows that each function is a parametrization of $C_1(\mathbf{0})$. ■

Definition 13.6. *Two parametrizations* $\mathbf{r}$ *and* $\boldsymbol{\rho}$ *are* ***equal*** *(written* $\mathbf{r} = \boldsymbol{\rho}$*) if* $\mathrm{Dom}(\mathbf{r}) = \mathrm{Dom}(\boldsymbol{\rho})$*, and* $\mathbf{r}(t) = \boldsymbol{\rho}(t)$ *for each* t *in the common domain.*

Notice that this definition is just the definition for what it means to say *any* two functions (vector-valued or otherwise) are equal. Now, the parametrization (13.4) doesn't equal the others since its domain is $[0, 4\pi]$ while the domains of (13.1), (13.2) and (13.3) are all $[0, 2\pi]$. To see that the others differ from one another, note that $\mathbf{r}_1(\pi/2) = (0, 1, 0)$, $\mathbf{r}_2(\pi/2) = (0, -1, 0)$, and $\mathbf{r}_3(\pi/2) = (-1, 0, 0)$. All different! Or just look at the "big picture", and notice that (13.1) goes once counterclockwise around the circle $C_1(0, 0, 0)$ as t goes from 0 to 2π, (13.2) goes once clockwise, and (13.3) goes *twice* counterclockwise. Yes, (13.4) also goes twice counterclockwise like (13.3), but it does so more "slowly": viewing t as time in seconds, (13.3) runs its laps in 2π seconds whilst (13.4) takes 4π seconds.

The nuances of the terminology are delicate here, and before going further they should be sorted out. What we've just done is examine four different algebraic *parametrizations* ($\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$) for the same geometric *curve* $C_1(0, 0, 0)$. In general if $\mathbf{r}$ is a parametrization for a curve C, and the occasion warrants making a distinction between the two, we will refer to the set of points C as the "curve" and the function $\mathbf{r}$ as a "parametrization."

Let us now examine lines, remembering that two points uniquely determine a line. We begin with the case of a line ℓ in $\mathbb{R}^2$ containing points (x_0, y_0) and (x_1, y_1). Recall that such a line has slope

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

generally referred to as "rise over run." Starting from the point (x_0, y_0), we can arrive at any other point (x, y) on ℓ by adding real numbers Δx (the "run") and Δy (the "rise") to x_0 and y_0, respectively: $(x, y) = (x_0 + \Delta x, y_0 + \Delta y)$. But the rise over the run must equal m: $\Delta y/\Delta x = m$, which is to say $\Delta y = m\Delta x$. Thus

$$(x, y) = (x_0 + \Delta x,\, y_0 + m\Delta x) = \left(x_0 + \Delta x,\, y_0 + \frac{y_1 - y_0}{x_1 - x_0}\Delta x\right).$$

Now, if we let $t \in \mathbb{R}$ be such that $(x_1 - x_0)t = \Delta x$, we obtain

$$(x, y) = \left(x_0 + (x_1 - x_0)t,\, y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x_1 - x_0)t\right) = \big(x_0 + (x_1 - x_0)t,\, y_0 + (y_1 - y_0)t\big). \quad (13.5)$$

Since (x, y) is an arbitrary point on ℓ, we find that all points on ℓ must be expressible as the rightmost ordered pair in (13.5), and therefore

$$\ell = \Big\{ \big(x_0 + (x_1 - x_0)t,\, y_0 + (y_1 - y_0)t\big) : t \in \mathbb{R} \Big\}. \quad (13.6)$$

The findings above motivate the following definition, which naturally extends the pattern in (13.6) to lines in $\mathbb{R}^3$.

Definition 13.7. *The* ***line*** *in* $\mathbb{R}^3$ *containing points* (x_0, y_0, z_0) *and* (x_1, y_1, z_1) *is the set*

$$L = \Big\{ \big(x_0 + (x_1 - x_0)t,\, y_0 + (y_1 - y_0)t,\, z_0 + (z_1 - z_0)t\big) : t \in \mathbb{R} \Big\}.$$

Define the vectors

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle \quad \text{and} \quad \mathbf{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle.$$

Then, for any $t \in \mathbb{R}$,

$$\mathbf{r}_0 + t\mathbf{v} = \big\langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0) \big\rangle$$

is the position vector of the point in the set L in Definition 13.7, and therefore

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}, \quad (13.7)$$

is a parametrization for the line L. Behold how $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$, which is merely a rearrangement of (13.7), bears a striking resemblance to the familiar old function $f(x) = mx + b$ that models a line in the xy-plane! We see that $\mathbf{v}$, called the **direction vector** of the line L, takes the place of slope m. Indeed, L is seen to be fully determined by its direction vector $\mathbf{v}$ and the point (x_0, y_0, z_0) represented by $\mathbf{r}_0$.

An alternate parametrization for the line L that doesn't look as much akin to $f(x) = mx + b$ is given by

$$\tilde{\mathbf{r}}(t) = \mathbf{r}_0 + t^3\mathbf{v}, \quad -\infty < t < \infty,$$

and still another is

$$\check{\mathbf{r}}(t) = \mathbf{r}_0 - 2t\mathbf{v}, \quad -\infty < t < \infty.$$

Play with these, and notice that $\mathbf{r}$, $\tilde{\mathbf{r}}$ and $\check{\mathbf{r}}$ all trace out the same line. The first two, you might notice, move in the same direction along L, while the third parametrization moves in the opposite direction. Thus a parametrized curve that traces a given curve (like L) has what's called an **orientation**. So again, any given curve (a set of points) in $\mathbb{R}^3$ has innumerable parametrizations, which is to say it can be traced by innumerable parametrized curves, each parametrized curve having one of two possible orientations.

Related to a line is a line segment, which is a piece of a line with two endpoints.

Definition 13.8. *The **line segment** in $\mathbb{R}^3$ from $p_0 = (x_0, y_0, z_0)$ to $p_1 = (x_1, y_1, z_1)$ is the set*

$$[p_0, p_1] = \left\{ \big(x_0 + t(x_1 - x_0),\ y_0 + t(y_1 - y_0),\ z_0 + t(z_1 - z_0)\big) : t \in [0, 1] \right\}.$$

This definition corresponds to the usual idea of a line segment in geometry. If we let

$$\mathbf{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

and (as before) $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, then a fine parametrization for the line segment in Definition 13.8 is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in [0, 1].$$

Example 13.9. Find a parametrization for the line L passing through $(-1, 4, -3)$ and having direction given by $\langle 3, 2, 3 \rangle$.

Solution. We have $\mathbf{r}_0 = \langle -1, 4, -3 \rangle$ and $\mathbf{v} = \langle 3, 2, 3 \rangle$. So, referring to (13.7), we obtain

$$\mathbf{r}(t) = \langle -1, 4, -3 \rangle + t\langle 3, 2, 3 \rangle, \quad t \in \mathbb{R},$$

or alternatively

$$\mathbf{r}(t) = \langle -1 + 3t, 4 + 2t, -3 + 3t \rangle, \quad t \in \mathbb{R},$$

as a parametrization. ■

13.2 – The Calculus of Vector-Valued Functions

In keeping with the notion of derivative for a real-valued function of a single variable, the **derivative** of a vector function $\mathbf{r}(t) = \langle x_1(t), \ldots, x_n(t) \rangle$ at t_0 is defined to be

$$\mathbf{r}'(t_0) = \lim_{t \to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}, \tag{13.8}$$

provided that the limit exists. The following theorem furnishes an easy way to find the derivative of $\mathbf{r}$ wherever it exists.

Theorem 13.10. *Let $\mathbf{r}(t) = \langle x_1(t), \ldots, x_n(t) \rangle$. Then $\mathbf{r}'(t_0)$ exists if and only if $x_k'(t_0)$ exists for all $1 \leq k \leq n$, in which case*

$$\mathbf{r}'(t_0) = \langle x_1'(t_0), \ldots, x_n'(t_0) \rangle.$$

Proof. Suppose that $\mathbf{r}'(t_0)$ exists. Since

$$\frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0} = \left\langle \frac{x_1(t) - x_1(t_0)}{t - t_0}, \ldots, \frac{x_n(t) - x_n(t_0)}{t - t_0} \right\rangle,$$

by (13.8) and Proposition 13.2 we obtain

$$\mathbf{r}'(t_0) = \left\langle \lim_{t \to t_0} \frac{x_1(t) - x_1(t_0)}{t - t_0}, \ldots, \lim_{t \to t_0} \frac{x_n(t) - x_n(t_0)}{t - t_0} \right\rangle, \tag{13.9}$$

and so the existence of $\mathbf{r}'(t_0)$ implies the existence of each limit on the right-hand side. That is,

$$x_i'(t_0) = \lim_{t \to t_0} \frac{x_i(t) - x_i(t_0)}{t - t_0} \tag{13.10}$$

exists for each $1 \leq i \leq n$, and equations (13.9) and (13.10) taken together yield

$$\mathbf{r}'(t_0) = \langle x_1'(t_0), \ldots, x_n'(t_0) \rangle$$

as desired.

The proof of the converse is straightforward and so is left as an exercise. ∎

Example 13.11. The derivative of

$$\mathbf{r}(t) = \big\langle (t+1)^{-1}, \arctan(2t), \ln(t+1) \big\rangle$$

is

$$\mathbf{r}'(t) = \left\langle -\frac{1}{(t+1)^2}, \frac{2}{1+4t^2}, \frac{1}{t+1} \right\rangle,$$

recalling that $\arctan'(x) = 1/(1+x^2)$. ∎

We say that $\mathbf{r}$ is **differentiable** on an interval I if $\mathbf{r}'(t)$ exists for all $t \in I$. If, in addition, $\mathbf{r}'$ is continuous on I, then we say $\mathbf{r}$ is **continuously differentiable** on I. If I is $[a, b]$, or some other interval that is not open, then we define $\mathbf{r}'$ at the endpoints a and b by appropriate one-sided limits,

$$\mathbf{r}'(a) = \lim_{t \to a^+} \frac{\mathbf{r}(t) - \mathbf{r}(a)}{t - a} \quad \text{and} \quad \mathbf{r}'(b) = \lim_{t \to b^-} \frac{\mathbf{r}(t) - \mathbf{r}(b)}{t - b},$$

provided the limits exist. If $\mathbf{r}'(a)$ exists we say $\mathbf{r}$ is **right-differentiable** at a, and if $\mathbf{r}'(b)$ exists we say $\mathbf{r}$ is **left-differentiable** at b. (Sometimes the symbols $\mathbf{r}'_+(a)$ and $\mathbf{r}'_-(b)$ to denote right-hand and left-hand derivatives, respectively.)

Definition 13.12. *Let I be an interval, and let $\mathbf{r} : I \to \mathbb{R}^n$ be a continuous function. We say $\mathbf{r}$ is **smooth** on I if it is continuously differentiable on I, and $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in \operatorname{Int}(I)$. We say $\mathbf{r}$ is **piecewise smooth** on I if I may be partitioned into subintervals $I_1, \dots, I_n$ such that $\mathbf{r}$ is smooth on I_k for each $1 \leq k \leq n$.*

A curve C is called **smooth** if it admits a smooth parametrization, and **piecewise smooth** if it admits a piecewise smooth parametrization. Whenever a curve is said to be smooth it is assumed that any parametrization given for the curve is itself smooth.

Definition 13.13. *If $\mathbf{r}'(t) \neq \mathbf{0}$, then $\mathbf{r}'(t)$ is called the **tangent vector** for $\mathbf{r}$ at the point corresponding to $\mathbf{r}(t)$. The **unit tangent vector** for $\mathbf{r}$ at $\mathbf{r}(t)$ is*

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Example 13.14. Find the unit tangent vector for the curve given by $\mathbf{r}(t) = \langle \sin t, \cos t, e^{-t} \rangle$, $-\pi \leq t \leq \pi$, at the point $\mathbf{r}(0)$.

Solution. From $\mathbf{r}'(t) = \langle \cos t, -\sin t, -e^{-t} \rangle$ and the observation that $t = 0$ at the point $\mathbf{r}(0)$, we obtain

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\langle 1, 0, -1 \rangle}{\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle$$

as the unit tangent vector. ■

Example 13.15. Consider the curve $\mathbf{r}(t) = \langle \sqrt{t}, 1, t \rangle$, $t > 0$. Find all the points on the curve at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

Solution. We have

$$\mathbf{r}'(t) = \left\langle \frac{1}{2\sqrt{t}}, 0, 1 \right\rangle$$

for $t > 0$. By definition $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal when $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, so we must find all $t > 0$ for which

$$\left(\sqrt{t}\right)\left(\frac{1}{2\sqrt{t}}\right) + (1)(0) + (t)(1) = \frac{1}{2} + t = 0.$$

Clearly this only occurs for $t = -1/2$; but $-1/2$ is not in the domain of $\mathbf{r}$, so we conclude that $\mathbf{r}$ and $\mathbf{r}'$ are never orthogonal. ■

If $\mathbf{u}, \mathbf{v} : I \to \mathbb{R}^n$ are vector functions defined on some interval I, $f : I \to \mathbb{R}$ is a scalar function, and $c \in \mathbb{R}$, then we define $c\mathbf{u}$, $f\mathbf{u}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ to be the functions given by

$$(c\mathbf{u})(t) = c\mathbf{u}(t), \quad (f\mathbf{u})(t) = f(t)\mathbf{u}(t), \quad (\mathbf{u} + \mathbf{v})(t) = \mathbf{u}(t) + \mathbf{v}(t), \quad (\mathbf{u} \cdot \mathbf{v})(t) = \mathbf{u}(t) \cdot \mathbf{v}(t),$$

and

$$(\mathbf{u} \times \mathbf{v})(t) = \mathbf{u}(t) \times \mathbf{v}(t)$$

for all $t \in I$. Notice that only $\mathbf{u} \cdot \mathbf{v}$ is not a vector function!

Theorem 13.16. *Let $I \subseteq \mathbb{R}$ be an interval. Let $\mathbf{u}, \mathbf{v} : I \to \mathbb{R}^n$ be vector functions and $f : I \to \mathbb{R}$ a scalar function such that $\mathbf{u}$, $\mathbf{v}$, and f are differentiable at $t \in \operatorname{Int}(I)$.*

1. *Constant Rule: If $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ is a constant vector function, then $\mathbf{c}' = \mathbf{0}$.*
2. *Constant Multiple Rule: If $c \in \mathbb{R}$, then $(c\mathbf{u})'(t) = c\mathbf{u}'(t)$.*
3. *Sum Rule: $(\mathbf{u} + \mathbf{v})'(t) = \mathbf{u}'(t) + \mathbf{v}'(t)$.*
4. *Product Rule: $(f\mathbf{u})'(t) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$.*
5. *Dot Product Rule: $(\mathbf{u} \cdot \mathbf{v})'(t) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$.*
6. *Cross Product Rule: $(\mathbf{u} \times \mathbf{v})'(t) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.*

Theorem 13.17 (**Chain Rule**). *Let $I, J \subseteq \mathbb{R}$ be intervals. Let $f : I \to J$ and $\mathbf{r} : J \to \mathbb{R}^n$. If f is differentiable at $t \in \operatorname{Int}(I)$ and $\mathbf{r}$ is differentiable at $f(t) \in \operatorname{Int}(J)$, then $\mathbf{r} \circ f$ is differentiable at t, and*

$$(\mathbf{r} \circ f)'(t) = \mathbf{r}'(f(t))f'(t).$$

Proof. Since $\mathbf{r}$ maps to $\mathbb{R}^n$ there exist scalar functions $x_1, \ldots, x_n : J \to \mathbb{R}$ for which

$$\mathbf{r}(\tau) = \langle x_1(\tau), \ldots, x_n(\tau) \rangle$$

for all $\tau \in J$. Suppose that $t \in \operatorname{Int}(I)$, $f(t) \in \operatorname{Int}(J)$, f is differentiable at t, and $\mathbf{r}$ is differentiable at $f(t)$. From the last supposition it follows by Theorem 13.10 that x_i is differentiable at $f(t)$ for each $1 \le i \le n$, and so by Theorem 3.21 (the Chain Rule for scalar functions) we have

$$(x_i \circ f)'(t) = x_i'(f(t))f'(t) \tag{13.11}$$

for each i. Now, for any $\tau \in I$ we have

$$(\mathbf{r} \circ f)(\tau) = \mathbf{r}(f(\tau)) = \big\langle x_1(f(\tau)), \ldots, x_n(f(\tau)) \big\rangle = \big\langle (x_1 \circ f)(\tau), \ldots, (x_n \circ f)(\tau) \big\rangle,$$

and hence by another application of Theorem 13.10 together with equation (13.11),

$$\begin{aligned}(\mathbf{r} \circ f)'(t) &= \big\langle (x_1 \circ f)'(t), \ldots, (x_n \circ f)'(t) \big\rangle = \big\langle x_1'(f(t))f'(t), \ldots, x_n'(f(t))f'(t) \big\rangle \\ &= \big\langle x_1'(f(t)), \ldots, x_n'(f(t)) \big\rangle f'(t) = \mathbf{r}'(f(t))f'(t).\end{aligned}$$

This completes the proof. ∎

13.3 – Objects in Motion

Definition 13.18. *Let I be an interval, and let the* ***position*** *of an object $\mathcal{O}$ in $\mathbb{R}^3$ at time t be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for all $t \in I$. The* ***velocity*** *of $\mathcal{O}$ at time t is*

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle,$$

with the ***speed*** *being the scalar $\|\mathbf{v}(t)\|$. The* ***acceleration*** *of $\mathcal{O}$ at time t is*

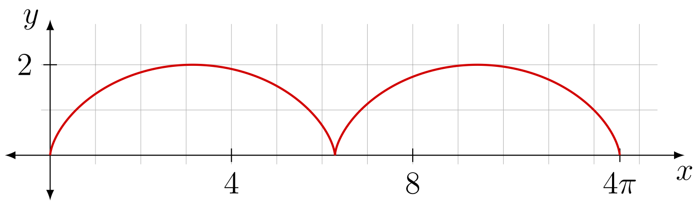
$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle x''(t), y''(t), z''(t)\rangle.$$

Example 13.19. Consider an object in motion along a cycloid trajectory as given by the position function $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$, $t \in [0, 4\pi]$.

(a) Graph the trajectory.
(b) Find the velocity and speed of the object. At what points on the trajectory does the object move fastest? Slowest?
(c) Find the acceleration of the object and show that $\|\mathbf{a}(t)\|$ is constant.

Solution.

(a) The trajectory is seen to have a cusp at $x = 2\pi$, which happens to be the point cor-responding to $t = 2\pi$.



(b) The velocity of the object is given by $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - \cos t, \sin t\rangle$, and the speed is

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}.$$

The object is moving fastest at the points where $2 - 2\cos t = 4$, which implies that $\cos t = -1$. Solutions in the interval $[0, 4\pi]$ are $t = \pi, 3\pi$, with corresponding points being $(\pi, 2)$ and $(3\pi, 2)$.

The object is moving slowest when $2 - 2\cos t = 0$, which implies that $\cos t = 1$. Solutions are $t = 0, 2\pi, 4\pi$, with corresponding points being $(0, 0)$, $(2\pi, 0)$ and $(4\pi, 0)$.

(c) Here $\mathbf{a}(t) = \mathbf{v}'(t) = \langle \sin t, \cos t\rangle$, so that

$$a(t) = \|\mathbf{a}(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1,$$

a constant. ■

13.4 – Arc Length

In all that follows in this section, and in any situation when the length of a curve C is under consideration, we assume that C is given a parametrization $\mathbf{r}(t)$, $t \in I$, that either never passes through the same point on C more than once (i.e. $\mathbf{r}$ is one-to-one), or else passes through only a finite number of points on C more than once. Thus, we wouldn't entertain a parametrization of an ellipse that runs twice around the ellipse if we're interested in determining the length of the elliptical path.

Suppose a curve C in $\mathbb{R}^2$ has a parametrization given by the vector-valued position function $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, $t \in [a, b]$, as depicted in Figure 53. What is the length $\mathcal{L}(C)$ of the curve? We could get a reasonable approximation by choosing points $p_0, p_1, \ldots, p_n$, connecting them by line segments $[p_{i-1}, p_i]$ for $1 \leq i \leq n$, and then summing the lengths of the line segments:

$$\mathcal{L}(C) \approx \sum_{i=1}^{n} \mathcal{L}([p_{i-1}, p_i]). \tag{13.12}$$

This so-called **polygonal approximation** approach, also depicted in Figure 53, should be expected to give "better" results as the number of approximating line segments is increased.

The general procedure for arriving at a value for $\mathcal{L}(C)$ given a curve C that lies in a plane now follows. Assume C to be a smooth curve in $\mathbb{R}^2$ with two endpoints (i.e. C is a path with a "beginning" and an "end"). Then C admits a smooth parametrization $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, $t \in [a, b]$. Let $P \in \mathcal{P}[a, b]$, which is to say that P is a partition of $[a, b]$ by points $t_0, t_1, \ldots, t_n$, with

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b,$$

to give subintervals $[t_{i-1}, t_i]$ for $1 \leq i \leq n$. Define $x_i = f(t_i)$ and $y_i = g(t_i)$ for each i, so that $p_i(x_i, y_i)$ is the point corresponding to $\mathbf{r}(t_i)$:

$$\mathbf{r}(t_i) = \langle f(t_i), g(t_i)\rangle = \langle x_i, y_i\rangle \mapsto p_i$$

If we let $\Delta x_i = |x_i - x_{i-1}|$ and $\Delta y_i = |y_i - y_{i-1}|$, then from Figure 54 it can be seen that

$$\mathcal{L}([p_{i-1}, p_i]) = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2},$$

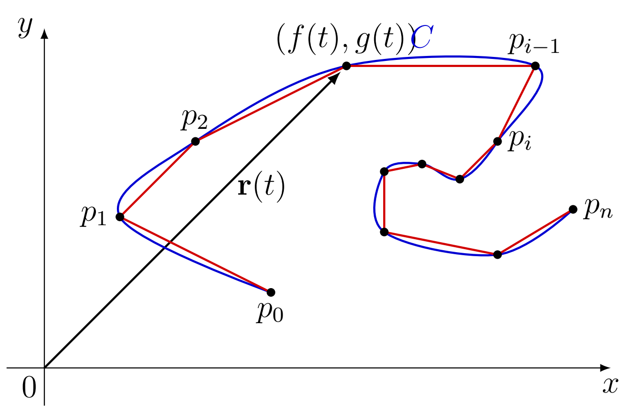


Figure 53. A polygonal approximation of the curve C.

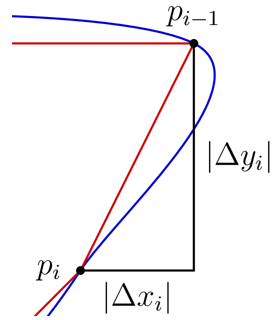


FIGURE 54. Determining the distance between p_{i-1} and p_i.

and so from (13.12)

$$\mathcal{L}(C) \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

As before, let $\|P\|$ denote the mesh of the partition P; that is, if $\Delta t_i = |t_i - t_{i-1}|$ then $\|P\| = \max_{1\le i\le n} \Delta t_i$. We define the **arc length of $\boldsymbol{C}$** (often just called the **length**) to be given by

$$\mathcal{L}(C) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \mathcal{L}([p_{i-1}, p_i]) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}. \tag{13.13}$$

Referring to Definition 13.12, the smoothness of $\mathbf{r}$ on $[a, b]$ easily implies that $\mathbf{r}$ is continuous on $[a, b]$ and differentiable on (a, b), which in turn implies that the component functions f and g are both continuous on $[a, b]$ and differentiable on (a, b). Thus f and g are continuous on $[t_{i-1}, t_i]$ and differentiable on (t_{i-1}, t_i) for each i. By the Mean Value Theorem, then, there exists some $\tau_i \in (t_{i-1}, t_i)$ such that

$$f'(\tau_i) = \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}},$$

which gives $x_i - x_{i-1} = f'(\tau_i)(t_i - t_{i-1})$, or equivalently $\Delta x_i = f'(\tau_i)\Delta t_i$. Similarly, there exists some $\hat{\tau}_i \in (t_{i-1}, t_i)$ such that $\Delta y_i = g'(\hat{\tau}_i)\Delta t_i$. Now (13.13) becomes

$$\mathcal{L}(C) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{[f'(\tau_i)]^2 + [g'(\hat{\tau}_i)]^2}\, \Delta t_i. \tag{13.14}$$

A quick inspection of this equation suggests that

$$\mathcal{L}(C) = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt, \tag{13.15}$$

but the sum on the right-hand side of (13.14) is not actually a Riemann sum since, in general, $\tau_i \neq \hat{\tau}_i$. It turns out that (13.15) is in fact true, as given in the theorem that follows. The proof of the theorem is included for the sake of completeness, but it will utilize the concept of uniform continuity, which is not included in a standard calculus sequence. It can safely be skipped.

Theorem 13.20. *If $\mathbf{r}(t)$, $t \in [a, b]$, is a smooth parametrization of a curve C, then*

$$\mathcal{L}(C) = \int_a^b \|\mathbf{r}'(t)\|\, dt$$

It will suffice to prove the result for $C \subseteq \mathbb{R}^2$, so that $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ and $\mathcal{L}(C)$ is given by (13.15), since the treatment in the case when $C \subseteq \mathbb{R}^3$ and $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is formally the same.

Proof. We start by showing that

$$\lim_{\|P\|\to 0}\left(\sum_{i=1}^{n}\sqrt{[f'(\tau_i)]^2 + [g'(\hat{\tau}_i)]^2}\,\Delta t_i - \sum_{i=1}^{n}\sqrt{[f'(\tau_i)]^2 + [g'(\tau_i)]^2}\,\Delta t_i\right) = 0. \tag{13.16}$$

To accomplish this we show that, for every $\epsilon > 0$, there exists some $\delta > 0$ such that, if $P \in \mathcal{P}[a,b]$ with $0 < \|P\| < \delta$, then

$$\left|\sum_{i=1}^{n}\sqrt{[f'(\tau_i)]^2 + [g'(\hat{\tau}_i)]^2}\,\Delta t_i - \sum_{i=1}^{n}\sqrt{[f'(\tau_i)]^2 + [g'(\tau_i)]^2}\,\Delta t_i\right| < \epsilon.$$

Fix $\epsilon > 0$. Define $F : [a,b]\times[a,b] \to \mathbb{R}$ by

$$F(u,v) = \sqrt{[f'(u)]^2 + [g'(v)]^2}.$$

Since $\mathbf{r}$ is smooth on $[a,b]$, both f' and g' are continuous on this interval, and thus F is continuous on $[a,b]\times[a,b]$. Now, since $[a,b]\times[a,b]$ is a closed and bounded set, it follows that F is uniformly continuous on this set, and so there exists some $\delta > 0$ such that, for any $(u_1,v_1),(u_2,v_2) \in [a,b]\times[a,b]$ with $|(u_1,v_1)-(u_2,v_2)| < \delta$, we obtain

$$|F(u_1,v_1) - F(u_2,v_2)| < \frac{\epsilon}{(b-a)}.$$

Now, suppose that $P \in \mathcal{P}[a,b]$ is such that $0 < \|P\| < \delta$. For each $1 \le i \le n$ we choose sample points $\tau_i, \hat{\tau}_i \in [t_{i-1}, t_i] \subseteq [a,b]$, so $(\tau_i,\tau_i),(\tau_i,\hat{\tau}_i) \in [a,b]\times[a,b]$, and by the distance formula

$$|(\tau_i,\hat{\tau}_i) - (\tau_i,\tau_i)| = \sqrt{(\tau_i-\tau_i)^2 + (\hat{\tau}_i-\tau_i)^2} = |\hat{\tau}_i - \tau_i| \le \Delta t_i \le \|P\| < \delta.$$

Hence $|F(\tau_i,\hat{\tau}_i) - F(\tau_i,\tau_i)| < \epsilon/(b-a)$ holds for each i, and so

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\sqrt{[f'(\tau_i)]^2 + [g'(\hat{\tau}_i)]^2}\,\Delta t_i - \sum_{i=1}^{n}\sqrt{[f'(\tau_i)]^2 + [g'(\tau_i)]^2}\,\Delta t_i\right| \\
&\le \sum_{i=1}^{n}\left|\sqrt{[f'(\tau_i)]^2 + [g'(\hat{\tau}_i)]^2} - \sqrt{[f'(\tau_i)]^2 + [g'(\tau_i)]^2}\right|\Delta t_i \\
&= \sum_{i=1}^{n}|F(\tau_i,\hat{\tau}_i) - F(\tau_i,\tau_i)|\,\Delta t_i \\
&< \sum_{i=1}^{n}\frac{\epsilon}{b-a}\Delta t_i = \frac{\epsilon}{b-a}\cdot(b-a) = \epsilon,
\end{aligned}$$

which proves (13.16).

Now, the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ is continuous and therefore integrable on $[a, b]$, which means the limit

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{[f'(\tau_i)]^2 + [g'(\tau_i)]^2}\,\Delta t_i = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt \tag{13.17}$$

exists in $\mathbb{R}$. Thus, adding equations (13.16) and (13.17) together and employing the appropriate limit law, we obtain

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{[f'(\tau_i)]^2 + [g'(\hat{\tau}_i)]^2} = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt,$$

whereupon a reference to equation (13.14) completes the proof. ∎

Example 13.21. Find the length of the curve C given by $\mathbf{r}(t) = \langle t^2, 2t, \ln(t)\rangle$, $1 \le t \le e$.

Solution. First, we have $\mathbf{r}'(t) = \langle 2t, 2, 1/t\rangle$, and so

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{(2t)^2 + 2^2 + (1/t)^2} = \sqrt{4t^2 + 4 + 1/t^2} = \sqrt{\frac{1}{t^2}\,(4t^4 + 4t^2 + 1)} \\ &= \frac{1}{t}\sqrt{(2t^2+1)^2} = \frac{2t^2+1}{t} = 2t + \frac{1}{t}.\end{aligned}$$

Next, we compute

$$\mathcal{L}(C) = \int_1^e \|\mathbf{r}'(t)\|\,dt = \int_1^e \left(2t + \frac{1}{t}\right) dt = \left[t^2 + \ln|t|\right]_1^e = e^2,$$

using Theorem 13.20. ∎

Example 13.22. Find the length of the curve C given by

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, \frac{8}{3}(t+1)^{3/2} \right\rangle, \quad 0 \le t \le 2.$$

Solution. First we have

$$\begin{aligned}\mathcal{L}(C) &= \int_0^2 \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{t^2 + 16t + 16}\,dt \\ &= \int_0^2 \sqrt{(t+8)^2 + 48}\,dt = \int_8^{10} \sqrt{x^2 + a^2}\,dx,\end{aligned} \tag{13.18}$$

where we make the substitution $x = t + 8$ and let $a = \sqrt{48}$. Let's forget the limits of integration for the moment and make the substitution $x = a\sec\theta$, so that

$$\int \sqrt{x^2 + a^2}\,dx = \int \sqrt{a^2\sec^2\theta + a^2} \cdot a\sec\theta\tan\theta\,d\theta = a^2 \int |\tan\theta|\sec\theta\tan\theta\,d\theta,$$

where the identity $\sec^2\theta + 1 = \tan^2\theta$ is used. In our integral for $\mathcal{L}(C)$ we have $8 \le x \le 10$, which implies that $8/a \le \sec\theta \le 10/a$ and therefore $\sec\theta > 0$. Thus, regarded as an angle,

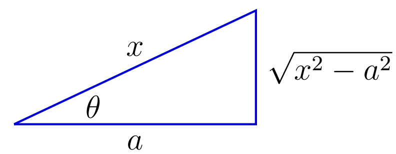
θ is ensconced in Quadrant I (i.e. $0 < \theta < \pi/2$), which informs us that $\tan\theta > 0$ and hence $|\tan\theta| = \tan\theta$. We press on...

$$\int \sqrt{x^2 + a^2}\,dx = a^2 \int \tan\theta \sec\theta \tan\theta\, d\theta = 48 \int \tan^2\theta \sec\theta\, d\theta.$$

The integral on the right-hand side was determined in an example in section 8.2, so that

$$\int \sqrt{x^2 + a^2}\,dx = 48\left[\frac{\sec\theta\tan\theta}{2} - \frac{1}{2}\ln|\sec\theta + \tan\theta| + K\right]$$

for arbitrary constant K. From $\sec\theta = x/a$ we construct the triangle



which reveals that $\tan\theta = \sqrt{x^2 - a^2}/a$. Noting that $a = \sqrt{48} = 4\sqrt{3}$, we progress...

$$\begin{aligned}\int \sqrt{x^2 + a^2}\,dx &= 24\sec\theta\tan\theta - 24\ln(\sec\theta + \tan\theta) + K \\ &= \frac{24x\sqrt{x^2 - a^2}}{a^2} - 24\ln\left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right) + K \\ &= \frac{x\sqrt{x^2 - 48}}{2} - 24\ln\left(\frac{x + \sqrt{x^2 - 48}}{4\sqrt{3}}\right) + K \qquad (13.19)\end{aligned}$$

Let $F(x)$ be (13.19) with $K = 0$. Referring back to (13.18), we obtain

$$\begin{aligned}\mathcal{L}(C) &= F(10) - F(8) \\ &= \left[\frac{10\sqrt{52}}{2} - 24\ln\left(\frac{10 + \sqrt{52}}{4\sqrt{3}}\right)\right] - \left[\frac{8\sqrt{16}}{2} - 24\ln\left(\frac{8 + \sqrt{16}}{4\sqrt{3}}\right)\right] \\ &= 10\sqrt{13} - 24\ln\left(\frac{5 + \sqrt{13}}{2\sqrt{3}}\right) - 16 + 24\ln\sqrt{3} \\ &= 24\ln\left(\sqrt{3}\cdot\frac{2\sqrt{3}}{5 + \sqrt{13}}\right) + 10\sqrt{13} - 16 = 10\sqrt{13} - 16 - 24\ln\left(\frac{5 + \sqrt{13}}{6}\right),\end{aligned}$$

using the property that $\ln(u/v) = -\ln(v/u)$. ■

A **geometric** property of a smooth curve C is a property that is independent of the smooth vector function $\mathbf{r}$ that is chosen to parametrize C. That is, different smooth parametrizations of C either result in no change in the property's value, or else a mere change in the value's sign. The unit tangent vector $\mathbf{T}$ of section 12.6 is one such property. Another is the arc length $\mathcal{L}(C)$ we have just defined.

Often the most convenient way to parametrize a curve C is by arc length. A curve C is **parametrized by arc length** if it has parametrization $\mathbf{r}(s)$, $s \in I$, such that for every $s_0, s_1 \in I$ with $s_0 < s_1$ we have

$$\mathcal{L}\big(\mathbf{r}([s_0, s_1])\big) = \mathcal{L}\big([s_0, s_1]\big);$$

that is, the piece of the curve given by $\mathbf{r}(s)$, $s \in [s_0, s_1]$, has arc length equal to the length of $[s_0, s_1]$, which of course is just $s_1 - s_0$. In particular, if $I = [a, \infty)$ and $s \geq a$, then the curve given by $\mathbf{r}(u)$, $u \in [a, s]$, has arc length of $s - a$. Finally, if $a = 0$ as is most typical, then the curve given by $\mathbf{r}(u)$, $u \in [0, s]$, has arc length s.

The next objective is to fashion a way to take any given smooth parametrization $\mathbf{r}(t)$, $t \in I$, for a curve C and obtain a parametrization in terms of arc length s. Such a "re-parametrization" procedure is common in the study of curves.

By Theorem 13.20, if $\mathbf{r}(t)$, $t \in [a, \infty)$, is a smooth parametrization for a curve C, then for each $t \geq a$ the arc length $s(t)$ of the curve given by $\mathbf{r}(u)$, $u \in [a, t]$, is

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du. \tag{13.20}$$

From this we obtain

$$s'(t) = \frac{d}{dt} \int_a^t \|\mathbf{r}'(u)\|\, du = \|\mathbf{r}'(t)\| \tag{13.21}$$

by the Fundamental Theorem of Calculus, since the smoothness of $\mathbf{r}$ on $[a, \infty)$ implies the continuity of $\|\mathbf{r}'\|$ on $[a, \infty)$. Now, if $\mathbf{r}(t)$, $t \in [a, \infty)$, happens to already be a parametrization for C in terms of arc length, then for each $t \geq a$ the arc length $s(t)$ of the curve $\mathbf{r}(u)$, $u \in [a, t]$ is $t - a$. That is,

$$t - a = \int_a^t \|\mathbf{r}'(u)\|\, du$$

by (13.20), and differentiation leads by the Fundamental Theorem of Calculus to

$$\|\mathbf{r}'(t)\| = \frac{d}{dt} \int_a^t \|\mathbf{r}'(u)\|\, du = (t - a)' = 1.$$

Replacing t with s, we have proven the forward implication of the following result.

Proposition 13.23. *Let C be a smooth curve. Then $\mathbf{r}(s)$, $s \in I$ is a parametrization for C in terms of arc length if and only if $\|\mathbf{r}'(s)\| = 1$ for all $s \in I$.*

Proof. Only the reverse implication remains to be proven. Suppose $\|\mathbf{r}'(s)\| = 1$ for all $s \in I$. Let $s_0, s_1 \in I$ with $s_0 < s_1$. By Theorem 13.20 the length of the piece of the curve C given by $\mathbf{r}(s)$, $s \in [s_0, s_1]$, denoted by $\mathbf{r}([s_0, s_1])$, is

$$\mathcal{L}\big(\mathbf{r}([s_0, s_1])\big) = \int_{s_0}^{s_1} \|\mathbf{r}'(s)\|\, ds = \int_{s_0}^{s_1} ds = s_1 - s_0 = \mathcal{L}\big([s_0, s_1]\big).$$

Therefore $\mathbf{r}(s)$, $s \in I$ is a parametrization for C in terms of arc length. ∎

Equation (13.20) is the key to finding a parametrization of C in terms of arc length s given a parametrization in terms of some other parameter t, as the next example illustrates.

Example 13.24. A curve C is given by the parametrization

$$\mathbf{r}(t) = \langle t^2, 2t^2, 4t^2 \rangle, \quad t \in [1, 4]. \tag{13.22}$$

Find a parametrization of C in terms of arc length.

Solution. For each $1 \leq t \leq 4$, the arc length $s(t)$ of the curve given by $\mathbf{r}(u) = \langle u^2, 2u^2, 4u^2 \rangle$, $u \in [1, t]$, is

$$\begin{aligned} s(t) &= \int_1^t \|\mathbf{r}'(u)\|\, du = \int_1^t \|\langle 2u, 4u, 8u \rangle\|\, du = \int_1^t \sqrt{84u^2}\, du \\ &= \sqrt{84} \int_1^t u\, du = 2\sqrt{21} \left[\tfrac{1}{2} u^2 \right]_1^t = \sqrt{21}(t^2 - 1), \end{aligned}$$

by equation (13.20). Thus when $t = 1$ we have $s = 0$, and when $t = 4$ we have $s = 15\sqrt{21}$. That is, $1 \leq t \leq 4$ corresponds to $0 \leq s \leq 15\sqrt{21}$. Solving

$$s = \sqrt{21}(t^2 - 1)$$

for t^2 yields

$$t^2 = \frac{s}{\sqrt{21}} + 1,$$

and so from the original parametrization (13.22) for C we obtain a new parametrization in terms of arc length s:

$$\boldsymbol{\rho}(s) = \left\langle \frac{s}{\sqrt{21}} + 1, \frac{2s}{\sqrt{21}} + 2, \frac{4s}{\sqrt{21}} + 4 \right\rangle, \quad s \in \left[0, 15\sqrt{21}\right],$$

which is a new function entirely and so is named $\boldsymbol{\rho}$ instead of $\mathbf{r}$. ■

13.5 – Curvature and Normal Vectors

The unit tangent vector $\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$ introduced in section 12.6 is a vector function that indicates the direction that a curve C parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is heading at each point where $\mathbf{r}'(t)$ is defined and nonzero (i.e. at any point where C is "smooth"). So at the point $(x(t), y(t), z(t)) \in C$ the curve is heading in the direction given by the vector $\mathbf{T}(t)$. As an encore, we'd like a way to quantify a couple other properties of C wherever possible: (1) How "bent" is C at a given point; and (2) in what direction is $\mathbf{r}(t)$ "bending" at that point? The measure of the first property is called curvature, while the second property is given by what is called the principal unit normal vector.

Definition 13.25. *Let C be a smooth curve parametrized in terms of arc length by $\mathbf{r}(s)$, $s \in I$. The **curvature** of C at $\mathbf{r}(s)$ is*

$$\kappa(s) = \|\mathbf{T}'(s)\|.$$

More generally, if C is a curve with smooth parametrization $\mathbf{r}(t)$, $t \in I$, then we will use the symbol $\kappa(t)$ to denote the curvature of C at $\mathbf{r}(t)$. Then, if $\boldsymbol{\rho}(s)$, $s \in J$, is a parametrization for C in terms of arc length, and we define $s : I \to J$ by $s = s(t)$ as in equation (13.20) such that $\mathbf{r}(t) = \boldsymbol{\rho}(s(t))$ for all $t \in I$, then it becomes clear that

$$\kappa(s) = \underbrace{\kappa(s(t))}_{\text{Curvature at } \boldsymbol{\rho}(s(t))} = \underbrace{\kappa(t).}_{\text{Curvature at } \mathbf{r}(t)} \tag{13.23}$$

What remains to do is develop a useful formula for computing $\kappa(t)$ when t is not the arc length parameter.

Theorem 13.26. *Let C be a curve with smooth parametrization $\mathbf{r}(t)$, $t \in I$. Then*

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

for any $t \in I$ such that $\|\mathbf{r}'(t)\| \neq 0$.

Note from Definition 13.12 that if $\mathbf{r}(t)$, $t \in I$, is given to be smooth, then $\|\mathbf{r}'(t)\| \neq 0$ for all $t \in \text{Int}(I)$. That is, $\|\mathbf{r}'(t)\| = 0$ may only be the case if t is an endpoint of the interval I.

Proof. Let $\boldsymbol{\rho}(s)$, $s \in J$, be a parametrization for C in terms of arc length. Define unit tangent vector functions $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

for $t \in I$, and

$$\widetilde{\mathbf{T}}(s) = \frac{\boldsymbol{\rho}'(s)}{\|\boldsymbol{\rho}'(s)\|}$$

for $s \in J$. By (13.20) arc length s can be cast as a function of the parameter t so that $\mathbf{r}(t) = \boldsymbol{\rho}(s(t))$ for all $t \in I$, and in particular

$$\mathbf{T}(t) = \widetilde{\mathbf{T}}(s(t)) = (\widetilde{\mathbf{T}} \circ s)(t).$$

Differentiating and employing Theorem 13.17 gives

$$\mathbf{T}'(t) = (\widetilde{\mathbf{T}} \circ s)'(t) = \widetilde{\mathbf{T}}'(s(t))s'(t).$$

From this we obtain $\mathbf{T}'(t) = \widetilde{\mathbf{T}}'(s(t))\|\mathbf{r}'(t)\|$, since $s'(t) = \|\mathbf{r}'(t)\|$ by equation (13.21). Now, if $t \in I$ is such that $\|\mathbf{r}'(t)\| \neq 0$, then

$$\widetilde{\mathbf{T}}'(s(t)) = \frac{\mathbf{T}'(t)}{\|\mathbf{r}'(t)\|},$$

and by equation (13.23) we finally have

$$\kappa(t) = \kappa(s(t)) = \|\widetilde{\mathbf{T}}'(s(t))\| = \left\| \frac{\mathbf{T}'(t)}{\|\mathbf{r}'(t)\|} \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

as was to be shown. ■

Example 13.27. Find the curvature of the circle with radius R.

Solution. The circle may be centered at the origin and thus parametrized by

$$\mathbf{r}(t) = \langle R\cos t, R\sin t\rangle, \quad t \in [0, 2\pi].$$

We have

$$\mathbf{r}'(t) = \langle -R\sin t, R\cos t\rangle,$$

so that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle -R\sin t, R\cos t\rangle}{\sqrt{R^2\sin^2 t + R^2\cos^2 t}} = \frac{R\langle -\sin t, \cos t\rangle}{R} = \langle -\sin t, \cos t\rangle$$

and hence

$$\mathbf{T}'(t) = \langle -\cos t, -\sin t\rangle.$$

Finally we obtain

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{\cos^2 t + \sin^2 t}}{R} = \frac{1}{R}$$

for all $0 \leq t \leq 2\pi$. ■

In light of Example 13.27, we see that if a curve C has curvature κ at some point p, then that means the curve "bends" to the same degree as a circle of radius $1/\kappa$ at p. Hence a greater curvature value corresponds to a circle of smaller radius; that is, the larger κ is at some point p on C, the more "bent" C is at p.

Example 13.28. Let C be a curve with parametrization

$$\mathbf{r}(t) = \langle \cosh t, \sinh t, t\rangle, \quad t \in (-\infty, \infty).$$

Find the curvature of C at $\mathbf{r}(t)$ for all $t \in (-\infty, \infty)$.

Solution. Let $-\infty < t < \infty$ be arbitrary. We have

$$\mathbf{r}'(t) = \langle \sinh t, \cosh t, 1 \rangle,$$

and thus

$$\|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t,$$

using the hyperbolic identity $\cosh^2 t - \sinh^2 t = 1$ and recalling that $\cosh t > 0$ for all $t \in \mathbb{R}$. Now,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle \sinh t, \cosh t, 1 \rangle}{\sqrt{2}\cosh t} = \frac{1}{\sqrt{2}}\langle \tanh t, 1, \operatorname{sech} t \rangle,$$

whence

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle \operatorname{sech}^2 t, 0, -\tanh t \operatorname{sech} t \rangle = \frac{\operatorname{sech} t}{\sqrt{2}}\langle \operatorname{sech} t, 0, -\tanh t \rangle,$$

and finally

$$\|\mathbf{T}'(t)\| = \frac{\operatorname{sech} t}{\sqrt{2}}\sqrt{\operatorname{sech}^2 t + \tanh^2 t} = \frac{\operatorname{sech} t}{\sqrt{2}} = \frac{1}{\sqrt{2}\cosh t},$$

using the hyperbolic identity $\operatorname{sech}^2 t + \tanh^2 t = 1$. By Theorem 13.26 the curvature of C at $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \left(\frac{1}{\sqrt{2}\cosh t}\right)\left(\frac{1}{\sqrt{2}\cosh t}\right) = \frac{1}{2\cosh^2 t} = \frac{\operatorname{sech}^2 t}{2},$$

or equivalently

$$\kappa(t) = \frac{2}{(e^t + e^{-t})^2},$$

recalling the definition $\operatorname{sech} t = 2/(e^t + e^{-t})$. ■

Example 13.29. Find the point at which $f(x) = e^x$ attains a maximum curvature, and then find the maximum curvature.

Solution. The vector function $\mathbf{r}(t) = \langle t, e^t \rangle$, $t \in (-\infty, \infty)$, yields the same curve. We have

$$\mathbf{r}'(t) = \langle 1, e^t \rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + e^{2t}},$$

so that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+e^{2t}}}\langle 1, e^t \rangle = \left\langle \frac{1}{\sqrt{1+e^{2t}}}, \frac{e^t}{\sqrt{1+e^{2t}}} \right\rangle,$$

which yields

$$\mathbf{T}'(t) = \left\langle -\frac{e^{2t}}{(1+e^{2t})^{3/2}}, \frac{e^t}{(1+e^{2t})^{3/2}} \right\rangle = \frac{e^t}{(1+e^{2t})^{3/2}}\langle -e^t, 1 \rangle,$$

and thus

$$\|\mathbf{T}'(t)\| = \frac{e^t}{(1+e^{2t})^{3/2}}\sqrt{e^{2t}+1} = \frac{e^t}{1+e^{2t}}.$$

Finally we obtain

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{e^t}{1+e^{2t}} \cdot \frac{1}{\sqrt{1+e^{2t}}} = \frac{e^t}{(1+e^{2t})^{3/2}}$$

as the curvature of the curve at the point (t, e^t).

To find the value of t for which $\kappa(t)$ attains a maximum value, we first find $\kappa'(t)$:

$$\kappa'(t) = \frac{(1+e^{2t})^{3/2}e^t - e^t \cdot \frac{3}{2}(1+e^{2t})^{1/2} \cdot 2e^{2t}}{(1+e^{2t})^3} = \frac{e^t - 2e^{3t}}{(1+e^{2t})^{5/2}}.$$

Now we set $\kappa'(t) = 0$ to obtain

$$\frac{e^t - 2e^{3t}}{(1+e^{2t})^{5/2}} = 0,$$

and hence

$$e^t - 2e^{3t} = 0.$$

From this comes the equation $e^{2t} = 1/2$, which has solution

$$t = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{2}.$$

Thus the curve has maximum curvature at the point

$$\mathbf{r}\left(-\frac{\ln(2)}{2}\right) = \left\langle -\frac{\ln(2)}{2},\ e^{-\ln(2)/2}\right\rangle = \left\langle -\frac{\ln(2)}{2},\ \frac{1}{\sqrt{2}}\right\rangle.$$

The value of the maximum curvature is

$$\kappa\left(-\tfrac{1}{2}\ln(2)\right) = \frac{e^{-\ln(2)/2}}{\left[1+e^{-\ln(2)}\right]^{3/2}} = \frac{1/\sqrt{2}}{(1+1/2)^{3/2}} = \frac{2\sqrt{3}}{9}.$$

This is about 0.385, which is a curvature corresponding to a circle with radius approximately 2.60. ■

13.6 – Planes and Quadric Surfaces

The solution set $S \subseteq \mathbb{R}^3$ of an equation of the form

$$A_1x^2 + A_2y^2 + A_3z^2 + A_4xy + A_5xz + A_6yz + A_7x + A_8y + A_9z + A_{10} = 0 \tag{13.24}$$

is called a **quadric surface** in $\mathbb{R}^3$. Here $A_1, \dots, A_{10}$ represent constant coefficients, with at least one of $A_1, \dots, A_9$ not equal to 0. (If $A_1, \dots, A_9$ are all 0, then the solution set for (13.24) is either $\mathbb{R}^3$ or $\varnothing$ depending on whether A_{10} is 0 or not.)

Example 13.30. In §12.1 we found that the sphere with center (a, b, c) and radius r is the set of points $(x, y, z) \in \mathbb{R}^3$ that satisfy the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

As this equation may be written as

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2) - r^2 = 0,$$

we see that a sphere is a quadric surface for which $A_1 = A_2 = A_3 = 1$, $A_4 = A_5 = A_6 = 0$, $A_7 = -2a$, $A_8 = -2b$, $A_9 = -2c$, and $A_{10} = a^2 + b^2 + c^2 - r^2$. ■

Definition 13.31. *The **plane** in $\mathbb{R}^3$ containing the point $p_0 = (x_0, y_0, z_0)$ and having **normal vector** $\mathbf{n} = \langle a, b, c\rangle \neq \mathbf{0}$ is the set*

$$\left\{p \in \mathbb{R}^3 : \mathbf{n} \cdot \overrightarrow{p_0p} = 0\right\}.$$

If plane P has normal vector $\mathbf{n}_1$ and plane Q has normal vector $\mathbf{n}_2$, then P is **parallel** to Q (written $P \parallel Q$) if $\mathbf{n}_1 \parallel \mathbf{n}_2$, and P is **perpendicular** to Q (written $P \perp Q$) if $\mathbf{n}_1 \perp \mathbf{n}_2$. Also, a vector $\mathbf{v}$ is **parallel** to a plane with normal vector $\mathbf{n}$ if $\mathbf{v} \perp \mathbf{n}$, and $\mathbf{v}$ is **orthogonal** to a plane with normal vector $\mathbf{n}$ if $\mathbf{v} \parallel \mathbf{n}$. We see that by definition a normal vector for a plane is orthogonal to the plane. Finally, it is straightforward to check that if P is a plane containing points p and q, then $\mathbf{v}$ is parallel to P if and only if $\mathbf{v}$ is parallel to $\overrightarrow{pq}$.

Proposition 13.32. *The plane determined by the point $p_0 = (x_0, y_0, z_0)$ and normal vector $\mathbf{n} = \langle a, b, c\rangle$ is given by the equation*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{13.25}$$

This means that the solution set of the equation (13.25) is equal to the set of points given in Definition 13.31. It also means that a set $P \subseteq \mathbb{R}^3$ is a plane if and only if it is the solution set to an equation of the form $ax + by + cz = d$, where a, b, c, and d are constants such that a, b, and c are not all zero. Thus planes, like lines in Chapter 12, could be defined in terms of point sets without any mention of vectors.

Proof. For any $p = (x, y, z)$ we have

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{p_0p} = 0 &\Leftrightarrow \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0 \\ &\Leftrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \end{aligned}$$

and we're done. ■

The equation $x = 0$, which defines a line in $\mathbb{R}^2$, defines a plane in $\mathbb{R}^3$. If we write it as

$$1(x-0)+0(y-0)+0(z-0)=0,$$

it can be seen that $x = 0$ is the plane determined by the point $(0,0,0)$ and normal vector $\langle 1,0,0\rangle$. This plane is the set of points $\{(x,y,z) \mid x = 0\}$, which is otherwise known as the yz-plane. Similarly, $y = 0$ is the xz-plane, and $z = 0$ is the xy-plane. See Figure 55.

In general three noncollinear points uniquely determine a plane, as illustrated in the example that follows.

Example 13.33. Find an equation of the plane that contains the points $p_0 = (-1,1,1)$, $p_1 = (0,0,2)$, and $p_2 = (3,-1,-2)$.

Solution. We need to find a normal vector $\mathbf{n}$ for the plane, which is a vector such that $\mathbf{n}\cdot\overrightarrow{p_0p} = 0$ for all points p in the plane. Recall that the cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ yields a vector $\mathbf{w}$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. The vectors of interest here are $\overrightarrow{p_0p_1} = \langle 1,-1,1\rangle$ and $\overrightarrow{p_0p_2} = \langle 4,-2,-3\rangle$, where

$$\begin{aligned}\overrightarrow{p_0p_1}\times\overrightarrow{p_0p_2} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 1\\ 4 & -2 & -3\end{vmatrix} = \begin{vmatrix}-1 & 1\\ -2 & -3\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 4 & -3\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & -1\\ 4 & -2\end{vmatrix}\mathbf{k}\\ &= 5\mathbf{i} - 7\mathbf{j} + 2\mathbf{k} = \langle 5,-7,2\rangle\end{aligned}$$

We let $\mathbf{n} = \langle a,b,c\rangle = \langle 5,-7,2\rangle$, which, together with $(a,b,c) = (-1,1,1)$ and Proposition 13.32, enables us to obtain the equation

$$5(x+1)-7(y-1)+2(z-1)=0,$$

or $5x - 7y + 2z = -10$. ■

It might be asked how we know that the vector $\mathbf{n}$ found in this example really is "the" normal vector for the plane. The reasoning is as follows. If vector $\mathbf{n}'$ is the normal vector for the given

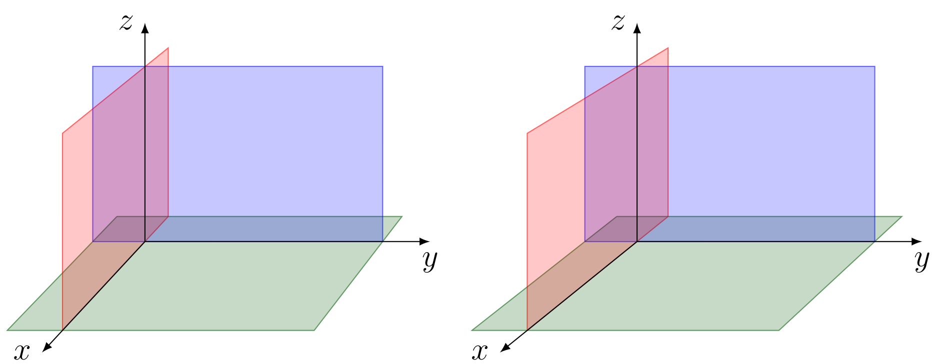


FIGURE 55. Parts of the xy-, xz- and yz-planes in green, red and blue, respectively, as viewed from Octant I.

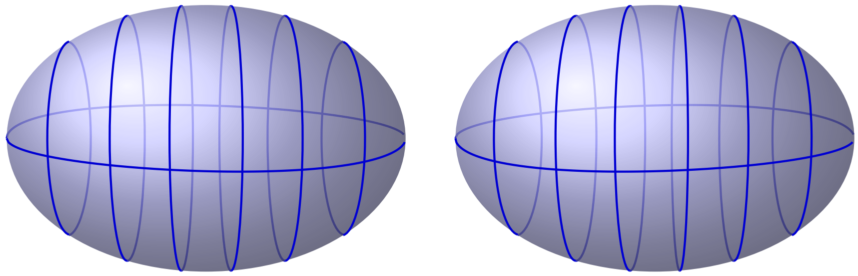


FIGURE 56. An ellipsoid.

plane, then $\mathbf{n}'$ must (by definition) be orthogonal to both $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_2}$, and thus $\mathbf{n}'$ must be parallel to $\mathbf{n} = \langle 5, -7, 2\rangle$. This means that $\mathbf{n}' = k\mathbf{n}$ for some $k \neq 0$. Now, the set of points p which satisfy $\mathbf{n}' \cdot \overrightarrow{p_0p} = 0$ must be the same set that satisfies $\mathbf{n} \cdot \overrightarrow{p_0p} = 0$, since

$$\mathbf{n}' \cdot \overrightarrow{p_0p} = 0 \quad \Leftrightarrow \quad (k\mathbf{n}) \cdot \overrightarrow{p_0p} = 0 \quad \Leftrightarrow \quad k(\mathbf{n} \cdot \overrightarrow{p_0p}) = 0 \quad \Leftrightarrow \quad \mathbf{n} \cdot \overrightarrow{p_0p} = 0.$$

Corollary to this analysis is that, given a point p_0 and nonzero vector $\mathbf{n}$, the plane that is determined is not altered if $\mathbf{n}$ is replaced by some parallel vector $k\mathbf{n}$ ($k \neq 0$ of course).

From a geometrical standpoint it should be clear that the intersection $P \cap Q$ of two planes P and Q in $\mathbb{R}^3$ can only be one of three things: $P \cap Q$ can be a plane if $P = Q$, and then $P \cap Q = P = Q$; or $P \cap Q = \varnothing$ if $P \neq Q$ but P is parallel to Q; or $P \cap Q$ can be a line if P and Q are not parallel. The next examples illustrates this third (and most common) situation.

Example 13.34. Find an equation of the line where the planes $P : x + 2y - 3z = 1$ and $Q : x + y + z = 2$ intersect.

Solution. It's easy to see that P and Q are not parallel, so they must intersect on a line. The intersection of P and the plane $z = 0$ is the set of points on the line $\ell_0 : x + 2y = 1$, and the intersection of Q and $z = 0$ is the line $\ell_0' : x + y = 2$. So the point that is an element of $\ell_0 \cap \ell_0'$ must be a point in $P \cap Q$. We find this point by finding the solution to the system $x + 2y = 1$, $x + y = 2$, which is $(3, -1)$. Thus $(3, -1, 0) \in P \cap Q$, since we are on the plane where z is 0.

Next, the intersection of P and the plane $z = 1$ is the line $\ell_1 : x + 2y = 4$, and the intersection of Q and $z = 1$ is the line $\ell_1' : x + y = 1$. Again, a point in $\ell_1 \cap \ell_1'$ is a point in $P \cap Q$. The system $x + 2y = 4$, $x + y = 1$ has solution $(-2, 3)$, and thus $(-2, 3, 1) \in P \cap Q$ (recall we're now on the plane where z is 1).

So the line of intersection for P and Q contains points $r_0 = (3, -1, 0)$ and $r_1 = (-2, 3, 1)$. Let $\mathbf{v} = \overrightarrow{r_0r_1} = \langle -5, 4, 1\rangle$. An equation for the line is thus

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 3, -1, 0\rangle + t\langle -5, 4, 1\rangle,$$

or simply

$$\mathbf{r}(t) = \langle -5t + 3, 4t - 1, t\rangle,$$

for $-\infty < t < \infty$. ■

14

Partial Derivatives

14.1 – Multivariable Functions

A real-valued function of several variables, or **multivariable function**, is a function f for which $\mathrm{Dom}(f) = D$ for some $D \subseteq \mathbb{R}^n$ (where $n \geq 2$), and $\mathrm{Ran}(f) \subseteq \mathbb{R}$. Thus, for each $(x_1, x_2, \ldots, x_n) \in D$ we have $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$, and we write

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}.$$

Unless other considerations are in play that give us cause to restrict the domain of a function f to some smaller set, we take $\mathrm{Dom}(f)$ to be the set

$$\mathrm{Dom}(f) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in \mathbb{R}\},$$

where we make use of the symbol $\mathbf{x}$ to denote $(x_1, \ldots, x_n)$ as described in §13.0. Thus

$$\begin{aligned}\mathrm{Ran}(f) &= \{y \in \mathbb{R} : y = f(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathrm{Dom}(f)\} \\ &= \{f(\mathbf{x}) : \mathbf{x} \in \mathrm{Dom}(f)\}.\end{aligned}$$

Most times we will be working with either a function of two independent variables x and y (so $\mathrm{Dom}(f)$ is in $\mathbb{R}^2$), or three independent variables x, y and z (so $\mathrm{Dom}(f) \subseteq \mathbb{R}^3$).

Example 14.1. The function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is defined by the rule

$$f(x, y) = \sqrt{36 - 9x^2 - 4y^2}.$$

Find the set D for which $D = \mathrm{Dom}(f)$, and then find $\mathrm{Ran}(f)$.

Solution. By definition we have

$$\mathrm{Dom}(f) = \left\{(x, y) \in \mathbb{R}^2 \;:\; \sqrt{36 - 9x^2 - 4y^2} \in \mathbb{R}\right\},$$

which requires that (x, y) be such that $36 - 9x^2 - 4y^2 \geq 0$, and hence $9x^2 + 4y^2 \leq 36$. Dividing by 36 gives

$$\frac{x^2}{4} + \frac{y^2}{9} \leq 1,$$

and since $x^2/4 + y^2/9 = 1$ is an ellipse centered at the origin with vertices at $(-2, 0)$, $(2, 0)$, $(0, -3)$, and $(0, 3)$, it's concluded that $\mathrm{Dom}(f)$ consists of all points lying either on the ellipse or in the region bounded by the ellipse, as shown in Figure 57.

As for the range of f, first note that $9x^2 + 4y^2 \geq 0$ for any $(x, y) \in \mathbb{R}^2$, and then

$$-9x^2 - 4y^2 \leq 0 \;\Rightarrow\; 36 - 9x^2 - 4y^2 \leq 36 \;\Rightarrow\; 0 \leq \sqrt{36 - 9x^2 - 4y^2} \leq 6.$$

That is, $0 \leq f(x, y) \leq 6$ is always the case, which shows that $\mathrm{Ran}(f) \subseteq [0, 6]$. Now, suppose that $z \in [0, 6]$. We would like to find some $(x, y) \in \mathrm{Dom}(f)$ such that $f(x, y) = z$; that is,

$$\sqrt{36 - 9x^2 - 4y^2} = z,$$

or equivalently

$$z^2 = 36 - 9x^2 - 4y^2.$$

To simplify matters suppose that $y = x$, so that $z^2 = 36 - 13x^2$. Because $z \in [0, 6]$ implies that $36 - z^2 \geq 0$, it is possible to solve $z^2 = 36 - 13x^2$ to obtain real-valued solutions for x:

$$z^2 = 36 - 13x^2 \;\Rightarrow\; x^2 = \frac{36 - z^2}{13} \;\Rightarrow\; x = \pm\sqrt{\frac{36 - z^2}{13}}.$$

Therefore

$$(x, y) = \left(\sqrt{\frac{36 - z^2}{13}}, \sqrt{\frac{36 - z^2}{13}}\right)$$

is a point in the domain of f that is easily verified to give $f(x, y) = z$, so $z \in \mathrm{Ran}(f)$ and we have shown that $[0, 6] \subseteq \mathrm{Ran}(f)$. Therefore $\mathrm{Ran}(f) = [0, 6]$. ■

Example 14.2. Find the domain and range of $g(x, y) = \arcsin(x + 2y)$.

Solution. The domain of g will be dictated by the domain of the arcsine function, which is $[-1, 1]$. Thus,

$$\mathrm{Dom}(g) = \{(x, y) \;:\; -1 \leq x + 2y \leq 1\} = \left\{(x, y) \;:\; -\tfrac{1}{2}x - \tfrac{1}{2} \leq y \leq -\tfrac{1}{2}x + \tfrac{1}{2}\right\}, \tag{14.1}$$

as shown in Figure 58.

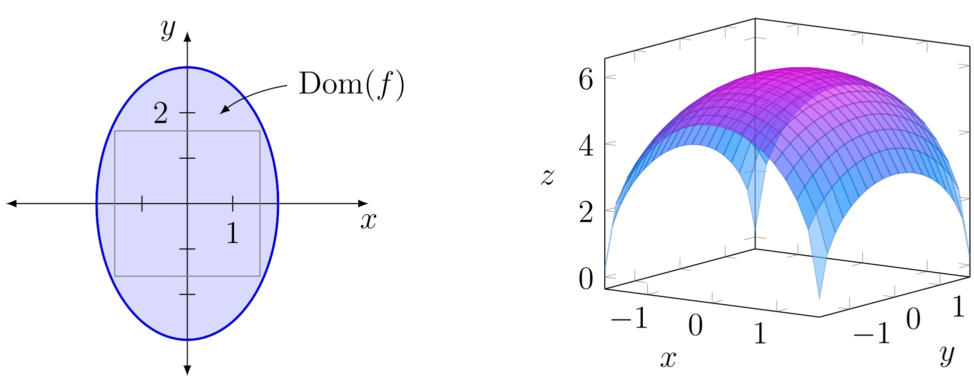


Figure 57. Left: The domain of f. Right: The part of the surface $z = f(x, y)$ that lies over the square pictured on the left.

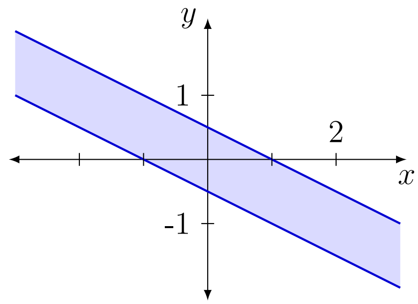


FIGURE 58. The domain of g.

As for the range of g, consider the cross-section of the domain that we obtain when we set $y = 0$. We obtain $g(x, 0) = \arcsin(x)$, and from (14.1) it's seen that $-1 \leq x \leq 1$. It's known that as x increases from -1 to 1, $\arcsin(x)$ increases from $-\pi/2$ to $\pi/2$. Thus $g(x, y)$ attains all values in the interval $[-\pi/2, \pi/2]$, and since it's also known that this constitutes the entirety of the arcsine function's possible output, we conclude that $\mathrm{Ran}(g) = [-\pi/2, \pi/2]$. ■

Example 14.3. Find the domain and range of

$$h(x, y) = \frac{\sqrt{x + y} - 3}{x + y - 9}.$$

Solution. To avoid division by zero and square roots of negatives, we need

$$\mathrm{Dom}(h) = \{(x, y) \ : \ x + y \geq 0 \text{ and } x + y \neq 9\},$$

which is the region in $\mathbb{R}^2$ shown in Figure 59.

To find the range of h, it helps to carry out the following manipulation:

$$h(x, y) = \frac{\sqrt{x + y} - 3}{(\sqrt{x + y}\,)^2 - 3^2} = \frac{\sqrt{x + y} - 3}{(\sqrt{x + y} - 3)(\sqrt{x + y} + 3)} = \frac{1}{\sqrt{x + y} + 3},$$

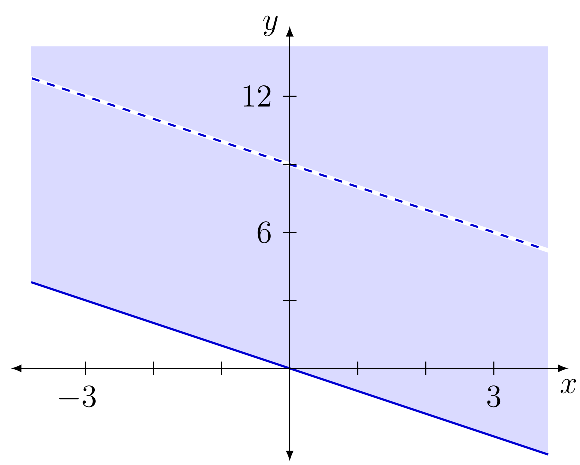


FIGURE 59. The domain of h.

for $x+y\neq 9$. It can be seen that the global maximum value of h is $h(0,0)=1/3$, and then as $x+y\to\infty$ we have $h(x,y)\to 0^+$; however, since $x+y\neq 9$, it follows that $h(x,y)$ cannot ever equal $1/(\sqrt{9}+3)=1/6$. Hence $\operatorname{Ran}(h)=\left(0,\frac{1}{6}\right)\cup\left(\frac{1}{6},\frac{1}{3}\right]$. ∎

14.2 – Limits and Continuity

The definition of a limit for a function of two variables is similar to that for a limit of a single-variable function. Remember that to say a limit "exists" means that it equals some real number.[10]

Definition 14.4. *Let* (a, b) *be a limit point of* $\mathrm{Dom}(f) \subseteq \mathbb{R}^2$*. Then* f *has* ***limit*** $L \in \mathbb{R}$ *as* (x, y) *approaches* (a, b)*, written*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for any $\epsilon > 0$ *there exists some* $\delta > 0$ *such that, for any* $(x, y) \in \mathrm{Dom}(f)$*,*

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \Rightarrow \quad |f(x, y) - L| < \epsilon.$$

It is necessary to require that (a, b) be a limit point of the domain of the function f so as to avoid undesirable situations where, say, $f(x, y) = \sqrt{x+y}$ has *all real numbers* as its limit at the point $(-1, -1)$, even though f isn't defined anywhere near $(-1, -1)$! (This can happen because, in formal logic, the statement "If P, then Q" is considered true, by default, whenever P is false.)

Using the notation established at the beginning of §13.0, we now define the limit of a real-valued function f of n independent variables $x_1, \ldots, x_n$, which is to say $\mathrm{Dom}(f) \subseteq \mathbb{R}^n$ and $f(x_1, \ldots, x_n) \in \mathbb{R}$.

Definition 14.5. *Let* $\mathbf{c}$ *be a limit point of* $\mathrm{Dom}(f) \subseteq \mathbb{R}^n$*. Then* f *has* ***limit*** $L \in \mathbb{R}$ *as* $\mathbf{x}$ *approaches* $\mathbf{c}$*, written*

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L,$$

if for any $\epsilon > 0$ *there exists some* $\delta > 0$ *such that, for any* $\mathbf{x} \in \mathrm{Dom}(f)$*,*

$$0 < \|\mathbf{x} - \mathbf{c}\| < \delta \quad \Rightarrow \quad |f(\mathbf{x}) - L| < \epsilon$$

If $n = 1$ in Definition 14.5, then we obtain the definition of limit given in §2.2 (for which $\mathbf{x}$ becomes x and $\mathbf{c}$ becomes c). If $n = 2$, then we obtain Definition 14.4.

To evaluate a limit of a function of two independent variables using Definition 14.4 directly requires constructing a so-called ϵ-δ argument, illustrated in the following example.

Example 14.6. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

Solution. First observe that the domain of the function

$$f(x, y) = \frac{3x^2y}{x^2+y^2}$$

is $\mathbb{R}^2 \setminus \{(0, 0)\}$ (i.e. all of $\mathbb{R}^2$ except for the origin), so as (x, y) approaches $(0, 0)$ it always remains in $\mathrm{Dom}(f)$.

[10] Sometimes we say a limit "exists in $\mathbb{R}$" to emphasize this, or to be more specific in situations when limits might equal other kinds of numbers. We do not entertain other kinds of numbers here.

Let $\epsilon > 0$. Choose $\delta = \epsilon/3$. Suppose that

$$0 < \sqrt{x^2 + y^2} < \delta.$$

Then, observing that

$$\frac{x^2}{x^2 + y^2} \le 1$$

for $(x, y) \neq (0, 0)$, we obtain

$$|f(x, y)| = \frac{3x^2|y|}{x^2 + y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2 + y^2} < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

We have now shown that, for any $(x, y) \neq (0, 0)$,

$$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta \quad \Rightarrow \quad \left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| < \epsilon.$$

Therefore

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2} = 0$$

by Definition 14.4. ■

Two things that make an ϵ-δ argument difficult, in particular, is that the value L of a limit must be known ahead of time, and so too must the value of δ (which will always depend on ϵ in some way). In this section we will develop various tools that will enable us to evaluate limits without resorting to ϵ-δ arguments. The first such tool is as follows.

Theorem 14.7. *Let $\mathbf{c} \in \mathbb{R}^n$ be a limit point of* $\mathrm{Dom}(f)$*, suppose there is an open set U and function φ such that $\mathbf{c} \in U$ and $f(\mathbf{x}) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in U \cap \mathrm{Dom}(f)$ with $\mathbf{x} \neq \mathbf{c}$. If $\lim_{\mathbf{x}\to\mathbf{c}} \varphi(\mathbf{x}) = L$, then $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$.*

Proof. Let $\epsilon > 0$. Since $\mathbf{c} \in U$ and U is open, there exists some $\delta_1 > 0$ such that $B_{\delta_1}(\mathbf{c}) \subseteq U$. Since $\lim_{\mathbf{x}\to\mathbf{c}} \varphi(\mathbf{x}) = L$, there exists some $\delta_2 > 0$ such that, for any $\mathbf{x} \in \mathrm{Dom}(\varphi)$,

$$0 < \|\mathbf{x} - \mathbf{c}\| < \delta_2 \quad \Rightarrow \quad |\varphi(\mathbf{x}) - L| < \epsilon.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $\mathbf{x} \in \mathrm{Dom}(f)$ be arbitrary, and suppose that $0 < \|\mathbf{x} - \mathbf{c}\| < \delta$. From $0 < \|\mathbf{x} - \mathbf{c}\| < \delta_1$ comes $\mathbf{x} \in U \cap \mathrm{Dom}(f)$ with $\mathbf{x} \neq \mathbf{c}$, and thus $\varphi(\mathbf{x}) = f(\mathbf{x})$. Since $\mathbf{x} \in \mathrm{Dom}(\varphi)$ with $0 < \|\mathbf{x} - \mathbf{c}\| < \delta_2$, we have $|\varphi(\mathbf{x}) - L| < \epsilon$. Combining these results, we obtain $|f(\mathbf{x}) - L| < \epsilon$. ■

Limits of real-valued multivariable functions obey the same basic laws as limits of single-variable functions, with largely the same proofs. For this reason the proof of the following theorem is omitted, though the reader is invited to compare it with Theorem 2.12.

Theorem 14.8 (Laws of Limits). *For any $\mathbf{c} \in \mathbb{R}^n$ and $r, L, M \in \mathbb{R}$, if $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = M$, then*

1. $\displaystyle\lim_{\mathbf{x}\to\mathbf{c}} r = r$
2. $\displaystyle\lim_{\mathbf{x}\to\mathbf{c}} rf(\mathbf{x}) = rL = r\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})$

3. $\lim_{\mathbf{x}\to\mathbf{c}} \big[f(\mathbf{x}) + g(\mathbf{x})\big] = L + M = \lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x})$
4. $\lim_{\mathbf{x}\to\mathbf{c}} \big[f(\mathbf{x})g(\mathbf{x})\big] = LM = \lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) \cdot \lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x})$
5. *Provided that* $M \neq 0$,

$$\lim_{\mathbf{x}\to\mathbf{c}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{L}{M} = \frac{\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x})}.$$

6. *For any integer* $n > 0$,

$$\lim_{\mathbf{x}\to\mathbf{c}} [f(\mathbf{x})]^n = L^n = \Big[\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})\Big]^n.$$

7. *For any integer* $m > 0$,

$$\lim_{\mathbf{x}\to\mathbf{c}} \sqrt[m]{f(\mathbf{x})} = \sqrt[m]{L} = \sqrt[m]{\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})},$$

provided $L > 0$ *if* m *is even.*

An important property of functions that can be a great aid in evaluating limits is continuity, which in the case of multivariable functions is defined as follows.

Definition 14.9. *Let* f *be a function and* $\mathbf{c} \in \mathrm{Dom}(f)$*. Then* f *is* ***continuous at*** $\mathbf{c}$ *if for every* $\epsilon > 0$ *there exists some* $\delta > 0$ *such that, for any* $\mathbf{x} \in \mathrm{Dom}(f)$,

$$\|\mathbf{x} - \mathbf{c}\| < \delta \quad \Rightarrow \quad |f(\mathbf{x}) - f(\mathbf{c})| < \epsilon.$$

If f *is continuous at every point in a set* S*, then* f *is said to be* ***continuous on*** S*. If* f *is continuous at every point in its domain, then* f *is said to be* ***continuous on its domain*** *or simply* ***continuous****.*

So a function cannot be continuous at any point not in its domain, which makes sense. An immediate consequence of this definition is the following theorem.

Theorem 14.10. *If* $\mathbf{c}$ *is a limit point of* $\mathrm{Dom}(f)$*, then* f *is continuous at* $\mathbf{c}$ *if and only if*

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$$

Typically calculus textbooks give the *definition* for continuity as "The function f is continuous at $\mathbf{c} \in \mathrm{Dom}(f)$ if $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$," which is not equivalent to Definition 14.9 since it requires that $\mathbf{c}$ be a limit point of $\mathrm{Dom}(f)$ whereas Definition 14.9 does not. In particular Definition 14.9 implies that a function is continuous, by default, at any isolated point of its domain. However, the discrepancy does not present us with any problems since in these pages we shall only ever be considering functions with domains consisting entirely of limit points.

Proposition 14.11. *Rational, radical, trigonometric, exponential and logarithmic functions of two or more variables are continuous on their domains, as are functions that are combinations of these.*

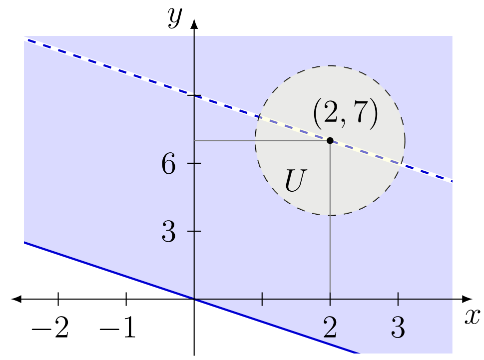


FIGURE 60. A neighborhood for the point $(2, 7)$.

The proof of this proposition is not of great concern at this moment, but the implications are far-reaching: if f is any one of the five types of functions mentioned in the proposition, and a point $\mathbf{c}$ is both in the domain of f and is a limit point of $\mathrm{Dom}(f)$, then the limit of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{c}$ can be evaluated by direct substitution: $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$.

Example 14.12. Evaluate

$$\lim_{(x,y)\to(2,7)} \frac{\sqrt{x+y}-3}{x+y-9}.$$

Solution. Let

$$h(x,y) = \frac{\sqrt{x+y}-3}{x+y-9}.$$

The domain of h was determined in Example 14.3 and illustrated in Figure 59. It can be seen from Figure 60 that, although $(2,7) \notin \mathrm{Dom}(h)$, $(2,7)$ *is* a limit point of $\mathrm{Dom}(h)$. Consider any open set U containing $(2,7)$, such as the one depicted in Figure 60. For all $(x,y) \in U' = U \cap \mathrm{Dom}(h)$ we have

$$\frac{\sqrt{x+y}-3}{x+y-9} = \frac{\sqrt{x+y}-3}{(\sqrt{x+y}-3)(\sqrt{x+y}+3)} = \frac{1}{\sqrt{x+y}+3},$$

since $\sqrt{x+y}-3 \neq 0$ on U'. Thus if we define the function

$$\varphi(x,y) = \frac{1}{\sqrt{x+y}+3},$$

we see that $h(x,y) = \varphi(x,y)$ for all $(x,y) \in U'$. Now, if $\lim_{(x,y)\to(2,7)} \varphi(x,y)$ exists, then

$$\lim_{(x,y)\to(2,7)} h(x,y) = \lim_{(x,y)\to(2,7)} \varphi(x,y)$$

by Theorem 14.7.

The function φ is a combination of rational and radical functions, and so since $(2,7) \in \mathrm{Dom}(\varphi)$, Proposition 14.11 implies that

$$\lim_{(x,y)\to(2,7)} \varphi(x,y) = \varphi(2,7) = \frac{1}{6}.$$

Therefore

$$\lim_{(x,y)\to(2,7)} \frac{\sqrt{x+y}-3}{x+y-9} = \lim_{(x,y)\to(2,7)} h(x,y) = \frac{1}{6},$$

and we're done. ■

It may seem that the process for evaluating the limit in Example 14.12 is rather lengthy and laborious, but this is only because every step is presented, along with the full justification for each step. The next example goes a little quicker.

Example 14.13. Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}.$$

Solution. The domain of the function is

$$\{(x,y)\ :\ (x,y)\neq(0,0)\};$$

that is, all of $\mathbb{R}^2$ except for the origin. For any $(x,y)\neq(0,0)$,

$$\begin{aligned}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}\\ &= \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2} = \sqrt{x^2+y^2+1}+1,\end{aligned}$$

and so by Theorem 14.7 we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)}\left(\sqrt{x^2+y^2+1}+1\right) = \sqrt{0+0+1}+1 = 2,$$

recalling that a radical function is continuous on its domain. ■

Yet another weapon that is of great worth in the quest to slay limits without recourse to ϵ-δ arguments is the following squeeze theorem, which is much the same as the Squeeze Theorem given in §2.3.

Theorem 14.14 (**Multivariable Squeeze Theorem**). *Let* $\mathbf{c}\in\mathbb{R}^n$ *be a limit point of* $\mathrm{Dom}(f)$. *Suppose there are functions* φ *and* ψ, *and a neighborhood* U *of* $\mathbf{c}$, *such that*

$$\varphi(\mathbf{x})\leq f(\mathbf{x})\leq\psi(\mathbf{x})$$

for all $\mathbf{x}\in U\cap\mathrm{Dom}(f)$ *with* $\mathbf{x}\neq\mathbf{c}$. *If*

$$\lim_{\mathbf{x}\to\mathbf{c}}\varphi(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{c}}\psi(\mathbf{x}) = L$$

for some $L\in\mathbb{R}$, *then*

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L.$$

Proof. Suppose that $\varphi(\mathbf{x}),\psi(\mathbf{x})\to L$ as $\mathbf{x}\to\mathbf{c}$. Since $\mathbf{c}\in U$ and U is open, there exists some $r>0$ such that $B_r(\mathbf{c})\subseteq U$.

Let $\epsilon>0$. Since $\lim_{\mathbf{x}\to\mathbf{c}}\varphi(\mathbf{x})=L$, there exists some $\delta_1>0$ such that, for any $\mathbf{x}\in\mathrm{Dom}(\varphi)$,

$$0<\|\mathbf{x}-\mathbf{c}\|<\delta_1 \ \text{ implies } \ |\varphi(\mathbf{x})-L|<\epsilon.$$

Since $\lim_{\mathbf{x}\to\mathbf{c}} \psi(\mathbf{x}) = L$, there exists some $\delta_2 > 0$ such that, for any $\mathbf{x} \in \mathrm{Dom}(\psi)$,

$$0 < \|\mathbf{x} - \mathbf{c}\| < \delta_2 \text{ implies } |\psi(\mathbf{x}) - L| < \epsilon.$$

Choose $\delta = \min\{r, \delta_1, \delta_2\}$ and suppose $\mathbf{x} \in \mathrm{Dom}(f)$ is such that $0 < \|\mathbf{x} - \mathbf{c}\| < \delta$; that is, $\mathbf{x} \in B_\delta(\mathbf{c}) \cap \mathrm{Dom}(f)$ with $\mathbf{x} \neq \mathbf{c}$. Since $B_\delta(\mathbf{c}) \subseteq B_r(\mathbf{c}) \subseteq U$, it follows that $\mathbf{x} \in U \cap \mathrm{Dom}(f)$ with $\mathbf{x} \neq \mathbf{c}$, and therefore $\varphi(\mathbf{x}) \leq f(\mathbf{x}) \leq \psi(\mathbf{x})$ holds. Moreover we have

$$0 < \|\mathbf{x} - \mathbf{c}\| < \delta_1 \quad \text{and} \quad 0 < \|\mathbf{x} - \mathbf{c}\| < \delta_2$$

since $\delta \leq \delta_1, \delta_2$, and so

$$-\epsilon < \varphi(\mathbf{x}) - L < \epsilon \quad \text{and} \quad -\epsilon < \psi(\mathbf{x}) - L < \epsilon \tag{14.2}$$

both hold. From $\varphi(\mathbf{x}) \leq f(\mathbf{x}) \leq \psi(\mathbf{x})$ we obtain

$$\varphi(\mathbf{x}) - L \leq f(\mathbf{x}) - L \leq \psi(\mathbf{x}) - L,$$

which together with (14.2) gives

$$-\epsilon < f(\mathbf{x}) - L < \epsilon,$$

or equivalently $|f(\mathbf{x}) - L| < \epsilon$.

We have now shown that, for any $\mathbf{x} \in \mathrm{Dom}(f)$, $0 < \|\mathbf{x} - \mathbf{c}\| < \delta$ implies $|f(\mathbf{x}) - L| < \epsilon$, and therefore

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$$

as desired. ■

Example 14.15. Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}}.$$

Solution. Define

$$f(x,y) = \frac{|xy|}{\sqrt{x^2+y^2}}, \quad \varphi(x,y) = 0, \quad \text{and} \quad \psi(x,y) = |x|.$$

Note that the domain of f is $D = \mathbb{R}^2 \setminus \{(0,0)\}$ (i.e. all of $\mathbb{R}^2$ except for the origin). Clearly $\varphi(x,y) \leq f(x,y)$ for all $(x,y) \in D$. If $x \in \mathbb{R}$ and $y \neq 0$, then

$$f(x,y) = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{|xy|}{\sqrt{y^2}} = \frac{|x||y|}{|y|} = |x| = \psi(x,y);$$

and if $x \neq 0$ and $y = 0$, then

$$f(x,0) = \frac{0}{\sqrt{x^2}} = 0 < |x| = \psi(x,0).$$

Thus $f(x,y) \leq \psi(x,y)$ for all $(x,y) \in D$. Now, since $\varphi(x,y) \leq f(x,y) \leq \psi(x,y)$ for all $(x,y) \in D$,

$$\lim_{(x,y)\to(0,0)} \varphi(x,y) = \lim_{(x,y)\to(0,0)} (0) = 0,$$

and

$$\lim_{(x,y)\to(0,0)} \psi(x,y) = \lim_{(x,y)\to(0,0)} |x| = 0,$$

we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}} = 0$$

by the Multivariable Squeeze Theorem. ■

Recall from first-semester calculus that

$$\lim_{x\to c} f(x) = L \quad \text{iff} \quad \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L;$$

that is, the two-sided limit of f equals L if and only if the two one-sided limits both equal L. The value of $f(x)$ must not converge on different values depending on whether c is approached from the left or the right! If f is a two-variable function, however, things are a little more complicated, because the domain of f will be some region in a plane, and any point (a, b) on the xy-plane can be approached along an *infinite* number of possible paths. A consequence of Definition 14.4 is that, in order for the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

to exist as some real number L, $f(x, y)$ must approach L as (x, y) approaches (a, b) *regardless of the path that is taken!* If two paths can be found that result in $f(x, y)$ approaching two different real numbers (or if *one* path can be found that results in $f(x, y)$ approaching no real number whatsoever), then it can be shown that the limit cannot exist in $\mathbb{R}$.

The next example illustrates the method, known as the **Two-Path Test**, for showing that a limit does not exist. The idea is simple: if $f(\mathbf{x})$ approaches two different values L_1 and L_2 when two different paths γ_1 and γ_2 to a point $\mathbf{c}$ are taken, then $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})$ cannot exist because the value of a limit must be unique. In applying the test we must, of course, consider only points on each path which lie in the domain of f. The formal statement of the test is as follows.

Theorem 14.16 (Two-Path Test). *Suppose $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, with $\mathbf{c}$ a limit point of D. Let $L_1, L_2 \in \mathbb{R}$ such that $L_1 \neq L_2$. If for each $\epsilon > 0$ there exists some $\mathbf{x}_1, \mathbf{x}_2 \in D \cap B'_\epsilon(\mathbf{c})$ such that $f(\mathbf{x}_1) = L_1$ and $f(\mathbf{x}_2) = L_2$, then $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})$ does not exist.*

Example 14.17. Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^2}{x^3 + y^2}$$

does not exist.

Solution. Let

$$f(x, y) = \frac{x^3 - y^2}{x^3 + y^2}.$$

The domain of f is consists of all of $\mathbb{R}^2$ except the origin. Let $\epsilon > 0$ be arbitrary. Now, within the set $B'_\epsilon(0, 0)$ (i.e. all points that are a distance less than ϵ from $(0, 0)$ except for $(0, 0)$ itself) we can choose two points: a point $(x, 0) \neq (0, 0)$ on the x-axis that is less than ϵ away from $(0, 0)$, and a point $(0, y) \neq (0, 0)$ on the y-axis that is less than ϵ away from $(0, 0)$. For instance we could choose $(x, 0) = (\epsilon/2, 0)$ and $(0, y) = (0, \epsilon/2)$, but it is not necessary to be this explicit. Now,

$$f(x, 0) = \frac{x^3 - 0^2}{x^3 + 0^2} = \frac{x^3}{x^3} = 1,$$

and

$$f(0,y) = \frac{0^3 - y^2}{0^3 + y^2} = -\frac{y^2}{y^2} = -1.$$

We have now shown that, for each $\epsilon > 0$, there exists some $(x_1, y_1), (x_2, y_2) \in B'_\epsilon(0,0)$ such that $f(x_1, y_1) = 1$ and $f(x_2, y_2) = -1$. Therefore the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^2}{x^3 + y^2}$$

does not exist by the Two-Path Test. ■

In the example above, the two paths leading to the point $(0,0)$ that were chosen to show the given limit does not exist were the x-axis (i.e. the line $y = 0$) and the y-axis (i.e. the line $x = 0$). Any number of other paths could have been considered to achieve the same result, such as $x = y^{2/3}$.

Proposition 14.18. *Let f be a function with domain $D \subseteq \mathbb{R}$.*

1. *If $\varphi : D \times \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x,y) = f(x)$, then for any $a \notin D$ and $b \in \mathbb{R}$*

$$\lim_{x\to a} f(x) = L \quad \Rightarrow \quad \lim_{(x,y)\to(a,b)} \varphi(x,y) = L.$$

2. *If $\psi : \mathbb{R} \times D \to \mathbb{R}$ is given by $\psi(x,y) = f(y)$, then for any $a \in \mathbb{R}$ and $b \notin D$*

$$\lim_{y\to b} f(y) = L \quad \Rightarrow \quad \lim_{(x,y)\to(a,b)} \psi(x,y) = L.$$

Proof. We prove only part (1), the proof of part (2) being similar. Suppose that $\varphi : D \times \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x,y) = f(x)$. Fix $a \notin D$ and $b \in \mathbb{R}$, and suppose that $f(x) \to L$ as $x \to a$. Let $\epsilon > 0$ be arbitrary. Then there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in \mathrm{Dom}(f)$ for which $0 < |x - a| < \delta$. Let $(x,y) \in \mathrm{Dom}(\varphi) = D \times \mathbb{R}$ be such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta. \tag{14.3}$$

Now, $x \neq a$ since $a \notin D$, so (14.3) implies $0 < |x - a| < \delta$, which in turn implies $|f(x) - L| < \epsilon$. Hence $|\varphi(x,y) - L| < \epsilon$, and therefore

$$\lim_{(x,y)\to(a,b)} \varphi(x,y) = L$$

by Definition 14.4. ■

The purpose of Proposition 14.18 is to provide a firm theoretical underpinning for evaluating a limit of the form

$$\lim_{(x,y)\to(a,b)} g(x)h(y)$$

in the natural way:

$$\lim_{(x,y)\to(a,b)} g(x)h(y) = \Big(\lim_{x\to a} g(x)\Big)\Big(\lim_{y\to b} h(y)\Big). \tag{14.4}$$

It is not a straightforward matter of using Theorem 14.8(4), because the limits on the right-hand side of (14.4) are single-variable limits, which each are confined to the real number line. However, provided the hypotheses of Proposition 14.18 are satisfied, then the equality (14.4) is true. The advantage of passing from a single multivariable limit to multiple single-variable limits is clear:

many techniques for evaluating limits, such as L'Hôpital's Rule, can only be applied to limits of a single-variable function. The next example puts the theory into practice.

Example 14.19. Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{y\sin 3x}{8x\sin 5y}$$

Solution. Let $D = (-\infty, 0) \cup (0, \infty)$, and define $\varphi : D \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x, y) = \frac{\sin 3x}{8x}.$$

Applying L'Hôpital's Rule, we have

$$\lim_{x\to 0} \frac{\sin 3x}{8x} = \lim_{x\to 0} \frac{3\cos 3x}{8} = \frac{3}{8}.$$

Now, observing that the function $f(x) = \sin 3x/8x$ has domain D and $0 \notin D$, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{\sin 3x}{8x} = \lim_{x\to 0} \frac{\sin 3x}{8x} = \frac{3}{8}$$

by Proposition 14.18(1).

Next, define $\psi : \mathbb{R} \times D \to \mathbb{R}$ by

$$\psi(x, y) = \frac{y}{\sin 5y}.$$

Again using L'Hôpital's Rule,

$$\lim_{y\to 0} \frac{y}{\sin 5y} = \lim_{y\to 0} \frac{1}{5\cos 5y} = \frac{1}{5}.$$

Observing that the function $g(y) = y/\sin 5y$ has domain D and $0 \notin D$, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{y}{\sin 5y} = \lim_{y\to 0} \frac{y}{\sin 5y} = \frac{1}{5}$$

by Proposition 14.18(2).

Finally,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{y\sin 3x}{8x\sin 5y} &= \lim_{(x,y)\to(0,0)} \left(\frac{\sin 3x}{8x} \cdot \frac{y}{\sin 5y}\right) \\ &= \left(\lim_{(x,y)\to(0,0)} \frac{\sin 3x}{8x}\right)\left(\lim_{(x,y)\to(0,0)} \frac{y}{\sin 5y}\right) = \left(\frac{3}{8}\right)\left(\frac{1}{5}\right) = \frac{3}{40}\end{aligned}$$

by Theorem 14.8(4). ■

Theorem 14.20. *Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : I \subseteq \mathbb{R} \to \mathbb{R}$. If f is continuous at $\mathbf{c}$ and g is continuous at $f(\mathbf{c})$, then $g \circ f$ is continuous at $\mathbf{c}$.*

Proof. Suppose f is continuous at $\mathbf{c}$ and g is continuous at $f(\mathbf{c})$. Fix $\epsilon > 0$. There exists some $\gamma > 0$ such that, for all $y \in \mathrm{Dom}(g)$,

$$|y - f(\mathbf{c})| < \gamma \quad \Rightarrow \quad |g(y) - g(f(\mathbf{c}))| < \epsilon. \tag{14.5}$$

Now, the continuity of f at $\mathbf{c}$ implies there exists some $\delta > 0$ such that, for all $\mathbf{x} \in \mathrm{Dom}(f)$,

$$\|\mathbf{x} - \mathbf{c}\| < \delta \quad \Rightarrow \quad |f(\mathbf{x}) - f(\mathbf{c})| < \gamma. \tag{14.6}$$

Let $\mathbf{x} \in \mathrm{Dom}(g \circ f)$, so that $\mathbf{x} \in \mathrm{Dom}(f)$ and $f(\mathbf{x}) \in \mathrm{Dom}(g)$. Suppose that $\|\mathbf{x} - \mathbf{c}\| < \delta$. Then $|f(\mathbf{x}) - f(\mathbf{c})| < \gamma$ by (14.6), and since $f(\mathbf{x}) \in \mathrm{Dom}(g)$ it follows from (14.5) that

$$|g(f(\mathbf{x})) - g(f(\mathbf{c}))| < \epsilon.$$

That is,

$$|(g \circ f)(\mathbf{x}) - (g \circ f)(\mathbf{c})| < \epsilon,$$

and therefore $g \circ f$ is continuous at $\mathbf{c}$. ■

Example 14.21. Determine the set of points in $\mathbb{R}^2$ where the function

$$h(x, y) = \sqrt{x - y^2}$$

is continuous.

Solution. Define $f(x, y) = x - y^2$, which is a polynomial function, and $g(x) = \sqrt{x}$, which is a radical function. By Proposition 14.11 both f and g are continuous on their domains, where $\mathrm{Dom}(f) = \mathbb{R}^2$, and $\mathrm{Dom}(g) = [0, \infty)$. Now, since

$$\begin{aligned}
\mathrm{Dom}(g \circ f) &= \{(x, y) \in \mathbb{R}^2 : (x, y) \in \mathrm{Dom}(f) \text{ and } f(x, y) \in \mathrm{Dom}(g)\} \\
&= \{(x, y) \in \mathbb{R}^2 : x - y^2 \in [0, \infty)\} = \{(x, y) \in \mathbb{R}^2 : x - y^2 \geq 0\} \\
&= \{(x, y) \in \mathbb{R}^2 : x \geq y^2\} = \mathrm{Dom}(h),
\end{aligned}$$

and $h(x, y) = (g \circ f)(x, y)$ for all (x, y) in the common domain, we see that $h = g \circ f$.

Let $(a, b) \in \mathrm{Dom}(h)$. Since $(a, b) \in \mathbb{R}^2$ we know that f is continuous at (a, b). Moreover,

$$f(a, b) = a - b^2 \geq 0$$

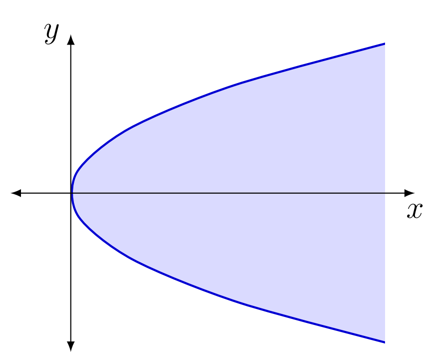


FIGURE 61.

shows that $f(a,b) \in [0,\infty) = \mathrm{Dom}(g)$, and so g is continuous at $f(a,b)$. Hence $g \circ f$ is continuous at (a,b) by Theorem 14.20, and because $h = g \circ f$ it follows that h is continuous at (a,b). Since (a,b) is an arbitrary point in $\mathrm{Dom}(h)$, we conclude that h is continuous on its domain. That is, h is continuous on precisely the set $\{(x,y) : x \geq y^2\}$, shown in Figure 61. ■

There will be frequent occasions when a vector function

$$\mathbf{r}(t) = \big\langle x_1(t), \ldots, x_n(t) \big\rangle$$

will be composed with a real-valued function $f(x_1, \ldots, x_n) = y$. Thus, some results concerning the function $f \circ \mathbf{r}$ should be developed. Since $f \circ \mathbf{r}$ is just a real-valued function of a single real variable, we can expect the results (and their proofs) to strongly resemble developments in single-variable calculus. The following proposition, for instance, has an analog in §2.6.

Proposition 14.22. *Let* $\mathbf{r} : I \subseteq \mathbb{R} \to \mathbb{R}^n$ *and* $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. *If*

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{c}$$

for some $\mathbf{c}$ *in the interior of* D *and* f *is continuous at* $\mathbf{c}$, *then*

$$\lim_{t \to a} f(\mathbf{r}(t)) = f\left(\lim_{t \to a} \mathbf{r}(t)\right) = f(\mathbf{c}).$$

Proof. Suppose that $\lim_{t \to a} \mathbf{r}(t) = \mathbf{c}$ for some $\mathbf{c} \in \mathrm{Int}(D)$. Let $\epsilon > 0$. Since f is continuous at $\mathbf{c}$ and $\mathbf{c} \in \mathrm{Int}(D)$, there exists some $\delta_0 > 0$ such that $\|\mathbf{x} - \mathbf{c}\| < \delta_0$ implies that $|f(\mathbf{x}) - f(\mathbf{c})| < \epsilon$. Since

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{c}$$

there exists some $\delta > 0$ such that $0 < |t-a| < \delta$ implies that $\|\mathbf{r}(t) - \mathbf{c}\| < \delta_0$. Thus, $0 < |t-a| < \delta$ implies that $|f(\mathbf{r}(t)) - f(\mathbf{c})| < \epsilon$, and therefore

$$\lim_{t \to a} f(\mathbf{r}(t)) = f(\mathbf{c}).$$

as desired. ■

14.3 – Partial Derivatives

A function f of two independent variables x and y, conventionally denoted by $f(x, y)$, can be differentiated with respect to either variable. The way this is done is to treat one of the variables as a constant and differentiate with respect to the other variable in the usual manner developed in first-semester calculus.

Definition 14.23. *Suppose $(x, y) \in \mathbb{R}^2$ is an interior point of $\operatorname{Dom}(f)$. The **partial derivative of f with respect to x** at (x, y) is*

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}, \tag{14.7}$$

*and the **partial derivative of f with respect to y** at (x, y) is*

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h},$$

provided these limits exist.

Both limits above have only one variable in play, and so they can be interpreted using the definition of limit in Chapter 2; however, since (x, y) is in the interior of $\operatorname{Dom}(f)$, Definition 14.5 is wholly equivalent and thus equally appropriate. So, with a little bit of thought, the limit (14.7) can be translated into the following statement: For every $\epsilon > 0$ there exists some $\delta > 0$ such that, if $0 < |h| < \delta$ and $(x + h, y) \in \operatorname{Dom}(f)$, then

$$\left| \frac{f(x + h, y) - f(x, y)}{h} - f_x(x, y) \right| < \epsilon.$$

The functions f_x and f_y are together referred to as the "first-order partial derivatives" of f, or simply the "first partials" of f. If f is a function of three variables x, y, and z, then there are three first partials of f: f_x, f_y, and f_z, where

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and so on. In general, for a function f of n variables $x_1, \ldots, x_n$, we have

$$f_{x_i}(\mathbf{x}) = f_{x_i}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

for each $1 \leq i \leq n$ and $\mathbf{x} \in \operatorname{Int}(\operatorname{Dom}(f))$.

Besides the symbol f_x (called "subscript notation"), the partial derivative of f with respect to x can be denoted by $\partial_x f$ (operator notation) or $\partial f / \partial x$ (Leibniz notation). Correspondingly $f_x(x, y)$ can be denoted by

$$\frac{\partial f}{\partial x}(x, y) \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{(x,y)} \quad \text{or} \quad \partial_x f(x, y).$$

Notation. In these notes only two notations will be employed to any significant extent to denote the partial derivative of f with respect to x: f_x and $\partial_x f$. Both notations extend naturally into higher order partial derivatives, with $f_{xx} = (f_x)_x$ being the "second partial derivative of f with respect to x," and $f_{xy} = (f_x)_y$ being the partial derivative of f_x with respect to y (otherwise known as a "mixed partial derivative"). We can translate to operator notation as follows:

$$f_{xx} = (f_x)_x = \partial_x(\partial_x f) = \partial_{xx} f$$

and

$$f_{xy} = (f_x)_y = \partial_y(\partial_x f) = \partial_{yx} f.$$

Alternative operator symbols are ∂_1 for ∂_x, and ∂_2 for ∂_y. Also we have $\partial_x^2 = \partial_{xx}$, $\partial_x^3 = \partial_{xxx}$, and so on, with ∂_x^n used generally to denote the nth partial derivative of f with respect to x.

Example 14.24. Given

$$f(x,y) = 3x^2y^7 - 2xy + 5y - 8x^3,$$

find f_x and f_y.

Solution. To find f_x we treat y as a constant and consider x to be the only variable, enabling us to differentiate in the usual fashion.

$$\begin{aligned} f_x(x,y) &= \partial_x(3x^2y^7 - 2xy + 5y - 8x^3) \\ &= \partial_x(3x^2y^7) - \partial_x(2xy) + \partial_x(5y) - \partial_x(8x^3) \\ &= 6xy^7 - 2y - 24x^2. \end{aligned}$$

Notice that $\partial_x(5y) = 0$ since $5y$ is considered to be a constant.

To find f_y we treat x as a constant and consider y to be the only variable.

$$\begin{aligned} f_y(x,y) &= \partial_y(3x^2y^7 - 2xy + 5y - 8x^3) \\ &= \partial_y(3x^2y^7) - \partial_y(2xy) + \partial_y(5y) - \partial_y(8x^3) \\ &= 21x^2y^6 - 2x + 5, \end{aligned}$$

where $\partial_y(8x^3) = 0$. ■

Example 14.25. Given

$$f(x,y,z) = \frac{\sin xy - \ln yz}{x^2 + y^3 + z^4},$$

find f_x, f_y, and f_z.

Solution. To find f_x we treat y and z as constants and consider x to be the only variable.

$$f_x(x,y,z) = \frac{(x^2+y^3+z^4)(y\cos xy) - 2x(\sin xy - \ln yz)}{(x^2+y^3+z^4)^2}$$

To find f_y we treat x and z as constants.

$$f_y(x,y,z) = \frac{(x^2+y^3+z^4)\left(x\cos xy - \dfrac{1}{yz}\cdot z\right) - 3y^2(\sin xy - \ln yz)}{(x^2+y^3+z^4)^2}$$

$$= \frac{(x^2+y^3+z^4)(xy\cos xy - 1) - 3y^3(\sin xy - \ln y^2)}{y(x^2+y^3+z^4)^2}$$

Finally, to find f_z we treat x and y as constants.

$$\begin{aligned} f_z(x,y,z) &= \frac{(x^2+y^3+z^4)\left(-\frac{1}{yz}\cdot y\right) - 4z^3(\sin xy - \ln yz)}{(x^2+y^3+z^4)^2} \\ &= \frac{(x^2+y^3+z^4)\,(-1/z) + 4z^3(\ln yz - \sin xy)}{(x^2+y^3+z^4)^2} \\ &= \frac{4z^4(\ln yz - \sin xy) - (x^2+y^3+z^4)}{z(x^2+y^3+z^4)^2} \end{aligned}$$

■

Definition 14.26. *Let $U \subseteq \mathbb{R}^2$ be an open set. A function $f : U \to \mathbb{R}$ is* ***differentiable*** *at $(x,y) \in U$ if*

$$\lim_{(h,k)\to(0,0)} \frac{f(x+h,y+k) - f(x,y) - f_x(x,y)h - f_y(x,y)k}{\sqrt{h^2+k^2}} = 0. \tag{14.8}$$

If f is differentiable at every point in U, then f is said to be ***differentiable on*** *U.*

Note that by this definition, in order for f to be differentiable at (x,y) it is certainly necessary that the partial derivatives f_x and f_y both exist at (x,y)—*but that is not sufficient*, as part (c) of Example 14.32 illustrates.

We now define the gradient of a function f, which is a vector function that is often referred to as the **total derivative** (or simply the **derivative**) of f The reason for this will become gradually more apparent in the pages to come.

Definition 14.27. *If f is differentiable at $(x,y) \in \mathbb{R}^2$, then the* ***gradient*** *of f at (x,y) is*

$$\nabla f(x,y) = \big\langle f_x(x,y), f_y(x,y)\big\rangle.$$

More generally, if f is differentiable at $\mathbf{x} \in \mathbb{R}^n$, then the gradient of f at $\mathbf{x}$ is

$$\nabla f(\mathbf{x}) = \big\langle f_{x_1}(\mathbf{x}), f_{x_2}(\mathbf{x}), \ldots, f_{x_n}(\mathbf{x})\big\rangle.$$

So ∇f (read as "del f") is a vector function of two independent variables (or three, in $\mathbb{R}^3$) with domain consisting of all points $(x,y) \in \mathbb{R}^2$ where $f_x(x,y), f_y(x,y) \in \mathbb{R}$. As mentioned in §13.0, it can be convenient to regard the ordered pairs (x,y) in the domain of f as vectors as well (*position* vectors, in particular), to make use of operations such as vector addition and vector norms and thereby streamline the notation. We can represent (h,k) by $\mathbf{h} = \langle h,k\rangle$, and (x,y) by $\mathbf{x} = \langle x,y\rangle$, so that $(x+h,y+k)$ becomes

$$\langle x+h, y+k\rangle = \mathbf{x} + \mathbf{h}.$$

Now the limit in Definition 14.26 can be written as

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - f_x(\mathbf{x})h - f_y(\mathbf{x})k}{\|\mathbf{h}\|} = 0,$$

where we let $\mathbf{0} = \langle 0, 0\rangle$ replace $(0, 0)$. Next, noting that

$$f_x(\mathbf{x})h + f_y(\mathbf{x})k = \langle f_x(\mathbf{x}), f_y(\mathbf{x})\rangle \cdot \langle h, k\rangle = \nabla f(\mathbf{x}) \cdot \mathbf{h},$$

we can write

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0.$$

This version of equation (14.8) is easily extended to a general definition for differentiability in $\mathbb{R}^n$.

Definition 14.28. *Let $U \subseteq \mathbb{R}^n$ be an open set. A function $f : U \to \mathbb{R}$ is* ***differentiable*** *at $\mathbf{x} \in U$ if*

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0. \tag{14.9}$$

If f is differentiable at every point in U, then f is said to be ***differentiable on*** *U.*

Thus, for f in Definition 14.28 to be differentiable at $\mathbf{x}$, it is necessary (but not sufficient) to have $f_{x_i}(\mathbf{x}) \in \mathbb{R}$ for all i; that is, all first partials of f must exist at $\mathbf{x}$.

The advantage of (14.9) over the clunky equation presented in the book's definition of differentiability is its mild resemblance to the definition of differentiability for a real-valued function f of a *single* variable at some $x \in \mathbb{R}$, which states that f is differentiable at x if and only if the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{14.10}$$

exists. (You might notice that since the limit is two-sided—so h can be positive or negative as it approaches 0—it's necessary for x to be in some *open set* on which f is defined.)

But we can do better than a "mild resemblance." In fact (14.10) can exist *if and only if* there exists some $\ell \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - \ell \cdot h}{h} = 0 \tag{14.11}$$

To prove this, the argument goes as follows. Suppose (14.10) exists. Then $f'(x)$ is a real number, so we can choose $\ell = f'(x)$ and get

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x) - \ell \cdot h}{h} &= \lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x) \cdot h}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f'(x) \\
&= f'(x) - f'(x) = 0,
\end{aligned}$$

as desired.

Conversely if there exists some real number ℓ such that (14.11) holds, then since $\lim_{h\to 0} \ell = \ell$ a basic law of limits implies that

$$\begin{aligned}
0 + \ell &= \lim_{h\to 0} \frac{f(x+h) - f(x) - \ell \cdot h}{h} + \lim_{h\to 0} \ell \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x) - \ell \cdot h}{h} + \ell\right],
\end{aligned}$$

which readily yields

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\ell$$

and shows not only that (14.10) exists, but that ℓ must equal $f'(x)$. The ultimate conclusion: a single-variable function f is differentiable if and only if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)-f'(x)\cdot h}{h}=0. \tag{14.12}$$

Now, notice the *striking* resemblance between (14.12) and (14.9). Morever, an immediate consequence of Definition 14.28 is the following: A function f is differentiable at $\mathbf{c}$ if and only if

$$\lim_{\mathbf{x}\to\mathbf{c}}\frac{f(\mathbf{x})-f(\mathbf{c})-\nabla f(\mathbf{c})\cdot(\mathbf{x}-\mathbf{c})}{\|\mathbf{x}-\mathbf{c}\|}=0. \tag{14.13}$$

Notice that $\mathbf{x}\to\mathbf{c}$ implies that $\|\mathbf{x}-\mathbf{c}\|\to 0$ just as $\mathbf{h}\to\mathbf{0}$ implies $\|\mathbf{h}\|\to 0$, and compare (14.13) with

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)-f'(c)\cdot(x-c)}{x-c},$$

which derives from the familiar single-variable derivative formula

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

We see that ∇f takes the place of f' in the preceding comparisons, which gives us our first insight into why ∇f is often called the "total derivative" of f.

The next example shows that the continuity of a multivariable function f at $\mathbf{x}$, even if $\mathbf{x}$ is in the interior of $\mathrm{Dom}(f)$, is no guarantee that any of the partial derivatives of f will exist there.

Example 14.29. Show that the function

$$f(x,y)=\sqrt{x^2+y^2}$$

is continuous at $(0,0)$, but its first-order partial derivatives do not exist there.

Solution. This is a circular cone that opens along the positive z-axis with apex at $(0,0,0)$, as shown in Figure 62. By Proposition 14.11,

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}=\sqrt{0^2+0^2}=0=f(0,0),$$

which shows that f is continuous at $(0,0)$. On the other hand,

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{f(h,0)}{h}=\lim_{h\to 0}\frac{\sqrt{h^2}}{h}=\lim_{h\to 0}\frac{|h|}{h},$$

which does not exist since

$$\lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=1\quad\text{and}\quad\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1.$$

Much the same analysis will reveal that $f_y(0,0)$ also does not exist. ■

Thus, recalling that differentiability by definition requires existence of first partials, it follows that continuity does not imply differentiability. However, differentiability *does* imply continuity.

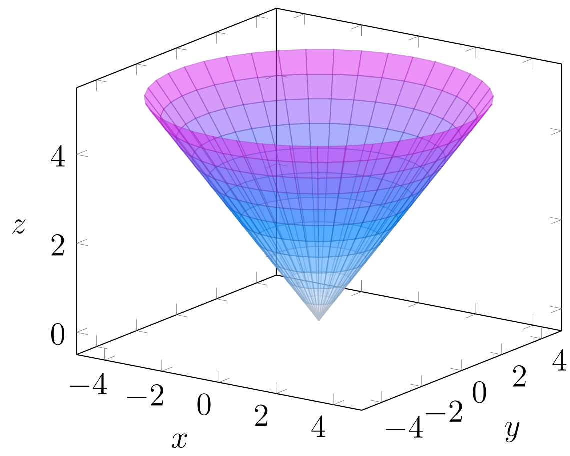


FIGURE 62. The cone $z = \sqrt{x^2 + y^2}$.

Theorem 14.30. *If f is differentiable at $\mathbf{c} \in \mathbb{R}^n$, then f is continuous at $\mathbf{c}$.*

Proof. Suppose that f is differentiable at $\mathbf{c} = \langle c_1, \ldots, c_n \rangle$. Then

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{c}+\mathbf{h}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0$$

holds, which, recalling (14.13), can be rewritten as

$$\lim_{\mathbf{x} \to \mathbf{c}} \frac{f(\mathbf{x}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c})}{\|\mathbf{x}-\mathbf{c}\|} = 0.$$

Now, since

$$\lim_{\mathbf{x} \to \mathbf{c}} \|\mathbf{x}-\mathbf{c}\| = \lim_{\mathbf{x} \to \mathbf{c}} \sqrt{(x_1-c_1)^2 + \cdots + (x_n-c_n)^2} = 0$$

by Proposition 14.11, it follows from Theorem 14.8(4) that

$$\begin{aligned}
&\lim_{\mathbf{x} \to \mathbf{c}} [f(\mathbf{x}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c})] \\
&\qquad = \lim_{\mathbf{x} \to \mathbf{c}} \left[\frac{f(\mathbf{x}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c})}{\|\mathbf{x}-\mathbf{c}\|} \cdot \|\mathbf{x}-\mathbf{c}\| \right] \\
&\qquad = \lim_{\mathbf{x} \to \mathbf{c}} \frac{f(\mathbf{x}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c})}{\|\mathbf{x}-\mathbf{c}\|} \cdot \lim_{\mathbf{x} \to \mathbf{c}} \|\mathbf{x}-\mathbf{c}\| \\
&\qquad = 0 \cdot 0 = 0.
\end{aligned}$$

Next,

$$\nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c}) = f_{x_1}(\mathbf{c})(x_1-c_1) + \cdots + f_{x_n}(\mathbf{c})(x_n-c_n),$$

and so

$$\lim_{\mathbf{x} \to \mathbf{c}} \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c}) = 0$$

follows from Proposition 14.11. Then, employing Theorem 14.8(3), we obtain

$$\lim_{\mathbf{x} \to \mathbf{c}} (f(\mathbf{x}) - f(\mathbf{c})) = \lim_{\mathbf{x} \to \mathbf{c}} \Big(\big[f(\mathbf{x}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c}) \big] + \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c}) \Big)$$

$$= \lim_{\mathbf{x}\to\mathbf{c}} \big[f(\mathbf{x}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c})\big] + \lim_{\mathbf{x}\to\mathbf{c}} \nabla f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c})$$
$$= 0 + 0 = 0,$$

from which it's readily concluded that

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}).$$

The continuity of f at $\mathbf{c}$ now follows from Theorem 14.10. ■

Example 14.31. Determine whether the function

$$f(x, y) = 1 - |xy|$$

is differentiable at $(0, 0)$.

Solution. If f is not continuous at $(0, 0)$ then we could use Theorem 14.30 to conclude that f is not differentiable at $(0, 0)$. However, since the polynomial function $p(x, y) = xy$ is continuous on $\mathbb{R}^2$, and the absolute value function $\text{abs}(t) = |t|$ was established in Chapter 2 to be continuous on $\mathbb{R}$, it follows that the composition $(\text{abs} \circ p)(x, y) = |xy|$ is continuous everywhere on $\mathbb{R}^2$. It should now be evident that f is likewise continuous everywhere on $\mathbb{R}^2$, including $(0, 0)$.

At this juncture we're left only with Definition 14.26 as a tool to investigate the differentiability of f. To use it, we must determine $f_x(0, 0)$ and $f_y(0, 0)$. We have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0}(0) = 0$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0}(0) = 0.$$

Now we evaluate the limit in Definition 14.26 directly. Referring to Example 14.15, we obtain

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{-|hk|}{\sqrt{h^2+k^2}} = 0,$$

and therefore f is differentiable at $(0, 0)$. ■

Example 14.32. Consider the function

$$f(x,y) = \begin{cases} \dfrac{2xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Is f continuous at $(0, 0)$?
(b) Is f differentiable at $(0, 0)$?
(c) Evaluate $f_x(0, 0)$ and $f_y(0, 0)$.
(d) Determine whether f_x and f_y are continuous at $(0, 0)$.

Solution.
(a) If we restrict f to the parabola $x = y^2$, $\lim_{(x,y)\to(0,0)} f(x,y)$ becomes

$$\lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} \frac{2y^2 \cdot y^2}{(y^2)^2 + y^4} = \lim_{y\to 0} \frac{2y^4}{2y^4} = \lim_{y\to 0}(1) = 1.$$

This already is enough to show that, if the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exists at all, then it must equal 1, and so in particular we have

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0).$$

Therefore f is not continuous at $(0,0)$.

(b) Since f is not continuous at $(0,0)$, by Theorem 14.30 f is not differentiable at $(0,0)$.

(c) By definition

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} = \lim_{h\to 0}(0) = 0$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)}{h} = \lim_{h\to 0}(0) = 0$$

So despite f not being differentiable at $(0,0)$, both of its first partial derivatives exist there!

(d) For $(x,y) \neq (0,0)$ we can find $f_x(x,y)$ and $f_y(x,y)$ by the usual differentiation rules, and thus, together with part (c), we obtain

$$f_x(x,y) = \begin{cases} \dfrac{2y^6 - 2x^2y^2}{(x^2+y^4)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

and

$$f_y(x,y) = \begin{cases} \dfrac{4x^3y - 4xy^5}{(x^2+y^4)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Now, restricting f_x to the line $y = x$, we find that $\lim_{(x,y)\to(0,0)} f_x(x,y)$ leads to

$$\lim_{x\to 0} f_x(x,x) = \lim_{x\to 0} \frac{2x^2 - 2}{(1+x^2)^2} = -2.$$

So if the limit $\lim_{(x,y)\to(0,0)} f_x(x,y)$ exists, it must equal -2. For continuity the limit must equal 0, and therefore f_x is not continuous at $(0,0)$. A similar analysis can be done for f_y. ■

It should be noted that Example 14.32(b) could have been done by direct application of Definition 14.26 if Example 14.32(c) had been done beforehand. Knowing $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, the limit (14.8) becomes:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \left(\frac{1}{\sqrt{h^2+k^2}} \cdot \frac{2hk^2}{h^2+k^4} \right). \tag{14.14}$$

If we tack toward the origin along the line $k = h$ in the first quadrant, the limit (14.14) leads to

$$\lim_{h\to 0}\left(\frac{1}{\sqrt{2h^2}}\cdot\frac{2h\cdot h^2}{h^2+h^4}\right)=\lim_{h\to 0}\left(\frac{1}{h\sqrt{2}}\cdot\frac{2h}{1+h^2}\right)=\lim_{h\to 0}\frac{\sqrt{2}}{1+h^2}=\sqrt{2},$$

where $\sqrt{h^2} = h$ since $h > 0$ in Quadrant I. Thus if the limit (14.14) exists at all, it must equal $\sqrt{2}$ and *not* 0. This implies that f is not differentiable at $(0,0)$.

Part (c) of Example 14.32 shows that the existence of a function's first partials at some point $\mathbf{x}$ is not sufficient to guarantee differentiability of the function at $\mathbf{x}$. However, if the function's first partials are *continuous* on an open set containing $\mathbf{x}$, then differentiability at $\mathbf{x}$ *does* follow. To prove this, we need the following mean value theorem for multivariable functions. A stronger version of the theorem will be given at the end of the section.

Theorem 14.33 (**Multivariable Mean Value Theorem 1**). *Let f have continuous first partials on $B_r(\mathbf{a}) \subseteq \mathbb{R}^n$. Then for any $\mathbf{x} \in B_r(\mathbf{a})$ there exist $\mathbf{c}_1,\ldots,\mathbf{c}_n \in B_r(\mathbf{a})$ such that*

$$f(\mathbf{x}) - f(\mathbf{a}) = f_{x_1}(\mathbf{c}_1)(x_1 - a_1) + \cdots + f_{x_n}(\mathbf{c}_n)(x_n - a_n).$$

So if f has continuous first-order partial derivatives in an open ball $B_r(a,b) \subseteq \mathbb{R}^2$, then for any $(x,y) \in B_r(a,b)$ there are points $(c_1,d_1),(c_2,d_2) \in B_r(a,b)$ such that

$$f(x,y) - f(a,b) = f_x(c_1,d_1)(x-a) + f_y(c_2,d_2)(y-b).$$

Theorem 14.34. *If the function f has continuous first partials on open set $U \subseteq \mathbb{R}^n$, then f is differentiable on U.*

Proof. Suppose f has continuous first partials on open set U. Fix $\mathbf{a} \in U$, and let $\epsilon > 0$. Since U is open there exists some $\delta_0 > 0$ such that $B_{\delta_0}(\mathbf{a}) \subseteq U$. Also, since the first partials $f_{x_1},\ldots,f_{x_n}$ are continuous on U, there exists some $0 < \delta_1,\ldots,\delta_n < \delta_0$ such that

$$\mathbf{x} \in B_{\delta_k}(\mathbf{a}) \quad\Rightarrow\quad \left|f_{x_k}(\mathbf{a}) - f_{x_k}(\mathbf{x})\right| < \frac{\epsilon}{n}$$

for all $1 \le k \le n$.

Choose $\delta = \min\{\delta_1,\ldots,\delta_n\}$. Suppose that $\mathbf{x} \in B_\delta(\mathbf{a})$ with $\mathbf{x} \ne \mathbf{a}$. Since f has continuous first partials on $B_\delta(\mathbf{a})$, by Theorem 14.33 there exists $\mathbf{c}_1,\ldots,\mathbf{c}_n \in B_\delta(\mathbf{a})$ such that

$$f(\mathbf{x}) - f(\mathbf{a}) = \sum_{k=1}^{n} f_{x_k}(\mathbf{c}_k)(x_k - a_k). \tag{14.15}$$

Now, since $\mathbf{c}_k \in B_{\delta_k}(\mathbf{a})$, we obtain

$$\left|f_{x_k}(\mathbf{a}) - f_{x_k}(\mathbf{c}_k)\right| < \frac{\epsilon}{n}$$

for all k. Hence

$$\sum_{k=1}^{n}\left|f_{x_k}(\mathbf{a}) - f_{x_k}(\mathbf{c}_k)\right| < \epsilon,$$

and since

$$\frac{|x_k - a_k|}{\|\mathbf{x}-\mathbf{a}\|} = \frac{|x_k - a_k|}{\sqrt{(x_1-a_1)^2+\cdots+(x_n-a_n)^2}} \le 1$$

for all k, it follows that
$$\sum_{k=1}^{n} \frac{\left|f_{x_k}(\mathbf{a}) - f_{x_k}(\mathbf{c}_k)\right||x_k - a_k|}{\|\mathbf{x}-\mathbf{a}\|} < \epsilon$$
and finally
$$\frac{1}{\|\mathbf{x}-\mathbf{a}\|}\left|\sum_{k=1}^{n}\big(f_{x_k}(\mathbf{a}) - f_{x_k}(\mathbf{c}_k)\big)(x_k - a_k)\right| < \epsilon \tag{14.16}$$
by the Triangle Inequality from §1.6. Combining equations (14.15) and (14.16) yields
$$\frac{1}{\|\mathbf{x}-\mathbf{a}\|}\left|\sum_{k=1}^{n} f_{x_k}(\mathbf{a})(x_k - a_k) - [f(\mathbf{x}) - f(\mathbf{a})]\right| < \epsilon,$$
and thus
$$\frac{|f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})|}{\|\mathbf{x}-\mathbf{a}\|} < \epsilon.$$
We have now shown that
$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} = 0,$$
or equivalently
$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot\mathbf{h}}{\|\mathbf{h}\|} = 0.$$

Therefore f is differentiable at $\mathbf{a}$, and since $\mathbf{a} \in U$ is arbitrary it follows that f is differentiable on U. ■

If we are only given that a function's first partials are continuous at the point $\mathbf{x}$, rather than on an open set containing $\mathbf{x}$, then the proof above fails since Theorem 14.33 cannot be used. This is why Theorem 14.34 is stated the way it is. Moreover the converse of Theorem 14.34 does not hold in general; that is, the differentiability of a function f at $\mathbf{x}$ does *not* necessarily imply the continuity of the partial derivatives of f at $\mathbf{x}$, as the next example illustrates.

Example 14.35. Show that
$$f(x,y) = \begin{cases} x^2\sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$
is differentiable at $(0,0)$, and yet f_x is discontinuous there.

Solution. For $x \neq 0$ we obtain
$$f_x(x,y) = 2x\sin(1/x) - \cos(1/x)$$
using the usual differentiation rules. For $x = 0$ the rules cannot be applied, and so we must repair to Definition 14.23 to get
$$\begin{aligned} f_x(0,y) &= \lim_{h\to 0}\frac{f(0+h,y) - f(0,y)}{h} \\ &= \lim_{h\to 0}\frac{h^2\sin(1/h)}{h} = \lim_{h\to 0} h\sin(1/h) = 0, \end{aligned}$$

where the last equality can be justified by use of the Squeeze Theorem. Thus,

$$f_x(x,y) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

However, it's easy to see that the limit

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \big[2x\sin(1/x) - \cos(1/x)\big]$$

does not exist, which is to say that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) \neq f_x(0,0) = 0,$$

and so f_x is not continuous at $(0,0)$.

Before showing that f is differentiable at $(0,0)$ we first need to compute $f_y(0,0)$:

$$f_y(0,0) = \lim_{(x,y)\to(0,0)} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{(x,y)\to(0,0)} \frac{f(0,h)}{h} = \lim_{(x,y)\to(0,0)} \frac{0}{h} = 0.$$

Now we show that

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = 0,$$

or equivalently

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = 0. \tag{14.17}$$

Let $\epsilon > 0$. Since $\lim_{x\to 0} x\sin(1/x) = 0$, there exists some $\delta > 0$ for which $0 < x < \delta$ implies that $|x\sin(1/x)| < \epsilon$. Suppose that (h,k) is such that

$$0 < \sqrt{h^2+k^2} < \delta.$$

Now, if $h = 0$ we immediately obtain

$$\frac{|f(h,k)|}{\sqrt{h^2+k^2}} = \frac{|f(0,k)|}{|k|} = 0 < \epsilon,$$

as desired. So assume that $h \neq 0$. Then

$$\frac{|f(h,k)|}{\sqrt{h^2+k^2}} = \frac{h^2|\sin(1/h)|}{\sqrt{h^2+k^2}} \leq \frac{h^2|\sin(1/h)|}{|h|} = |h\sin(1/h)| < \epsilon,$$

since

$$0 < |h| = \sqrt{h^2} \leq \sqrt{h^2+k^2} < \delta.$$

Therefore (14.17) holds, and so f is differentiable at $(0,0)$. ∎

The next theorem concerning when the mixed second partials of a function of two variables can be assumed to be equal may seem rather obscure, but beyond serving as a labor-saving device it will also be crucial in the proof of some results in Chapter 15.

Theorem 14.36 (Clairaut's Theorem). *Let $U \subseteq \mathbb{R}^2$ be an open set and $f : U \to \mathbb{R}$ a function. If f_{xy} and f_{yx} are continuous on U, then $f_{xy} = f_{yx}$ on U.*

As promised, we now give a stronger mean value theorem for multivariable functions, which can be seen to bear a resemblance to the single-variable Mean Value Theorem in §4.2. Recall that for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the symbol $[\mathbf{a}, \mathbf{b}]$ denotes the line segment that joins $\mathbf{a}$ to $\mathbf{b}$.

Theorem 14.37 (Multivariable Mean Value Theorem 2). *Let f have continuous first partials on $B_r(\mathbf{x}) \subseteq \mathbb{R}^n$. Then for any $\mathbf{a}, \mathbf{b} \in B_r(\mathbf{x})$ there exists some $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ such that*

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}).$$

This version is "stronger" in the sense that its conclusion involves only one quite specific additional point $\mathbf{c}$, rather than n points $\mathbf{c}_1, \ldots, \mathbf{c}_n$.

14.4 – Chain Rules

We're now in a position to consider the most basic multivariable chain rule, from which most all other chain rules can be derived.

Theorem 14.38 (Chain Rule 1). *Let $\mathbf{r}(t) = \langle x_1(t), \dots, x_n(t)\rangle$ be differentiable at t. If $U \subseteq \mathbb{R}^n$ is an open set such that $\mathbf{r}(t) \in U$ and $f : U \to \mathbb{R}$ is differentiable on U, then $f \circ \mathbf{r}$ is differentiable at t and*

$$(f \circ \mathbf{r})'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Proof. Since $\mathbf{r}$ is continuous at t, $\mathbf{r}(t) \in U$, and U is open, there exists a sufficiently small $\gamma > 0$ such that $\mathbf{r}(t+h) \in U$ for all $|h| < \gamma$. Hence t is an interior point of $\mathrm{Dom}(f \circ \mathbf{r})$ so it is legitimate to investigate the differentiability of $f \circ \mathbf{r}$ at t.

Define $\mathbf{x}_h = \mathbf{r}(t+h)$, $\mathbf{c} = \mathbf{r}(t)$, $\Delta x_i = x_i(t+h) - x_i(t)$ for $1 \le i \le n$, and $\mathbf{x}_h - \mathbf{c} = \langle \Delta x_1, \dots, \Delta x_n\rangle$. Also let

$$R(\mathbf{c}, \mathbf{x}_h) = f(\mathbf{x}_h) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot (\mathbf{x}_h - \mathbf{c}).$$

Now,

$$\begin{aligned}(f \circ \mathbf{r})'(t) &= \lim_{h\to 0} \frac{(f\circ\mathbf{r})(t+h) - (f\circ\mathbf{r})(t)}{h} = \lim_{h\to 0}\frac{f(\mathbf{r}(t+h)) - f(\mathbf{r}(t))}{h} \\ &= \lim_{h\to 0}\frac{f(\mathbf{x}_h) - f(\mathbf{c})}{h} = \lim_{h\to 0}\frac{\nabla f(\mathbf{c})\cdot(\mathbf{x}_h - \mathbf{c}) + R(\mathbf{c},\mathbf{x}_h)}{h},\end{aligned}$$

whence

$$(f \circ \mathbf{r})'(t) = \lim_{h\to 0}\left[f_{x_1}(\mathbf{c})\frac{\Delta x_1}{h} + \cdots + f_{x_n}(\mathbf{c})\frac{\Delta x_n}{h} + \frac{R(\mathbf{c},\mathbf{x}_h)}{h}\right]. \tag{14.18}$$

The differentiability of $\mathbf{r}$ at t implies

$$\lim_{h\to 0}\frac{\Delta x_i}{h} = x_i'(t) \in \mathbb{R}$$

for each i, so that

$$\lim_{h\to 0}\left|\frac{\mathbf{x}_h - \mathbf{c}}{h}\right| = \sqrt{[x_1'(t)]^2 + \cdots + [x_n'(t)]^2} = M \tag{14.19}$$

for some real number $M \ge 0$. Also the differentiability of f on U implies $f_{x_i}(\mathbf{c}) \in \mathbb{R}$, and we can define a function $\rho : U \to \mathbb{R}$ by

$$\rho(\mathbf{x}) = \begin{cases} R(\mathbf{c},\mathbf{x})/\|\mathbf{x}-\mathbf{c}\|, & \text{if } \mathbf{x} \ne \mathbf{c} \\ 0, & \text{if } \mathbf{x} = \mathbf{c}\end{cases}$$

Indeed ρ is continuous at $\mathbf{c}$ since

$$\lim_{\mathbf{x}\to\mathbf{c}}\rho(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{c}}\frac{R(\mathbf{c},\mathbf{x})}{\|\mathbf{x}-\mathbf{c}\|} = \lim_{\mathbf{x}\to\mathbf{c}}\frac{f(\mathbf{x}) - f(\mathbf{c}) - \nabla f(\mathbf{c})\cdot(\mathbf{x}-\mathbf{c})}{\|\mathbf{x}-\mathbf{c}\|} = 0 = \rho(\mathbf{c}),$$

and so, since $\mathbf{c}$ is in the interior of $\mathrm{Dom}(\rho)$ and

$$\lim_{h\to 0}\mathbf{x}_h = \lim_{h\to 0}\mathbf{r}(t+h) = \mathbf{r}(t) = \mathbf{c},$$

by Proposition 14.22

$$\lim_{h\to 0} \rho(\mathbf{x}_h) = \lim_{h\to 0} \rho(\mathbf{r}(t+h)) = \rho(\mathbf{c}) = 0. \tag{14.20}$$

We now show that $\lim_{h\to 0} R(\mathbf{c}, \mathbf{x}_h)/h = 0$. Let $\epsilon > 0$. By (14.19) there is some $\delta_1 > 0$ such that $0 < |h| < \delta_1$ implies

$$\left| \frac{\mathbf{x}_h - \mathbf{c}}{h} - M \right| < 1,$$

and so $\|\mathbf{x}_h - \mathbf{c}\|/|h| < M + 1$. By (14.20) there is some $\delta_2 > 0$ such that $0 < |h| < \delta_2$ implies $|\rho(\mathbf{x}_h)| < \epsilon/(M+1)$. Choose $\delta = \min\{\delta_1, \delta_2, \gamma\}$, and suppose that $0 < |h| < \delta$.

If $\|\mathbf{x}_h - \mathbf{c}\| = 0$, then $\mathbf{x}_h = \mathbf{c}$ so that

$$\left| \frac{R(\mathbf{c}, \mathbf{x}_h)}{h} \right| = \left| \frac{R(\mathbf{c}, \mathbf{c})}{h} \right| = 0 < \epsilon.$$

If $\|\mathbf{x}_h - \mathbf{c}\| \neq 0$, then $\mathbf{x}_h \neq \mathbf{c}$ so that

$$\left| \frac{R(\mathbf{c}, \mathbf{x}_h)}{h} \right| = \frac{|R(\mathbf{c}, \mathbf{x}_h)|}{\|\mathbf{x}_h - \mathbf{c}\|} \cdot \frac{\|\mathbf{x}_h - \mathbf{c}\|}{|h|} = |\rho(\mathbf{x}_h)| \cdot \left\| \frac{\mathbf{x}_h - \mathbf{c}}{h} \right\| < \frac{\epsilon}{M+1} \cdot (M+1) = \epsilon.$$

Therefore $\lim_{h\to 0} R(\mathbf{c}, \mathbf{x}_h)/h = 0$.

Finally, from (14.18) we obtain

$$\begin{aligned} (f \circ \mathbf{r})'(t) &= f_{x_1}(\mathbf{c}) \lim_{h\to 0} \frac{\Delta x_1}{h} + \cdots + f_{x_n}(\mathbf{c}) \lim_{h\to 0} \frac{\Delta x_n}{h} + \lim_{h\to 0} \frac{R(\mathbf{c}, \mathbf{x}_h)}{h} \\ &= f_{x_1}(\mathbf{c}) x_1'(t) + \cdots + f_{x_n}(\mathbf{c}) x_n'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \end{aligned}$$

which completes the proof. ■

It can be seen that Chain Rule 1 is similar in form to the chain rule from single-variable calculus, in which $F(t) = f(r(t))$ has derivative $F'(t) = f'(r(t))r'(t)$. Indeed this similarity lends further credence to the idea that ∇f is the "total derivative" of f.

In $\mathbb{R}^2$ we may have functions $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and $f(x, y)$, so that

$$(f \circ \mathbf{r})(t) = f(\mathbf{r}(t)) = f(x(t), y(t))$$

and by Chain Rule 1

$$(f \circ \mathbf{r})'(t) = f_x(\mathbf{r}(t))x'(t) + f_y(\mathbf{r}(t))y'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

In $\mathbb{R}^3$, given functions $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $f(x, y, z)$ so that $(f \circ \mathbf{r})(t) = f(x(t), y(t), z(t))$, we obtain

$$(f \circ \mathbf{r})'(t) = f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t).$$

Example 14.39. Let

$$f(x, y, z) = \ln(x^2 + y^2 + z^2) \quad \text{and} \quad \mathbf{r}(t) = \langle t, t^2, t^3 \rangle.$$

If $F = f \circ \mathbf{r}$, find F'.

Solution. Here
$$F(t) = (f \circ \mathbf{r})(t) = f(x(t), y(t), z(t)),$$
where $x(t) = t$, $y(t) = t^2$ and $z(t) = t^3$. Chain Rule 1 immediately gives
$$\begin{aligned}
F'(t) &= (f \circ \mathbf{r})'(t) \\
&= f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t) \\
&= \frac{2x(t)}{x^2(t) + y^2(t) + z^2(t)} \cdot 1 + \frac{2y(t)}{x^2(t) + y^2(t) + z^2(t)} \cdot 2t + \frac{2z(t)}{x^2(t) + y^2(t) + z^2(t)} \cdot 3t^2 \\
&= \frac{2t}{t^2 + t^4 + t^9} + \frac{4t^3}{t^2 + t^4 + t^9} + \frac{6t^5}{t^2 + t^4 + t^9} = \frac{6t^5 + 4t^3 + 2t}{t^2 + t^4 + t^9},
\end{aligned}$$
which can be interpreted as being the rate of change of f with respect to t along the curve generated by $\mathbf{r}$. ■

Example 14.40. The radius r of a right circular cone is increasing at a rate of 1.8 cm/s while its height h is decreasing at a rate of 2.5 cm/s. At what rate is the volume V of the cone changing at the time when the radius is 120 cm and the height is 140 cm?

Solution. All rates of change are understood to be with respect to time here. V is a function of r and h, while r and h themselves are each functions of time t. In notation similar to that in Theorem 14.38 we write $V(t) = f(r, h)$, with $r = r(t)$ and $h = h(t)$. Now, from geometry the volume of a right circular cone with radius r and height h is given as $\pi r^2 h/3$, which takes the place of $f(r, h)$ to yield
$$V(t) = \frac{\pi r^2 h}{3}.$$
By Chain Rule 1 the rate of change of V at time t is
$$V'(t) = f_r(r, h)r'(t) + f_h(r, h)h'(t) = \frac{2\pi rh}{3}r'(t) + \frac{\pi r^2}{3}h'(t),$$
where we're given that $r'(t) = 1.8$ cm/s and $h'(t) = -2.5$ cm/s. Letting t_0 represent the time when $r = 120$ cm and $h = 140$ cm (note it's not our job to determine t_0), we obtain
$$V'(t_0) = \frac{2\pi(120\text{ cm})(140\text{ cm})}{3}(1.8\text{ cm/s}) + \frac{\pi(120\text{ cm})^2}{3}(-2.5\text{ cm/s}) = 8160\pi\text{ cm}^3/\text{s},$$
or approximately 25,635.4 cm^3/s. ■

Theorem 14.41 (Chain Rule 2). *Let $F(s, t) = f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$ are functions that are differentiable at $q = (s_0, t_0) \in \mathbb{R}^2$, and f is differentiable at $p = (x(q), y(q))$. Then F is differentiable at q, where*
$$F_s(q) = f_x(p)x_s(q) + f_y(p)y_s(q) \quad \text{and} \quad F_t(q) = f_x(p)x_t(q) + f_y(p)y_t(q).$$

Since partial derivatives only allow for one variable to be in play, the proof of Chain Rule 2 can be done in the same manner as Chain Rule 1's proof, or Chain Rule 1 could be used to prove Chain Rule 2.

Example 14.42. Find z_s and z_t, where

$$z = xy - 2x + 3y$$

with $x = \sin s$ and $y = \tan t$.

Solution. Here

$$z(s,t) = f(x,y) = xy - 2x + 3y$$

with

$$x = x(s,t) = \sin s \quad \text{and} \quad y = y(s,t) = \tan t.$$

By Chain Rule 2,

$$\begin{aligned} z_s(s,t) &= f_x(x,y)x_s(s,t) + f_y(x,y)y_s(s,t) \\ &= (y-2)\cos s + (x+3)(0) = (\tan t - 2)\cos s, \end{aligned}$$

and

$$\begin{aligned} z_t(s,t) &= f_x(x,y)x_t(s,t) + f_y(x,y)y_t(s,t) \\ &= (y-2)(0) + (x+3)\sec^2 t = (\sin s + 3)\sec^2 t, \end{aligned}$$

expressing each partial derivative in terms of s and t. ■

An exceedingly important result in mathematical analysis is the Implicit Function Theorem, which will be furnished here, without proof, for the specific case when f is a real-valued function of two variables

Theorem 14.43 (**Implicit Function Theorem**). *Suppose f has continuous first partials on open set $U \subseteq \mathbb{R}^2$, and $f(a,b) = 0$ for $(a,b) \in U$.*

1. *If $f_y(a,b) \neq 0$, then there exists an open interval $I \subseteq \mathbb{R}$ and a continuously differentiable function $h : I \to \mathbb{R}$ such that $a \in I$, $h(a) = b$, and $f(x,h(x)) = 0$ for all $x \in I$.*
2. *If $f_x(a,b) \neq 0$, then there exists an open interval $J \subseteq \mathbb{R}$ and a continuously differentiable function $g : J \to \mathbb{R}$ such that $b \in J$, $g(b) = a$, and $f(g(y),y) = 0$ for all $y \in J$.*

In Part (1) it's understood that I is sufficiently small so that $(x,h(x)) \in U$ for all $x \in I$, and similarly in Part (2) J is such that $(g(y),y) \in U$ for all $y \in J$. The theorem is so named because it provides a means to determine when an equation of the form $f(x,y) = 0$ implicitly defines either y as a function of x or x as a function of y.

Example 14.44. Consider the relation R defined by the equation

$$x^3 + y^3 = 2xy.$$

On what interval for x can we expect R to implicitly define y as a function of x? To help clarify matters let f be the function given by

$$f(x,y) = x^3 + y^3 - 2xy,$$

and note that R can now be expressed as $f(x,y) = 0$. Certainly f has continuous first partials on $\mathbb{R}^2$, with $f_y(x,y) = 3y^2 - 2x$ in particular. Now, since

$$f(1,1) = 1^3 + 1^3 - 2(1)(1) = 0$$

and

$$f_y(1,1) = 3(1)^2 - 2(1) = 1 \neq 0,$$

the Implicit Function Theorem implies there is an open interval $I \subseteq \mathbb{R}$ containing 1, and a continuously differentiable function $h : I \to \mathbb{R}$, such that $h(1) = 1$ and $f(x, h(x)) = 0$ for all $x \in I$. That is, setting $y = h(x)$ will satisfy the equation $f(x, y) = 0$ for all $x \in I$, and therefore R implicitly defines y as a function x in a neighborhood of $x = 1$. ∎

Proposition 14.45. *Suppose f has continuous first partials on open set $U \subseteq \mathbb{R}^2$, and let $(a, b) \in U$.*

1. *If $f_y(a, b) \neq 0$, then there exists open interval I containing a and continuously differentiable function $h : I \to \mathbb{R}$ such that, for all $x \in I$,*
$$h'(x) = -\frac{f_x(x, h(x))}{f_y(x, h(x))}.$$
2. *If $f_x(a, b) \neq 0$, then there exists open interval J containing b and continuously differentiable function $g : J \to \mathbb{R}$ such that, for all $y \in J$,*
$$g'(y) = -\frac{f_y(g(y), y)}{f_x(g(y), y)}.$$

Proof. We prove Part (1) only, the proof of Part (2) being similar. Suppose $f_y(a, b) \neq 0$. Letting $c = f(a, b) = c$, the function $\varphi = f - c$ is such that $\varphi(a, b) = f(a, b) - c = 0$. Moreover φ has continuous first partials on U since it differs from f by a mere constant, and $\varphi_y(a, b) = f_y(a, b) \neq 0$. By the Implicit Function Theorem there is an open interval $I \subseteq \mathbb{R}$ and a continuously differentiable function $h : I \to \mathbb{R}$ such that $a \in I$, $h(a) = b$, and

$$\varphi(x, h(x)) = f(x, h(x)) - c = 0$$

for all $x \in I$. Hence $f(x, h(x)) = c$ for all $x \in I$.

Now, let $g : I \to \mathbb{R}$ be given by $g(x) = x$, and define $F : I \to \mathbb{R}$ by

$$F(x) = f(x, y) = f(g(x), h(x)),$$

for all $x \in I$. Since g and h are differentiable on I and f is differentiable at $(x, h(x)) \in U$ for every $x \in I$, Chain Rule 1 gives

$$F'(x) = f_x(x, y)g'(x) + f_y(x, y)h'(x) = f_x(x, y) + f_y(x, y)h'(x)$$

for all $x \in I$. On the other hand we also have

$$F(x) = f(x, h(x)) = c$$

for all $x \in I$, so that $F'(x) = 0$. Therefore

$$f_x(x, y) + f_y(x, y)h'(x) = 0 \tag{14.21}$$

for $x \in I$.

Since

$$f_y(a, h(a)) = f_y(a, b) \neq 0,$$

f_y is continuous at (a,b), and h is continuous at a, we can assume that I is sufficiently small so that
$$f_y(x,y) = f_y(x,h(x)) \neq 0$$
for all $x \in I$. Therefore (14.21) implies that
$$h'(x) = -\frac{f_x(x,y)}{f_y(x,y)}$$
for all $x \in I$, whereupon substitution of $h(x)$ for y gives the desired result. ■

The conclusion of Proposition 14.45(1) is often abbreviated as
$$y' = -\frac{f_x}{f_y} \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y},$$
where the fact that $y = h(x)$ for $x \in I$ allows for denoting $h'(x)$ by y' or dy/dx.

Example 14.46. Find the slope of the curve C given by
$$y^5 + 3x^2y^2 = 12 - 5x^4$$
at the point $(0, 12^{1/5})$. Also express y' as a function of x and y.

Solution. Let
$$f(x,y) = y^5 + 3x^2y^2 + 5x^4 - 12,$$
so $f(0,12^{1/5}) = 0$. We have
$$f_x(x,y) = 6xy^2 + 20x^3 \quad \text{and} \quad f_y(x,y) = 5y^4 + 6x^2y,$$
and in particular $f_y(0,12^{1/5}) \neq 0$. By Proposition 14.45(1) there is an open interval I containing 0 and a continuously differentiable function $h : I \to \mathbb{R}$ such that, for all $x \in I$,
$$h'(x) = -\frac{f_x(x,h(x))}{f_y(x,h(x))}. \tag{14.22}$$
According to the Implicit Function Theorem $f(x,h(x)) = 0$ on I, so the graph of $y = h(x)$ for $x \in I$ will be a small piece of the graph of the curve C given by $f(x,y) = 0$. Since $0 \in I$ and $h(0) = 12^{1/5}$, this small piece contains the point $(0,12^{1/5})$. It follows that the slope of $y = h(x)$ at $x = 0$, which is of course $h'(0)$, will equal the slope of C at $(0,12^{1/5})$. We have
$$h'(0) = -\frac{f_x(0,h(0))}{f_y(0,h(0))} = -\frac{f_x(0,12^{1/5})}{f_y(0,12^{1/5})} = -\frac{0}{5(12^{4/5})} = 0;$$
that is, the slope of C at $(0,12^{1/5})$ is 0.

Finally, since $y = h(x)$ and $y' = h'(x)$ for $x \in I$, we obtain
$$y' = -\frac{f_x(x,y)}{f_y(x,y)} = -\frac{6xy^2 + 20x^3}{5y^4 + 6x^2y}$$
from (14.22). ■

14.5 – Directional Derivatives

The partial derivatives of a real-valued multivariable function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ at a point $(a, b) \in D$ give information about the rate of change of the value of $f(x, y)$ as one variable is held fixed and the other is allowed to vary. Thus, we are considering the variation of $f(x, b)$ as x varies about a and y is fixed at b, or the variation of $f(a, y)$ as y varies about b and x is fixed at a. These are movements that run parallel to either the x-axis or the y-axis. What of the variation of $f(x, y)$ that occurs by varying (x, y) about (a, b) along some straight line that runs parallel to, say, $y = x$ or $y = -2x$? For that there is the directional derivative.

Definition 14.47. *Let $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ be a unit vector, and $\mathbf{x} \in \mathbb{R}^n$ an interior point of* $\text{Dom}(f)$*. The **directional derivative** of f in the direction $\mathbf{u}$ at $\mathbf{x}$ is*

$$\partial_{\mathbf{u}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h}, \tag{14.23}$$

provided the limit exists.

Using Definition 14.5 the limit (14.23) is found to be equivalent to the following statement: For every $\epsilon > 0$ there exists some $\delta > 0$ such that, if $0 < |h| < \delta$ and $\mathbf{x} + h\mathbf{u} \in \text{Dom}(f)$, then

$$\left| \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} - \partial_{\mathbf{u}} f(\mathbf{x}) \right| < \epsilon.$$

In $\mathbb{R}^2$ we write $\mathbf{u} = \langle u_1, u_2 \rangle$ and let $\mathbf{x}$ be (x, y), so that

$$\partial_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h} \tag{14.24}$$

If $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then (14.24) becomes

$$\partial_{\mathbf{i}} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = f_x(x, y) = \partial_x f(x, y),$$

and similarly

$$\partial_{\mathbf{j}} f(x, y) = f_y(x, y) = \partial_y f(x, y).$$

Thus it's seen that the partial derivatives of a function are just special instances of a directional derivative.

The most convenient way to compute a directional derivative in practice is to use the formula provided by the following theorem.

Theorem 14.48. *If f is differentiable at $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u}$ is a unit vector, then*

$$\partial_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$

Proof. Suppose f is differentiable at $\mathbf{x}$ and $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ with $\|\mathbf{u}\| = 1$. By Definition 14.28,

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0,$$

and so if we substitute $h\mathbf{u}$ for $\mathbf{h}$ and observe that $h\mathbf{u} \to \mathbf{0}$ if and only if $h \to 0$, then we obtain

$$\lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot (h\mathbf{u})}{|h|} = 0,$$

where $\|h\mathbf{u}\| = |h|\|\mathbf{u}\| = |h|$. From this we can conclude that

$$\lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{u}) - f(\mathbf{x}) - \nabla f(\mathbf{x})\cdot(h\mathbf{u})}{h} = 0. \tag{14.25}$$

Also we have

$$\lim_{h\to 0} \frac{\nabla f(\mathbf{x})\cdot(h\mathbf{u})}{h} = \lim_{h\to 0} \frac{h\,\nabla f(\mathbf{x})\cdot\mathbf{u}}{h} = \lim_{h\to 0} [\nabla f(\mathbf{x})\cdot\mathbf{u}] = \nabla f(\mathbf{x})\cdot\mathbf{u}. \tag{14.26}$$

Adding the limits (14.25) and (14.26) using the appropriate limit law gives

$$\begin{aligned} 0 + \nabla f(\mathbf{x})\cdot\mathbf{u} &= \lim_{h\to 0} \frac{f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x}) - \nabla f(\mathbf{x})\cdot(h\mathbf{u})}{h} + \lim_{h\to 0} \frac{\nabla f(\mathbf{x})\cdot(h\mathbf{u})}{h} \\ &= \lim_{h\to 0} \left[\frac{f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x}) - \nabla f(\mathbf{x})\cdot(h\mathbf{u})}{h} + \frac{\nabla f(\mathbf{x})\cdot(h\mathbf{u})}{h}\right] \\ &= \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{u}) - f(\mathbf{x})}{h} = \partial_{\mathbf{u}} f(\mathbf{x}), \end{aligned}$$

and therefore $\partial_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x})\cdot\mathbf{u}$. ■

Thus if $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(a,b) \in U$ and $\mathbf{u} = \langle u_1, u_2\rangle$, then

$$\begin{aligned} \partial_{\mathbf{u}} f(a,b) &= \nabla f(a,b)\cdot\mathbf{u} \\ &= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle \\ &= f_x(a,b)u_1 + f_y(a,b)u_2. \end{aligned}$$

This particular formulation will be used in the next example.

Example 14.49. Compute the directional derivative for the function

$$f(x,y) = \frac{x}{x-y}$$

at the point $(4,1)$ in the direction of $\langle -1, 2\rangle$.

Solution. The direction vector $\mathbf{u}$ must be a unit vector, and so

$$\mathbf{u} = \frac{\langle -1,2\rangle}{\|\langle -1,2\rangle\|} = \frac{\langle -1,2\rangle}{\sqrt{5}} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle.$$

Now we obtain expressions for $f_x(x,y)$ and $f_y(x,y)$:

$$f_x(x,y) = \frac{(x-y)\cdot\partial_x(x) - x\cdot\partial_x(x-y)}{(x-y)^2} = \frac{(x-y)(1) - x(1-0)}{(x-y)^2} = -\frac{y}{(x-y)^2}$$

and

$$f_y(x,y) = \frac{(x-y)\cdot\partial_y(x) - x\cdot\partial_y(x-y)}{(x-y)^2} = \frac{(x-y)(0) - x(0-1)}{(x-y)^2} = \frac{x}{(x-y)^2}.$$

We now reckon thusly:

$$\partial_{\mathbf{u}} f(4,1) = f_x(4,1)\left(-\frac{1}{\sqrt{5}}\right) + f_y(4,1)\left(\frac{2}{\sqrt{5}}\right) = \left(-\frac{1}{9}\right)\left(-\frac{1}{\sqrt{5}}\right) + \left(\frac{4}{9}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}.$$

No cities on the Moon yet, but we're getting there. ■

Proposition 14.50. *Let f be differentiable at point $\mathbf{x} \in \mathbb{R}^n$.*

1. *If $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then $\partial_{\mathbf{u}} f(\mathbf{x})$ is maximal for*
$$\mathbf{u} = \nabla f(\mathbf{x})/\|\nabla f(\mathbf{x})\|,$$
and the maximum value is $\|\nabla f(\mathbf{x})\|$.
2. *If $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then $\partial_{\mathbf{u}} f(\mathbf{x})$ is minimal for*
$$\mathbf{u} = -\nabla f(\mathbf{x})/\|\nabla f(\mathbf{x})\|,$$
and the minimum value is $-\|\nabla f(\mathbf{x})\|$.
3. *If $\mathbf{u} \perp \nabla f(\mathbf{x})$, then $\partial_{\mathbf{u}} f(\mathbf{x}) = 0$.*

Proof. For the proof of (1), suppose that $\nabla f(\mathbf{x}) \neq \mathbf{0}$. By Theorems 14.48 and 12.7 we have

$$\partial_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u} = \|\nabla f(\mathbf{x})\|\|\mathbf{u}\| \cos(\theta) = \|\nabla f(\mathbf{x})\| \cos(\theta),$$

where $\theta \in [0, \pi]$ is the angle between the vectors $\mathbf{u}$ and $\nabla f(\mathbf{x})$. Thus $\partial_{\mathbf{u}} f(\mathbf{x})$ is maximal when $\theta = 0$, and the maximum value is $\|\nabla f(\mathbf{x})\|$. Now, the fact that $\theta = 0$ implies $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$, meaning $\mathbf{u} = k\nabla f(\mathbf{x})$ for some $k > 0$. From $\|k\nabla f(\mathbf{x})\| = \|\mathbf{u}\| = 1$ we obtain $k = 1/\|\nabla f(\mathbf{x})\|$, and therefore $\partial_{\mathbf{u}} f(\mathbf{x})$ is maximized when $\mathbf{u} = \nabla f(\mathbf{x})/\|\nabla f(\mathbf{x})\|$ as claimed. This is easily verified:

$$\partial_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u} = \nabla f(\mathbf{x}) \cdot \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|} = \frac{1}{\|\nabla f(\mathbf{x})\|}[\nabla f(\mathbf{x}) \cdot \nabla f(\mathbf{x})] = \|\nabla f(\mathbf{x})\|$$

(recall the general property $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$). This proves (1).

For the proof of (2), observe that from $\partial_{\mathbf{u}} f(\mathbf{x}) = \|\nabla f(\mathbf{x})\| \cos(\theta)$ it follows that $\partial_{\mathbf{u}} f(\mathbf{x})$ is minimal when $\theta = \pi$, which implies that $\mathbf{u}$ points in the opposite direction as $\nabla f(\mathbf{x})$. The rest of the argument is similar to the proof of (1).

For the proof of (3), suppose that $\mathbf{u} \perp \nabla f(\mathbf{x})$. By definition this means $\nabla f(\mathbf{x}) \cdot \mathbf{u} = 0$, which immediately implies that $\partial_{\mathbf{u}} f(\mathbf{x}) = 0$. ∎

Example 14.51. Let C be the path of steepest descent on the surface S given by $f(x, y) = y + x^{-1}$, starting at the point $(1, 2, 3) \in S$. Find an equation for the path C_0 that is the projection of C onto the xy-plane.

Solution. By Proposition 14.50(2), at any point $(x, y, f(x, y)) \in S$, the direction of steepest descent is $-\nabla f(x, y) = \langle 1/x^2, -1\rangle$, which is a vector in the xy-plane. So if C_0 is given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$, then for any t the tangent vector to C_0 at the point $(x(t), y(t))$, which is $\mathbf{r}'(t)$, must be in the direction of $\langle 1/x^2(t), -1\rangle$. Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \left\langle \frac{1}{x^2(t)}, -1 \right\rangle,$$

from which we obtain the differential equations $x' = x^{-2}$ and $y' = -1$. The first equation can be solved by the Method of Separation of Variables,[11] which for our present purposes will simply

[11] Fully explained in §2.2 of my Differential Equations manuscript.

be characterized thusly:

$$\frac{dx}{dt} = \frac{1}{x^2} \quad \Rightarrow \quad x^2\,dx = dt \quad \Rightarrow \quad \int x^2\,dx = \int dt \quad \Rightarrow \quad \frac{1}{3}x^3 = t + K_1,$$

or $x(t) = \sqrt[3]{3t + K_1}$ for arbitrary constant K_1. The equation $y' = -1$ easily gives $y(t) = -t + K_2$ for arbitrary K_2.

Since C is given to start at $(1, 2, 3)$, we must have C_0 start at $(1, 2)$; that is,

$$\mathbf{r}(0) = \langle x(0), y(0)\rangle = \langle 1, 2\rangle.$$

From $\sqrt[3]{3(0) + K_1} = x(0) = 1$ we obtain $K_1 = 1$, and from $-0 + K_2 = y(0) = 2$ we obtain $K_2 = 2$. Therefore an equation for C_0 is

$$\mathbf{r}(t) = \left\langle \sqrt[3]{3t + 1},\ 2 - t \right\rangle, \quad t \geq 0.$$

Incidentally, this implies that

$$\boldsymbol{\rho}(t) = \left\langle (3t + 1)^{1/3},\ 2 - t,\ 2 - t + (3t + 1)^{-1/3} \right\rangle$$

for $t \geq 0$ is an equation for C itself. ∎

For what follows, recall that a vector $\mathbf{v}$ is said to be orthogonal to a curve C at a point $\mathbf{x} \in C$ where C is smooth if $\mathbf{v}$ is orthogonal to the tangent line to C at $\mathbf{x}$.

Proposition 14.52. *Suppose f has continuous first partials on an open set $U \subseteq \mathbb{R}^2$. If $(x_0, y_0) \in U$, $\nabla f(x_0, y_0) \neq \mathbf{0}$, and C is the level curve of f at (x_0, y_0), then the following hold.*
1. *C is smooth at (x_0, y_0).*
2. *$\nabla f(x_0, y_0)$ is orthogonal to C at (x_0, y_0).*

Proof.
Proof of Part (1). Suppose that $\nabla f(x_0, y_0) \neq \mathbf{0}$ for some $(x_0, y_0) \in U$, so either $f_x(x_0, y_0) \neq 0$ or $f_y(x_0, y_0) \neq 0$. Suppose that $f_y(x_0, y_0) \neq 0$. By Proposition 14.45(1) there exists an open interval $I \subseteq \mathbb{R}$ containing x_0 and continuously differentiable function $h : I \to \mathbb{R}$ such that, for all $x \in I$,

$$h'(x) = -\frac{f_x(x, h(x))}{f_y(x, h(x))}. \tag{14.27}$$

Let $c = f(x_0, y_0)$, so the curve C that is the level curve of f at (x_0, y_0) is given by $f(x, y) = c$. As shown in the proof of Proposition 14.45(1), the function h is such that $h(x_0) = y_0$ and $f(x, h(x)) = c$ for all $x \in I$, which is to say that C is given by $y = h(x)$ for all $x \in I$. Thus C admits the local parametrization

$$\mathbf{r}(x) = \langle x, h(x)\rangle, \quad x \in I,$$

where $x_0 \in I$ and $\mathbf{r}(x_0) = \langle x_0, y_0\rangle$. Since x and $h(x)$ are differentiable on I, and

$$h'(x) = \langle 1, h'(x)\rangle \neq \mathbf{0}$$

for all $x \in I$, we see that $\mathbf{r}(x)$ is a smooth parametrization of at least a part of C containing (x_0, y_0). Therefore C is smooth at (x_0, y_0).

Proof of Part (2). Next, by (14.27) the slope of the tangent line ℓ to C at the point (x_0, y_0) is

$$h'(x_0) = -\frac{f_x(x_0, h(x_0))}{f_y(x_0, h(x_0))} = -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}.$$

A vector parallel to ℓ is $\langle -f_y(x_0, y_0), f_x(x_0, y_0)\rangle$. Now,

$$\begin{aligned}\nabla f(x_0, y_0) \cdot \langle -f_y(x_0, y_0), f_x(x_0, y_0)\rangle &= \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \langle -f_y(x_0, y_0), f_x(x_0, y_0)\rangle \\ &= -f_x(x_0, y_0)f_y(x_0, y_0) + f_y(x_0, y_0)f_x(x_0, y_0) = 0\end{aligned}$$

shows that $\nabla f(x_0, y_0)$ is orthogonal to ℓ. Therefore $\nabla f(x_0, y_0)$ is orthogonal to C at (x_0, y_0).

A similar proof using Proposition 14.45(2) comes to the same conclusion if we assume that $f_x(x_0, y_0) \neq 0$. ■

The proof turns up a result that is important in its own right: If f has continuous first partials on open set $U \subseteq \mathbb{R}^2$, $(x_0, y_0) \in U$, $f(x_0, y_0) = c$, and $f_y(x_0, y_0) \neq 0$, then the slope of the level curve given by $f(x, y) = c$ at (x_0, y_0) is

$$-\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}.$$

From this the following obtains.

Corollary 14.53. *If f has continuous first partials on some open set in $\mathbb{R}^2$ containing the level curve C given by $f(x, y) = c$, then the slope of C is $-f_x(x, y)/f_y(x, y)$ for all $(x, y) \in C$ for which $f_y(x, y) \neq 0$.*

14.6 – Tangent Spaces and Differentials

Given a function F for which $\mathrm{Dom}(F) \subseteq \mathbb{R}^3$, the set of points (x, y, z) that satisfy the equation $F(x, y, z) = 0$ will quite typically form a surface S in space. It is often the case that, very near a point $\mathbf{x}_0 \in S$, the surface can be fairly well approximated by a plane. For instance, although the Earth is essentially a sphere, to the skipper of a ship in the middle of the ocean it appears to be essentially a plane (see Figure 63). The plane that "best" approximates a surface S at a point $\mathbf{x}_0$ is called the tangent plane to S at $\mathbf{x}_0$.

To be more precise, suppose that S is a surface in $\mathbb{R}^n$ given by $F(\mathbf{x}) = 0$, and $\mathbf{x}_0$ is a point on S. If F is differentiable at $\mathbf{x}_0$ then we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{F(\mathbf{x}) - F(\mathbf{x}_0) - \nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|} = 0,$$

and since $\mathbf{x}_0 \in S$ if and only if $F(\mathbf{x}_0) = 0$, we next obtain

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{F(\mathbf{x}) - \nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|} = 0,$$

which implies that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} |F(\mathbf{x}) - \nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)| = 0. \tag{14.28}$$

That is, the function $\mathbf{x} \mapsto \nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)$ approximates the function $\mathbf{x} \mapsto F(\mathbf{x})$ with decreasing error the closer $\mathbf{x}$ gets to $\mathbf{x}_0$. This in turn means, in particular, that the surface $F(\mathbf{x}) = 0$ is better approximated by $\nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) = 0$ the closer $\mathbf{x}$ gets to $\mathbf{x}_0$. Now, in the $\mathbb{R}^3$ setting, if it happens that $\nabla F(\mathbf{x}_0) \neq \mathbf{0}$, then the equation $\nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) = 0$ is the equation of a conventional plane known as a "tangent plane" (see Definition 13.31). In the general $\mathbb{R}^n$ setting the term "tangent space" is used.

Definition 14.54. *Let S be a surface in $\mathbb{R}^n$ given by $F(\mathbf{x}) = 0$, and suppose $\mathbf{x}_0 \in S$. If F is differentiable at $\mathbf{x}_0$ with $\nabla F(\mathbf{x}_0) \neq \mathbf{0}$, then the* ***tangent space*** *to S at $\mathbf{x}_0$ is the surface $T_{\mathbf{x}_0}$ given by*

$$\nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) = 0, \tag{14.29}$$

We say S is ***smooth*** *at $\mathbf{x}_0 \in S$ if there exists a tangent space to S at $\mathbf{x}_0$.*

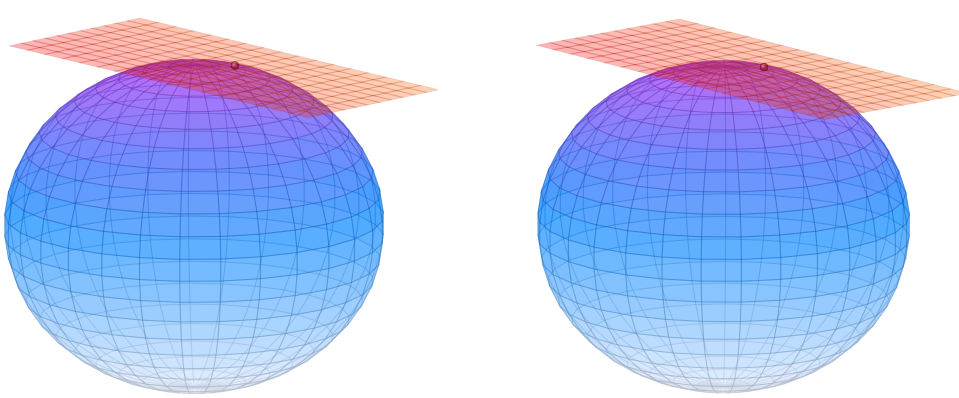


Figure 63. Stereoscopic image of a tangent plane to a surface in $\mathbb{R}^3$.

In $\mathbb{R}^2$ a tangent space may be called a **tangent line** (the very same kind of tangent line of Chapter 3 acquaintance), and as said before, in $\mathbb{R}^3$ a tangent space may be called a **tangent plane**. Any surface in $\mathbb{R}^n$ given by

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$$

for some constant $\mathbf{n}, \mathbf{c} \in \mathbb{R}^n$ with **normal vector** $\mathbf{n} \neq \mathbf{0}$ is known as a **hyperplane**, and so any tangent space in $\mathbb{R}^n$ is in fact a hyperplane.

A vector $\mathbf{v}$ is said to be **orthogonal** (resp. **parallel**) to a hyperplane $\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$ if $\mathbf{v}$ is parallel (resp. orthogonal) to $\mathbf{n}$. Moreover, a vector $\mathbf{v}$ is **orthogonal** (resp. **parallel**) to a surface S at $\mathbf{x}_0 \in S$ if $\mathbf{v}$ is orthogonal (resp. parallel) to the tangent space to S at $\mathbf{x}_0$, provided S is smooth at $\mathbf{x}_0$. Since such a tangent space is a hyperplane with normal vector $\nabla F(\mathbf{x}_0)$, it follows that $\mathbf{v}$ is orthogonal to S at $\mathbf{x}_0$ if $\mathbf{v}$ is parallel to $\nabla F(\mathbf{x}_0)$, and $\mathbf{v}$ is parallel to S at $\mathbf{x}_0$ if $\mathbf{v}$ is orthogonal to $\nabla F(\mathbf{x}_0)$. It can be instructive to compare these definitions with the findings of Proposition 14.52.

Let S be a surface in $\mathbb{R}^3$ given by $F(x, y, z) = 0$ that is smooth at (x_0, y_0, z_0). Then from (14.29) the tangent plane to S at (x_0, y_0, z_0) has equation

$$\big\langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\big\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0,$$

or equivalently

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0. \tag{14.30}$$

Define the function $L : \mathbb{R}^3 \to \mathbb{R}$ by

$$L(x, y, z) = F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0). \tag{14.31}$$

By equation (14.28) and the discussion following it, we find that

$$L(x, y, z) \approx F(x, y, z)$$

for (x, y, z) sufficiently near (x_0, y_0, z_0). More precisely, for any $\epsilon > 0$ there is some $\delta > 0$ such that

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta \quad \Rightarrow \quad |L(x, y, z) - F(x, y, z)| < \epsilon$$

for all $(x, y, z) \in \mathrm{Dom}(F)$. The function L given by (14.31) is called the **linearization** of F at (x_0, y_0, z_0).

The case when a surface S in $\mathbb{R}^3$ is given by $z = f(x, y)$ warrants special attention. First, we can write the equation for S as $z - f(x, y) = 0$, and thus characterize S as being given by $F(x, y, z) = 0$ for

$$F(x, y, z) = z - f(x, y).$$

Then

$$F_x(x_0, y_0, z_0) = -f_x(x_0, y_0), \quad F_y(x_0, y_0, z_0) = -f_y(x_0, y_0), \quad F_z(x_0, y_0, z_0) = 1,$$

which immediately makes clear that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, and so we only need F to be differentiable at (x_0, y_0, z_0) in order for S to be smooth there. *It is a fact that F will be differentiable at*

(x_0, y_0, z_0) *if and only if* f *is differentiable at* (x_0, y_0).[12] Now, observing that

$$(x_0, y_0, z_0) \in S \quad \Leftrightarrow \quad F(x_0, y_0, z_0) = 0 \quad \Leftrightarrow \quad z_0 = f(x_0, y_0),$$

we apply (14.30) to obtain the tangent plane to S at $(x_0, y_0, f(x_0, y_0))$:

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0). \tag{14.32}$$

This is quite similar in form to the equation of a tangent line to a curve give by $y = f(x)$ at the point $(x_0, f(x_0))$, which is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

provided $f'(x_0)$ exists.

Recalling (14.31), the estimate $L(x, y, z) \approx F(x, y, z)$ for (x, y, z) near (x_0, y_0, z_0) is equivalent to

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + z - f(x_0, y_0) \approx z - f(x, y),$$

and hence

$$f(x, y) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0) \tag{14.33}$$

for (x, y) near (x_0, y_0). If we denote the right-hand side of (14.33) by $L(x, y)$, then $f(x, y) \approx L(x, y)$ for (x, y) near (x_0, y_0). We make the following definition.

Definition 14.55. *If* $z = f(x, y)$ *is differentiable at* (x_0, y_0), *then the* ***linearization*** *of* f *at* (x_0, y_0) *is*

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0), \tag{14.34}$$

The next example illustrates how a tangent plane can be used as an approximation to a surface in space. In general the error of the approximation increases the further from the point of tangency one wanders.

[12]This would be a worthy exercise for the reader.

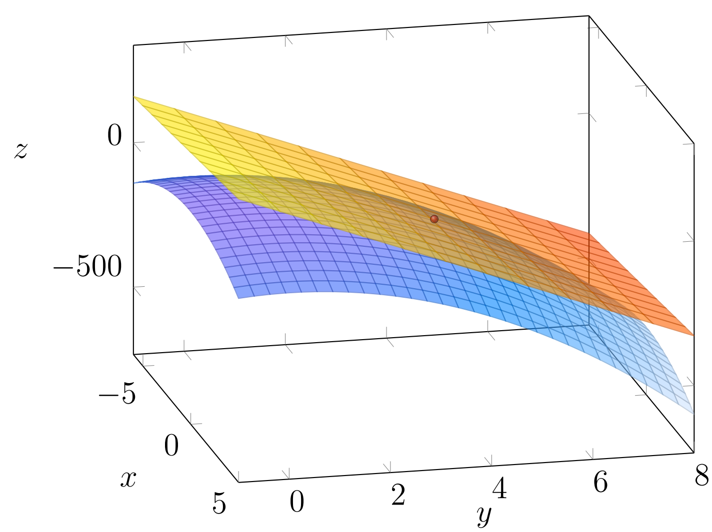


FIGURE 64.

Example 14.56. Consider the surface S given by

$$f(x,y) = 12 - 4x^2 - 8y^2.$$

(a) Find the tangent plane to S at $(-1,4,-120)$.
(b) Find the linearization of f at $(-1,4)$.
(c) Use the linearization L to approximate $f(-1.05, 3.95)$.

Solution.
(a) We can use equation (14.32) with $a = -1$ and $b = 4$. From $f_x(x,y) = -8x$ we get $f_x(-1,4) = 8$, and from $f_y(x,y) = -16y$ we get $f_y(-1,4) = -64$; then, since $f(-1,4) = -120$, the equation of the tangent plane is

$$z = 8(x+1) - 64(y-4) - 120,$$

or simply $z = 8x - 64y + 144$.

(b) Comparing equation (14.32) with equation (14.34), we can see that the work needed to find the linearization has already been done in part (a). We have

$$L(x,y) = 8x - 64y + 144.$$

(c) The function $L(x,y) = 8x - 64y + 144$ is a linear approximation of f for points (x,y) near $(-1,4)$. So, in particular we have

$$f(-1.05, 3.95) \approx L(-1.05, 3.95) = 8(-1.05) - 64(3.95) + 144 = -117.20,$$

which is quite close to the actual value $f(-1.05, 3.95) = -117.23$. See Figure 64 for an illustration of the surface S and the tangent plane to S at $(-1,4,-120)$. ■

If a surface S in $\mathbb{R}^3$ is given by $F(x,y,z) = 0$, then a tangent plane to S at the point $\mathbf{x}_0 = (x_0, y_0, z_0)$ is said to be **horizontal** if it has an equation of the form $z = d$ for some constant d. In general this is the case if and only if the normal vector $\mathbf{n}$ for the plane is $\langle 0,0,c\rangle$ for some $c \neq 0$, and since $\mathbf{n}$ may be taken to be $\nabla F(\mathbf{x}_0)$ in the present context, we see that a tangent plane to S at $\mathbf{x}_0$ is horizontal if and only if $\nabla F(\mathbf{x}_0) = \langle 0,0,c\rangle$ for some $c \neq 0$.

Example 14.57. We look for points on the surface S given by

$$x^2 + y^2 - z^2 - 2x + 2y + 3 = 0$$

where the tangent plane is horizontal. These will be points $(x,y,z) \in S$ such that $\nabla F(x,y,z) = \langle 0,0,c\rangle$ for some $c \neq 0$. Here

$$F(x,y,z) = x^2 + y^2 - z^2 - 2x + 2y + 3,$$

so $F_x(x,y,z) = 2x - 2$, $F_y(x,y,z) = 2y + 2$, and $F_z(x,y,z) = -2z$, and hence

$$\nabla F(x,y,z) = \langle 2x-2, 2y+2, -2z\rangle.$$

Now,

$$\nabla F(x,y,z) = \langle 0,0,c\rangle \quad \Rightarrow \quad \langle 2x-2, 2y+2, -2z\rangle = \langle 0,0,c\rangle,$$

which requires $x = 1$, $y = -1$, and $z = -c/2$. On the other hand $(1, -1, z) \in S$ requires

$$1^2 + (-1)^2 - z^2 - 2(1) + 2(-1) + 3 = 0,$$

giving $z = \pm 1$. Since $z = 1$ and $z = -1$ both result in $c \neq 0$ in $z = -c/2$, we find that there are precisely two points where S has a horizontal plane: $(1, -1, 1)$ and $(1, -1, -1)$. ■

Again consider a surface S given by $z = f(x, y)$, smooth at some point $(x_0, y_0, f(x_0, y_0))$. (As we found earlier, $z = f(x, y)$ is smooth at $(x_0, y_0, f(x_0, y_0))$ if and only if f is differentiable at (x_0, y_0).) Suppose $(x, y) \in \mathrm{Dom}(f)$. Letting $\Delta x = x - x_0$ and $\Delta y = y - y_0$, so that

$$(x, y) = (x_0 + \Delta x, y_0 + \Delta y),$$

we define Δf to be the change in the value of f in going from (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$; that is, we define

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

Since $z = f(x, y)$, a change in f corresponds to an identical change in z, and so we also define $\Delta z = \Delta f$.

Next, define the **total differential** (or simply the **differential**) of f at (x_0, y_0), denoted by df, to be the change in the value of the *linearization* L of f at (x_0, y_0) in going from (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$:

$$df = L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0).$$

Note that in our notation Δx and Δy denote a change in the value of x and y, respectively, just as Δz signifies change in z. However, in the current setting x and y are the independent variables, so we quite reasonably define the differential of x at x_0 to be $dx = \Delta x$, and the differential of y at y_0 to be $dy = \Delta y$. Thus

$$df = L(x_0 + dx, y_0 + dx) - L(x_0, y_0).$$

Now, from (14.34) we have $L(x_0, y_0) = f(x_0, y_0)$ and

$$L(x_0 + dx, y_0 + dx) = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + f(x_0, y_0),$$

and therefore

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{14.35}$$

If x_0 and y_0 are fixed, then df is a function only of dx and dy. Alternatively we could write

$$df = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and see df as a function of x and y, but (14.35) is nearer to the conventional notation in the sciences since dx and dy are often thought of as very small quantities or "infinitesimals." Such infinitesimals are generic, which is to say they are not regarded as equalling any particular real numbers, which is why the proper function notation $df(dx, dy)$ is not used in (14.35).

If the point (x_0, y_0) is allowed to vary, then more care with the notation must be taken. After all, the linearization L of f at one point is usually not the same as at another point. Letting $p = (x_0, y_0)$, we may let the symbol df_p denote the differential of f at p:

$$df_p = f_x(p)dx + f_y(p)dy.$$

Finally, if no point p is specified (or if a particular point is taken to be understood), then we write simply

$$df = f_x dx + f_y dy \quad \text{or} \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \tag{14.36}$$

both being common notations.

We now extend the idea of linearizations and differentials of functions to a more general setting.

Definition 14.58. *If $y = f(\mathbf{x})$ is a real-valued function that is differentiable at*

$$\mathbf{x}_0 = \langle x_{01}, \ldots, x_{0n} \rangle \in \mathbb{R}^n,$$

*then the **linearization** of f at $\mathbf{x}_0$ is*

$$L(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} f_{x_i}(\mathbf{x}_0)(x_i - x_{0i}), \tag{14.37}$$

*and the **total differential** (or simply the **differential**) of f at $\mathbf{x}_0$ is*

$$df_{\mathbf{x}_0} = L(\mathbf{x}) - L(\mathbf{x}_0) = \sum_{i=1}^{n} f_{x_i}(\mathbf{x}_0)dx_i, \tag{14.38}$$

where $dx_i = x_i - x_{0i}$ for each $1 \leq i \leq n$.

As we also saw in §4.6, differentials have properties that are formally the same as those of derivatives. For instance, for functions f and g, the differential of the product fg is

$$\begin{aligned}
d(fg)_{\mathbf{x}_0} &= \sum_{i=1}^{n}(fg)_{x_i}(\mathbf{x}_0)dx_i = \sum_{i=1}^{n}(f_{x_i}g + fg_{x_i})(\mathbf{x}_0)dx_i \\
&= \sum_{i=1}^{n}\big(f_{x_i}(\mathbf{x}_0)g(\mathbf{x}_0) + f(\mathbf{x}_0)g_{x_i}(\mathbf{x}_0)\big)dx_i \\
&= g(\mathbf{x}_0)\sum_{i=1}^{n} f_{x_i}(\mathbf{x}_0)dx_i + f(\mathbf{x}_0)\sum_{i=1}^{n} g_{x_i}(\mathbf{x}_0)dx_i \\
&= g(\mathbf{x}_0)df_{\mathbf{x}_0} + f(\mathbf{x}_0)dg_{\mathbf{x}_0},
\end{aligned}$$

using the product rule for partial differentiation.

Strictly speaking $df_{\mathbf{x}_0}$ in Definition 14.58 is a function of the n independent variables $dx_1, \ldots, dx_n$, but again, in the sciences the dx_i are taken to be generic infinitesimals: tiny changes in the coordinates of $\mathbf{x}_0$ to arrive at another point $\mathbf{x}$ nearby. This means two things: first, real numbers are not substituted for $dx_1, \ldots, dx_n$, so the role of the dx_i as "inputs" for the function $df_{\mathbf{x}_0}$ is largely lost; and second, the fact that each dx_i is technically a subtraction operation involving the ith coordinate of $\mathbf{x}_0$ is of little relevance. So, while (14.38) defines a proper mathematical function, the notation used more reflects its practical role as a tool for physicists and other specialists.

If we define $d\mathbf{x} = \langle dx_1, \ldots, dx_n \rangle$, which is to say $d\mathbf{x} = \mathbf{x} - \mathbf{x}_0$, then equation (14.38) may be written as

$$df_{\mathbf{x}_0} = \nabla f(\mathbf{x}_0) \cdot d\mathbf{x}.$$

Stripping away from this any explicit mention of a linearization point $\mathbf{x}_0$ leads to the form

$$df = \nabla f \cdot d\mathbf{x},$$

which is serviceable if context makes clear what the linearization point is supposed to be, or it's desired that no particular linearization point be specified. Finally, it's common practice to write something like

$$df = \nabla f(\mathbf{x}) \cdot d\mathbf{x}, \tag{14.39}$$

which certainly indicates that $\mathbf{x}$ is the linearization point, but in this case $d\mathbf{x}$ must be formulated somewhat differently as a vector whose coordinates are the incremental changes in the corresponding coordinates of $\mathbf{x}$. Thus we may write, say,

$$d\mathbf{x} = (\mathbf{x} + \Delta\mathbf{x}) - \mathbf{x} = \Delta\mathbf{x}.$$

We have seen this kind of treatment before: in the single-variable setting of §4.6 in which $y = f(x)$, the resultant differential formula was given as

$$dy = f'(x)\,dx,$$

which is formally the same as (14.39) in that x is the linearization point and $dx = \Delta x$.

14.7 – Multivariable Optimization

We start by assuming f to be a function of n independent variables for some $n \geq 2$, although the first definition is "backward-compatible" to functions of a single variable (see Chapter 4).

Definition 14.59. *For a function $f : D \to \mathbb{R}$ with domain $D \subseteq \mathbb{R}^n$, let $\mathbf{c} \in D$. If there exists an open set U containing $\mathbf{c}$ such that $f(\mathbf{x}) \leq f(\mathbf{c})$ for all $\mathbf{x} \in D \cap U$, then f has a **local maximum** at $\mathbf{c}$. If there exists an open set U containing $\mathbf{c}$ such that $f(\mathbf{x}) \geq f(\mathbf{c})$ for all $\mathbf{x} \in D \cap U$, then f has a **local minimum** at $\mathbf{c}$.*

*If $f(\mathbf{x}) \leq f(\mathbf{c})$ for all $\mathbf{x} \in D$, then f has a **global maximum** at $\mathbf{c}$. If $f(\mathbf{x}) \geq f(\mathbf{c})$ for all $\mathbf{x} \in D$, then f has a **global minimum** at $\mathbf{c}$.*

Proposition 14.60. *If f has a local extremum at $\mathbf{c}$ and $\nabla f(\mathbf{c})$ exists, then $\nabla f(\mathbf{c}) = \mathbf{0}$.*

The proof of this proposition will come later, but for now observe that by Definition 14.23, in order for $\nabla f(\mathbf{c})$ to exist, it must be that $\mathbf{c}$ is an interior point of $\mathrm{Dom}(f)$ and all first-order partial derivatives of f exist at $\mathbf{c}$.

Definition 14.61. *A point $\mathbf{c}$ in the interior of $\mathrm{Dom}(f)$ is a **critical point** of f if $\nabla f(\mathbf{c}) = \mathbf{0}$ or $\nabla f(\mathbf{c})$ does not exist.*

Definition 14.62. *A critical point $\mathbf{c}$ of f is a **saddle point** if, for every open set U containing $\mathbf{c}$, there are points $\mathbf{x}, \mathbf{y} \in U$ such that $f(\mathbf{x}) > f(\mathbf{c})$ and $f(\mathbf{y}) < f(\mathbf{c})$.*

Henceforth the focus in this section will be exclusively on functions of two independent variables x and y. Thus all functions will have domains in $\mathbb{R}^2$, and any point $\mathbf{x}$ will be understood to have coordinates (x, y), with $\mathbf{c}$ becoming (a, b) in particular. Then $\nabla f(a, b) = \mathbf{0}$ if and only if $f_x(a, b) = f_y(a, b) = 0$, and $\nabla f(a, b)$ does not exist if and only if either $f_x(a, b)$ or $f_y(a, b)$ does not exist.

We come now to the *pièce de résistance* of this section, a theorem whose proof is to be included later when time allows. First, given a function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, we define the **discriminant function** Φ by

$$\Phi(x, y) = f_{xx}(x, y) f_{yy}(x, y) - f_{xy}^2(x, y),$$

which figures prominently in the following.

Theorem 14.63 (Second Derivative Test). *Suppose the second-order partial derivatives of f are continuous on an open set U containing (a, b), and $f_x(a, b) = f_y(a, b) = 0$.*

1. *If $\Phi(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).*
2. *If $\Phi(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).*
3. *If $\Phi(a, b) < 0$, then f has a saddle point at (a, b).*

If $\Phi(a, b) = 0$ the test is inconclusive, in which case one should try the Method of Lagrange Multipliers in the next section.

Example 14.64. Use the Second Derivative Test to find the local extrema and saddle points, if any, of

$$f(x,y) = x^2 - x^4/2 - y^2 - xy.$$

Solution. First we obtain the first partials of f:

$$f_x(x,y) = 2x - 2x^3 - y \quad \text{and} \quad f_y(x,y) = -2y - x.$$

We now search for critical points. Since the first partials exist on all of $\mathbb{R}^2$, we look for points (x,y) where $f_x(x,y) = f_y(x,y) = 0$. This gives a system of equations

$$\begin{cases} 2x - 2x^3 - y = 0 \\ \quad - 2y \quad - x = 0 \end{cases}$$

Putting $y = -x/2$ from the second equation into the first yields $2x - 2x^3 + x/2 = 0$, which solves to give $x = 0, \pm\sqrt{5}/2$. Thus we obtain three critical points:

$$p_0 = (0,0), \quad p_1 = \left(\sqrt{5}/2, -\sqrt{5}/4\right), \quad \text{and} \quad p_2 = \left(-\sqrt{5}/2, \sqrt{5}/4\right).$$

We now obtain the second partials of f:

$$f_{xx}(x,y) = 2 - 6x^2, \quad f_{yy}(x,y) = -2, \quad \text{and} \quad f_{xy}(x,y) = -1.$$

From these we find that $\Phi(x,y) = 12x^2 - 5$. Calculating the values of the second partials and Φ at p_0, p_1, and p_2 gives:

	f_{xx}	f_{yy}	f_{xy}	Φ
p_0	2	-2	-1	-5
p_1	-5.5	-2	-1	10
p_2	-5.5	-2	-1	10

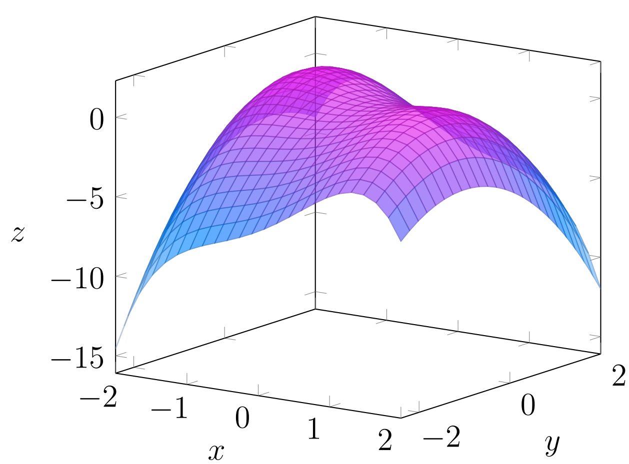


Figure 65. The surface $z = x^2 - x^4/2 - y^2 - xy$.

Since $\Phi(p_0) < 0$, we conclude that f has a saddle point at $(0,0)$. Since $\Phi(p_1), \Phi(p_2) > 0$ and $f_{xx}(p_1), f_{xx}(p_2) < 0$, we conclude that f has local maximums at $\left(\sqrt{5}/2, -\sqrt{5}/4\right)$ and $\left(-\sqrt{5}/2, \sqrt{5}/4\right)$. Drink in Figure 65 to see the lay of the land. ■

We turn now to the problem of finding the global extrema of a continuous function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ when D is a closed, bounded set. The procedure is as follows:

- Find all critical points for f that lie in D.
- Find the points where $f|_{\partial D}$ (i.e. f restricted to the boundary of D) may have an extremum.
- Evaluate f at all the points found above. The greatest value of f found is the global maximum value of f on D, and the smallest value is the global minimum value.

The second step may require use of the Closed Interval Method of Chapter 4 fame if restriction of f to some part of ∂D effectively reduces f to a function of a single variable.

Example 14.65. Find the global extrema of the function

$$f(x,y) = 3x^2 + y^2 - 3x - 2y + 4$$

restricted to the rectangle $R = \{(x,y) : -1 \le x \le 1, -1 \le y \le 2\}$.

Solution. First we obtain all necessary partial derivatives of f: $f_x(x,y) = 6x - 3$, $f_y(x,y) = 2y - 2$, $f_{xx}(x,y) = 6$, $f_{yy}(x,y) = 2$, and $f_{xy}(x,y) = 0$. Also we have $\Phi(x,y) = 12$.

Now we find all points (x,y) for which $f_x(x,y) = f_y(x,y) = 0$, which entails finding points that satisfy $6x - 3 = 0$ and $2y - 2 = 0$ simultaneously. This easily leads to the single critical point $(1/2, 1)$.

We now search for all the points on ∂D where $f|_{\partial D}$ may have an extremum. We do this by analyzing f on each side of the rectangle, one at a time.

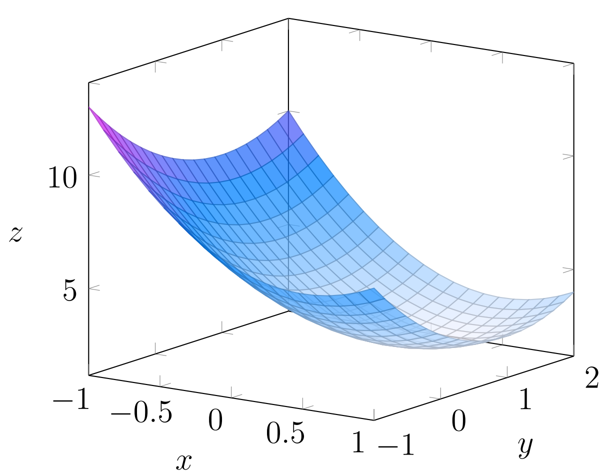


FIGURE 66. The surface $z = 3x^2 + y^2 - 3x - 2y + 4$.

On the top side of the rectangle R we have $y = 2$. Restricted to this line segment, the function f is given by

$$f(x, 2) = 3x^2 + 2^2 - 3x - 2(2) + 4 = 3x^2 - 3x + 4 = 3\left(x - \tfrac{1}{2}\right)^2 + \tfrac{13}{4}$$

for $-1 \leq x \leq 1$. The expression on the right should make it clear that $f(\,\cdot\,, 2)$ must have a minimum at $x = 1/2$ and a maximum at $x = -1$ (use the Closed Interval Method here, if necessary), which corresponds to points $(1/2, 2)$ and $(-1, 2)$.

Next, we consider f restricted to the bottom side of R, where $y = -1$. Here f is given by

$$f(x, -1) = 3x^2 + (-1)^2 - 3x - 2(-1) + 4 = 3x^2 - 3x + 7 = 3\left(x - \tfrac{1}{2}\right)^2 + \tfrac{25}{4}$$

for $-1 \leq x \leq 1$. From the expression on the right it's seen that $f(\,\cdot\,, -1)$ has a minimum at $x = 1/2$ and a maximum at $x = -1$, corresponding to points $(1/2, -1)$ and $(-1, -1)$.

On the left side of R where $x = -1$, f is given by

$$f(-1, y) = 3(-1)^2 + y^2 - 3(-1) - 2y + 4 = y^2 - 2y + 10 = (y - 1)^2 + 9$$

for $-1 \leq y \leq 2$. From this it's seen that $f(-1, \cdot\,)$ has a minimum at $y = 1$ and a maximum at $y = -1$, corresponding to points $(-1, 1)$ and $(-1, -1)$.

On the right side of R where $x = 1$, f is given by

$$f(1, y) = 3(1)^2 + y^2 - 3(1) - 2y + 4 = y^2 - 2y + 4 = (y - 1)^2 + 3$$

for $-1 \leq y \leq 2$. We find that $f(1, \cdot\,)$ has a minimum at $y = 1$ and a maximum at $y = -1$, corresponding to points $(1, 1)$ and $(1, -1)$.

Finally we evaluate f at all the points found above. We have $f(1/2, 1) = 2.25$, $f(1/2, 2) = 3.25$, $f(-1, 2) = 10$, $f(1/2, -1) = 6.25$, $f(-1, -1) = 13$, $f(-1, 1) = 9$, $f(1, 1) = 3$, and $f(1, -1) = 7$.

Therefore f has a global minimum on R at $(1/2, 1)$ which equals 2.25, and a global maximum on R at $(-1, -1)$ which equals 13. Gaze wonderingly at Figure 66. ∎

14.8 – Lagrange Multipliers

For what follows, recall that a vector $\mathbf{v}$ is said to be orthogonal to a curve $C \subseteq \mathbb{R}^n$ at a point $\mathbf{a} \in C$ where C is smooth if $\mathbf{v}$ is orthogonal to the tangent line to C at $\mathbf{a}$. By definition the tangent line to C at $\mathbf{a}$ is the line containing $\mathbf{a}$ that is parallel to the tangent vector to C at $\mathbf{a}$ given by any smooth parametrization of C. Thus, if $\mathbf{r}(t)$ is such a smooth parametrization, and $\mathbf{r}(t_0) = \mathbf{a}$, then the set

$$\{\mathbf{a} + t\mathbf{r}'(\mathbf{a}) : t \in \mathbb{R}\}$$

is the tangent line to C at $\mathbf{a}$.

Proposition 14.66. *Let C be a smooth curve in $\mathbb{R}^2$ such that $C = \{(x,y) : g(x,y) = 0\}$ for some function g, let U be an open set containing C, and let $f : U \to \mathbb{R}$ be differentiable on U. If $f|_C$ has a local extremum at $(a,b) \in C$, then the following hold.*

1. *$\nabla f(a,b)$ is orthogonal to C at (a,b).*
2. *If g has continuous first partials in a neighborhood of (a,b) and $\nabla g(a,b) \neq \mathbf{0}$, then there exists some $\lambda \in \mathbb{R}$ such that $\nabla f(a,b) = \lambda \nabla g(a,b)$.*

Proof.
Proof of Part (1). Let

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad t \in I,$$

be a smooth parametrization for C, so there exists some t_0 in the interval I such that $\mathbf{r}(t_0) = (a,b)$. With this parametrization the tangent vector to C at (a,b) is $\mathbf{r}'(t_0)$, and so to show that $\nabla f(a,b)$ is orthogonal to C at (a,b) it will suffice to show that

$$\nabla f(a,b) \cdot \mathbf{r}'(t_0) = 0.$$

By hypothesis $f|_C$ has an extremum at (a,b), and since $(a,b) = \mathbf{r}(t_0)$, it follows that $f|_C \circ \mathbf{r} : I \to \mathbb{R}$ has an extremum at $t_0 \in I$. Observing that $\mathbf{r}(I) = C$, we have $f \circ \mathbf{r} = f|_C \circ \mathbf{r}$, and hence $f \circ \mathbf{r} : I \to \mathbb{R}$ has an extremum at t_0. Now, since $\mathbf{r}$ is differentiable at t_0 and f is differentiable at $\mathbf{r}(t_0)$, by the Chain Rule (Theorem 14.38) we find that $f \circ \mathbf{r}$ is differntiable at t_0, with

$$(f \circ \mathbf{r})'(t_0) = \nabla f(\mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0).$$

On the other hand, since $f \circ \mathbf{r}$ has a local extremum at t_0 and $(f \circ \mathbf{r})'(t_0)$ exists, Fermat's Theorem (Theorem 4.5) implies that $(f \circ \mathbf{r})'(t_0) = 0$. Hence

$$\nabla f(a,b) \cdot \mathbf{r}'(t_0) = \nabla f(\mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = (f \circ \mathbf{r})'(t_0) = 0,$$

as desired.

Proof of Part (2). Suppose g has continuous first partials in a neighborhood of (a,b) and $\nabla g(a,b) \neq \mathbf{0}$. Then by Proposition 14.52 $\nabla g(a,b)$ is orthogonal to the level curve of g at (a,b), which is precisely the curve C, and so $\nabla g(a,b)$ is orthogonal to the tangent line to C at (a,b). Since $\nabla f(a,b)$ is orthogonal to the same line by part (1), it follows that $\nabla f(a,b)$ and $\nabla g(a,b)$ are parallel vectors if $\nabla f(a,b) \neq \mathbf{0}$. Thus there exists some scalar $\lambda \in \mathbb{R}$ such that $\nabla f(a,b) = \lambda \nabla g(a,b)$, with $\lambda = 0$ if $\nabla f(a,b) = \mathbf{0}$. ■

The function f in Proposition 14.66 is the **objective function**, and the function g is the **constraint function**. The proposition offers a means whereby local extrema of the objective function may be found subject to the constraint $g(x, y) = 0$.

Theorem 14.67 (**Method of Lagrange Multipliers in Two Variables**). *Let f be differentiable and g have continuous first partials on a open set $U \subseteq \mathbb{R}^2$ containing a closed level curve C given by $g(x, y) = 0$, and let $\nabla g(x, y) \neq \mathbf{0}$ for all $(x, y) \in C$. If*

$$S = \{(x, y) \in C : \nabla f(x, y) = \lambda \nabla g(x, y) \text{ for some } \lambda \in \mathbb{R}\},$$

then

$$\max_{(x,y)\in C} f(x, y) = \max_{(x,y)\in S} f(x, y) \quad \text{and} \quad \min_{(x,y)\in C} f(x, y) = \min_{(x,y)\in S} f(x, y).$$

Proof. Proposition 14.52 ensures that C is a smooth curve, so let

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle, \quad t \in I,$$

be any smooth parametrization for C. The functions $\mathbf{r} : I \to C$ and $f : C \to \mathbb{R}$ are continuous, and so $f \circ \mathbf{r} : I \to \mathbb{R}$ is also continuous. Since C is a closed curve, the interval $I \subseteq \mathbb{R}$ may be taken to be a closed, bounded interval, and hence by the Extreme Value Theorem $f \circ \mathbf{r} : I \to \mathbb{R}$ attains a global maximum value and a global minimum value. From this observation it follows that f attains a maximum value and a minimum value on $\mathbf{r}(I) = C$.

Now, the global extrema of $f|_C$ are of necessity local extrema of $f|_C$, and so the global maximum value of $f|_C$ must be the greatest local maximum value, and the global minimum value of $f|_C$ must be the least local minimum value. Suppose $f|_C$ has a local extremum at $(a, b) \in C$. Then by Proposition 14.66 there exists some $\lambda \in \mathbb{R}$ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$, and we conclude that $(a, b) \in S$. Thus S contains all the points on C where $f|_C$ has a local extremum. Therefore the maximum value of f on C will equal the maximum value of f on S, and the minimum value of f on C will equal the minimum value of f on S. ∎

In the definition of the set S above, observe that $(x, y) \in C$ if and only if $g(x, y) = 0$. Recalling the definition of the gradient operator ∇, the Method of Lagrange Multipliers in essence states that the extreme values of the objective function f restricted to the **constraint curve** $g(x, y) = 0$ will lie at those points $(x, y) \in \mathbb{R}^2$ for which there can be found some $\lambda \in \mathbb{R}$ such that (x, y, λ) is a solution to the system of equations

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = 0 \end{cases} \tag{14.40}$$

Some examples should help clarify the Lagrange Multiplier procedure, and also show how the procedure may be applied to more general situations.

Example 14.68. Find the maximum and minimum values of

$$f(x, y) = x^2 y + 10$$

subject to the constraint $x^2 + 2y^2 = 6$.

Solution. Defining
$$g(x, y) = x^2 + 2y^2 - 6,$$
we may write the constraint curve C given by $x^2 + 2y^2 = 6$ as $g(x, y) = 0$. Since
$$\nabla g(x, y) = \langle g_x(x, y), g_y(x, y)\rangle = \langle 2x, 4y\rangle,$$
we have $\nabla g(x, y) = \mathbf{0}$ if and only if $(x, y) = (0, 0)$, but $(0, 0)$ does not lie on the curve $g(x, y) = 0$ (that is, $g(0, 0) \neq 0$), and so $\nabla g(x, y) \neq \mathbf{0}$ for all $(x, y) \in C$. Since the polynomial function f is differentiable everywhere, and g has continuous first partials everywhere, all the hypotheses of Theorem 14.67 are satisfied. The system (14.40) here becomes
$$\begin{cases} 2xy = 2\lambda x \\ x^2 = 4\lambda y \\ x^2 + 2y^2 = 6 \end{cases} \tag{14.41}$$
The first equation in (14.41) offers two possibilities: either $x = 0$ or $\lambda = y$. If $x = 0$, then the third equation becomes $2y^2 = 6$, whence $y = \pm\sqrt{3}$, and we obtain two points: $(0, \sqrt{3})$ and $(0, -\sqrt{3})$. If $\lambda = y$, then the second and third equations become the system
$$\begin{cases} x^2 - 4y^2 = 0 \\ x^2 + 2y^2 = 6 \end{cases} \tag{14.42}$$
Adding the first equation in (14.42) to twice the second equation yields $3x^2 = 12$, and so $x = \pm 2$. Substituting this into the first equation, say, gives $4 - 4y^2 = 0$, and so $y = \pm 1$. We obtain four

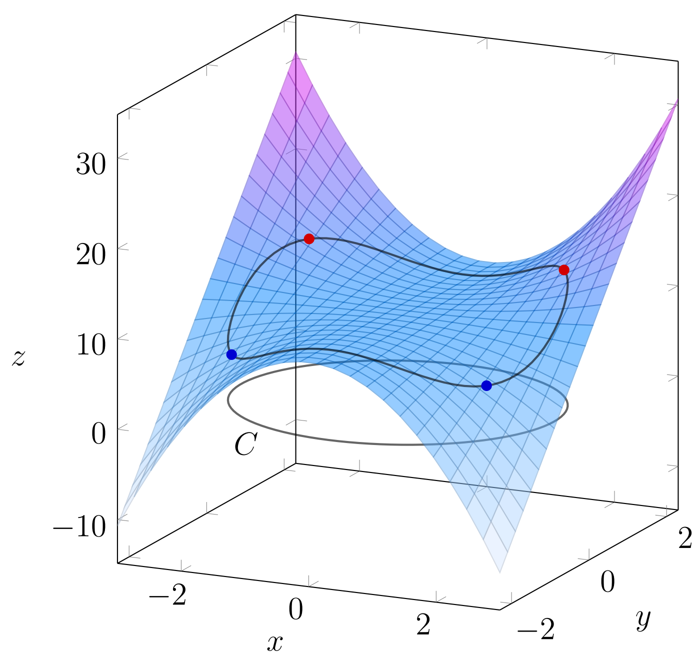


FIGURE 67. In black the graph of $f|_C$, with global maxima in red and global minima in blue.

points: $(2,1)$, $(2,-1)$, $(-2,1)$, and $(-2,-1)$. We evaluate $f(x,y)$ at each of the six points we found for which some $\lambda \in \mathbb{R}$ exists to satisfy (14.41):

$$f(0, \pm\sqrt{3}) = 10, \quad f(\pm 2, 1) = 14, \quad f(\pm 2, -1) = 6.$$

By Theorem 14.67 we conclude that $f|_C$ has maximum value 14 at $(\pm 2, 1)$, and minimum value 6 at $(\pm 2, -1)$. See Figure 67. ■

The Method of Lagrange Multipliers very easily extends to functions of three or more independent variables, with much that same proof as the two-variable version.

Theorem 14.69 (**Method of Lagrange Multipliers in n Variables**). *Let f be differentiable and g have continuous first partials on a open set $U \subseteq \mathbb{R}^n$ containing a closed level set Ω given by $g(\mathbf{x}) = 0$, and let $\nabla g(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \Omega$. If*

$$S = \{\mathbf{x} \in \Omega : \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \text{ for some } \lambda \in \mathbb{R}\},$$

then

$$\max_{\mathbf{x}\in\Omega} f(\mathbf{x}) = \max_{\mathbf{x}\in S} f(\mathbf{x}) \quad \text{and} \quad \min_{\mathbf{x}\in\Omega} f(\mathbf{x}) = \min_{\mathbf{x}\in S} f(\mathbf{x}).$$

This theorem becomes Theorem 14.67 in the case when $n = 2$, and the level set Ω becomes a level curve C in the xy-plane. When $n = 3$ the level set Ω becomes a level surface in xyz-space, which here will usually be denoted by the symbol Σ. When $n = 4$ the level set Ω inhabits $\mathbb{R}^4$ and becomes impractical to depict graphically.

In the definition of the set S in Theorem 14.69, observe that $\mathbf{x} \in \Omega$ if and only if $g(\mathbf{x}) = 0$. Thus the theorem essentially states that the extreme values of the objective function f restricted to the **constraint set** $g(\mathbf{x}) = 0$ will lie at those points $\mathbf{x} \in \mathbb{R}^n$ for which there can be found some $\lambda \in \mathbb{R}$ such that $(\mathbf{x}, \lambda)$ is a solution to the system of equations

$$\begin{cases} f_{x_1}(\mathbf{x}) = \lambda g_{x_1}(\mathbf{x}) \\ \qquad\vdots \\ f_{x_n}(\mathbf{x}) = \lambda g_{x_n}(\mathbf{x}) \\ \quad g(\mathbf{x}) = 0 \end{cases} \tag{14.43}$$

Example 14.70. Find the maximum and minimum values of

$$f(x,y,z) = xyz$$

subject to the constraint $x^2 + 2y^2 + 4z^2 = 9$.

Solution. Defining

$$g(x,y,z) = x^2 + 2y^2 + 4z^2 - 9,$$

we may write the constraint surface Σ given by $x^2 + 2y^2 + 4z^2 = 9$ as $g(x,y,z) = 0$. Since

$$\nabla g(x,y,z) = \langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z)\rangle = \langle 2x, 4y, 8z\rangle,$$

we have $\nabla g(x,y,z) = \mathbf{0}$ if and only if $(x,y,z) = (0,0,0)$, but $(0,0,0)$ does not lie on the curve $g(x,y,z) = 0$, and so $\nabla g(x,y,z) \neq \mathbf{0}$ for all $(x,y,z) \in \Sigma$. Since the polynomial function f is differentiable everywhere, and g has continuous first partials everywhere, all the hypotheses of Theorem 14.69 are satisfied. The system (14.43) here becomes

$$\begin{cases} yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \\ \;\; 9 = x^2 + 2y^2 + 4z^2 \end{cases} \tag{14.44}$$

The first equation in (14.44) offers two possibilities: either $x = 0$ or $\lambda = yz/2x$.

- Suppose $x = 0$. Then the system (14.44) becomes

$$\begin{cases} \;\; yz = 0 \\ 4\lambda y = 0 \\ 8\lambda z = 0 \\ \quad 9 = 2y^2 + 4z^2 \end{cases}$$

  From the first equation $yz = 0$ we find that either $y = 0$ or $z = 0$.
  - Suppose $y = 0$. The equations that do not reduce to $0 = 0$ are

$$8\lambda z = 0 \quad \text{and} \quad 4z^2 = 9.$$

    From $4z^2 = 9$ we find that $z = \pm 3/2$, whereupon $8\lambda z = 0$ yields $\lambda = 0$, and thus we obtain two points $(0, 0, \pm 3/2)$ for which there exists some $\lambda \in \mathbb{R}$ such that the system (14.44) is satisfied.
  - Suppose $z = 0$. The equations that do not reduce to $0 = 0$ are

$$4\lambda y = 0 \quad \text{and} \quad 2y^2 = 9.$$

    From $2y^2 = 9$ we find that $y = \pm 3/\sqrt{2}$, whereupon $4\lambda y = 0$ yields $\lambda = 0$, and thus we obtain two more points, $(0, \pm 3/\sqrt{2}, 0)$, for which there exists some $\lambda \in \mathbb{R}$ such that the system (14.44) is satisfied.
- Suppose $\lambda = yz/2x$, which implies that $x \neq 0$. Substituting $yz/2x$ for λ in the second and third equations of (14.44), we obtain the system

$$\begin{cases} x^2 z = 2y^2 z \\ x^2 y = 4yz^2 \\ \;\; 9 = x^2 + 2y^2 + 4z^2 \end{cases} \tag{14.45}$$

  - Suppose $z = 0$. Then the second equation in (14.45) becomes $x^2 y = 0$, which implies that $y = 0$ (since $x \neq 0$), whereupon the third equation becomes $x^2 = 9$ and we find that $x = \pm 3$. We obtain another two points, $(\pm 3, 0, 0)$, for which there exists some $\lambda \in \mathbb{R}$ such that the system (14.44) is satisfied.
  - Suppose $z \neq 0$. Then the first equation in (14.45) becomes $x^2 = 2y^2$, and since $x \neq 0$, it follows that $y \neq 0$ also. The second equation then becomes $x^2 = 4z^2$, which together with $x^2 = 2y^2$ gives $y^2 = 2z^2$. Putting $x^2 = 4z^2$ and $y^2 = 2z^2$ into

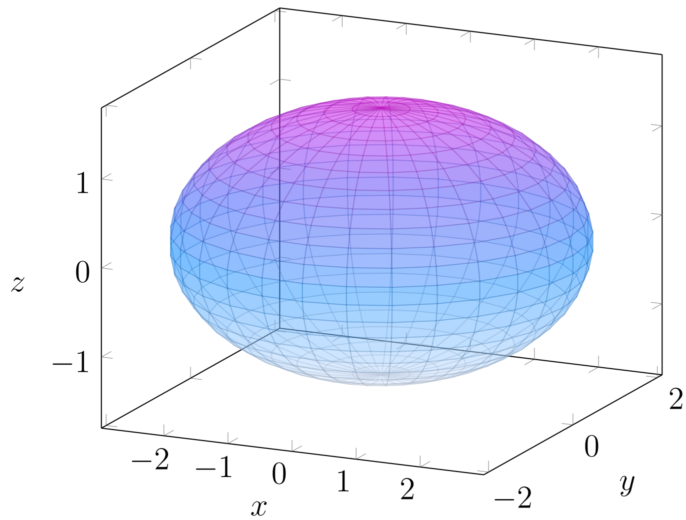


FIGURE 68. The constraint surface Σ: $x^2 + 2y^2 + 4z^2 = 9$.

the third equation in (14.45) then gives $12z^2 = 9$, and so $z = \pm\sqrt{3}/2$. Now,

$$x^2 = 4z^2 \quad \Rightarrow \quad x = \pm 2z = \pm\sqrt{3}$$

and

$$y^2 = 2z^2 \quad \Rightarrow \quad y = \pm\sqrt{2}z = \pm\sqrt{6}/2,$$

and we obtain eight more points (x, y, z) that satisfy

$$(|x|, |y|, |z|) = \big(\sqrt{3}, \sqrt{6}/2, \sqrt{3}/2\big).$$

We have found a total of fourteen points (x, y, z) for which there exists some $\lambda \in \mathbb{R}$ such that the system (14.44) is satisfied. At the points $(0, 0, \pm 3/2)$, $(0, \pm 3/\sqrt{2}, 0)$, and $(\pm 3, 0, 0)$ we find that $f(x, y, z) = 0$. At the remaining eight points we obtain $f(x, y, z) = -3\sqrt{6}/4$ at

$$\big(-\sqrt{3}, -\tfrac{\sqrt{6}}{2}, -\tfrac{\sqrt{3}}{2}\big), \quad \big(-\sqrt{3}, \tfrac{\sqrt{6}}{2}, \tfrac{\sqrt{3}}{2}\big), \quad \big(\sqrt{3}, -\tfrac{\sqrt{6}}{2}, \tfrac{\sqrt{3}}{2}\big), \quad \big(\sqrt{3}, \tfrac{\sqrt{6}}{2}, -\tfrac{\sqrt{3}}{2}\big),$$

and $f(x, y, z) = 3\sqrt{6}/4$ at

$$\big(\sqrt{3}, \tfrac{\sqrt{6}}{2}, \tfrac{\sqrt{3}}{2}\big), \quad \big(-\sqrt{3}, -\tfrac{\sqrt{6}}{2}, \tfrac{\sqrt{3}}{2}\big), \quad \big(\sqrt{3}, -\tfrac{\sqrt{6}}{2}, -\tfrac{\sqrt{3}}{2}\big), \quad \big(-\sqrt{3}, \tfrac{\sqrt{6}}{2}, -\tfrac{\sqrt{3}}{2}\big).$$

By Theorem 14.69 we conclude that $f|_\Sigma$ (i.e. the function f restricted to the set Σ given by $x^2 + 2y^2 + 4z^2 = 9$, shown in Figure 68) has maximum value $3\sqrt{6}/4$ and minimum value $-3\sqrt{6}/4$. ■

Appendix

Here we will prove Theorem 14.30 in the $\mathbb{R}^2$ case, using nothing other than the definition of limit. The technique can be easily extended to the $\mathbb{R}^3$ setting and beyond, but it can be seen to be rather laborious.

Claim. *If f is differentiable at $(a,b) \in \mathbb{R}^2$, then it is continuous there.*

Proof. Suppose f is differentiable at (a,b). By Definition 14.26 f is defined on some open set U containing (a,b), and $M_1 = f_x(a,b)$ and $M_2 = f_y(a,b)$ are also defined.

Let $\epsilon > 0$ be arbitrary. By (14.8) there is some $\delta_0 > 0$ such that

$$0 < \sqrt{h^2+k^2} < \delta_0$$

implies that

$$\left|\frac{f(a+h,b+k) - f(a,b) - M_1h - M_2k}{\sqrt{h^2+k^2}}\right| < \frac{\epsilon}{2}. \tag{14.46}$$

Choose

$$\delta = \min\left\{1, \delta_0, \frac{\epsilon}{4(|M_1|+1)}, \frac{\epsilon}{4(|M_2|+1)}\right\}.$$

If $\sqrt{h^2+k^2} = 0$, then $h = k = 0$ so that

$$f(a+h,b+k) = f(a,b),$$

and thus

$$|f(a+h,b+k) - f(a,b)| = 0 < \epsilon.$$

Suppose that

$$0 < \sqrt{h^2+k^2} < \delta.$$

Then (14.46) implies

$$|f(a+h,b+k) - f(a,b) - M_1h - M_2k| < \frac{\epsilon}{2},$$

from which the general property $|x| - |y| \le |x-y|$ gives us

$$|f(a+h,b+k) - f(a,b)| - |M_1h + M_2k| < \frac{\epsilon}{2},$$

and thus

$$|f(a+h,b+k) - f(a,b)| < \frac{\epsilon}{2} + |M_1h + M_2k|.$$

Applying the Triangle Inequality from §1.6 next yields

$$|f(a+h,b+k) - f(a,b)| < \frac{\epsilon}{2} + |M_1||h| + |M_2||k|. \tag{14.47}$$

Now,

$$|h| \le \sqrt{h^2+k^2} < \delta \le \frac{\epsilon}{4(|M_1|+1)}$$

implies that $|h|(|M_1|+1) < \epsilon/4$, and therefore $|M_1||h| < \epsilon/4$. In similar fashion

$$|k| \le \sqrt{h^2+k^2} < \delta \le \frac{\epsilon}{4(|M_2|+1)}$$

leads to $|M_2||k| < \epsilon/4$. Then from (14.47) we arrive at

$$|f(a+h, b+k) - f(a,b)| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

We have now shown that whenever

$$\sqrt{h^2 + k^2} < \delta$$

we obtain

$$|f(a+h, b+k) - f(a,b)| < \epsilon,$$

which is to say

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b).$$

This limit easily implies that

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b),$$

and therefore f is continuous at (a,b). ■

15

Multiple Integrals

15.1 – Double Integrals Over Rectangles

From here onward we take a **region** to be any set of points in $\mathbb{R}^n$. A region $R \subseteq \mathbb{R}^n$ is **open** if it is an open set, **closed** if it is a closed set, and **bounded** if there exists some $\mathbf{x} \in \mathbb{R}^n$ and $r \in (0, \infty)$ such that $R \subseteq B_r(\mathbf{x})$. Thus a region is bounded if it can be contained in an open ball of finite radius. A region that is both closed and bounded is said to be **compact**. Finally, regions in $\mathbb{R}^2$ will usually be denoted by R or S, and regions in $\mathbb{R}^3$ by D or E.

In this section we take the first step toward extending the notion of a definite integral to functions of two variables to obtain what are called double integrals. Instead of integrating a one-variable function over an interval $I \subseteq \mathbb{R}$, we integrate a two-variable function over a region $R \subseteq \mathbb{R}^2$. We start with R being a rectangle, and then progress in the next section to more general regions.

Before considering double integrals we should become acquainted with the notion of an **iterated integral**, which is an expression of the form $\int_c^d \int_a^b f(x,y)\,dx\,dy$, where by definition

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy. \tag{15.1}$$

Thus, $\int_c^d \int_a^b f(x,y)\,dx\,dy$ is evaluated by first evaluating $\int_a^b f(x,y)\,dx$, where the integration is done with respect to x and y is treated as a constant, and then integrating the result with respect to y. Similarly we have

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx. \tag{15.2}$$

Example 15.1. Evaluate

$$\int_0^\pi \int_1^3 \left(3x^2y - y\sin(xy)\right) dx\,dy$$

j

Solution. We obtain

$$\int_0^\pi \int_1^3 [3x^2y - y\sin(xy)]\,dx\,dy = \int_0^\pi \left[\int_1^3 \big(3x^2y - y\sin(xy)\big)\,dx\right] dy$$
$$= \int_0^\pi \left[x^3y + \cos(xy)\right]_1^3 dy = \int_0^\pi \big[\,(27y + \cos(3y)) - (y + \cos y)\,\big]\,dy$$
$$= \int_0^\pi [26y + \cos(3y) - \cos y]\,dy = \left[13y^2 + \frac{1}{3}\sin(3y) - \sin y\right]_0^\pi = 13\pi^2,$$

using equation (15.1). ■

The definition of a **double integral** of a function f over a rectangular region $R \subseteq \mathbb{R}^2$, which is denoted by any one of the symbols

$$\iint\limits_R f, \quad \iint\limits_R f\,dA, \quad \iint\limits_R f(x,y)\,dA, \quad \text{or} \quad \iint\limits_R f(x,y)\,dx\,dy$$

(in order of increasing specificity), involves a limit of a double Riemann sum. It will be included in these notes at a later date, but suffice it to say that one rarely evaluates a double integral using the definition, but rather employs the following result.

Theorem 15.2. *Let $R = [a,b] \times [c,d]$. If $f : R \to \mathbb{R}$ is continuous, then*

$$\iint\limits_R f = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Figure 1 depicts the rectangle $R = [a,b] \times [c,d]$ featured in the theorem in the usual case when $a, b, c, d > 0$.

Example 15.3. Evaluate

$$\iint\limits_R y^3 \sin(xy^2)\,dA$$

over the region

$$R = \left\{(x,y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{\pi/2}\right\},$$

choosing a convenient order.

Solution. We'll undertake evaluation using the order $dxdy$. We obtain

$$\iint\limits_R y^3 \sin(xy^2)\,dA = \int_0^{\sqrt{\pi/2}} \int_0^1 y^3 \sin(xy^2)\,dx\,dy = \int_0^{\sqrt{\pi/2}} \left[-\frac{y^3}{y^2}\cos(xy^2)\right]_0^1 dy$$
$$= \int_0^{\sqrt{\pi/2}} -y(\cos y^2 - 1)\,dy = \int_0^{\sqrt{\pi/2}} y\,dy - \int_0^{\sqrt{\pi/2}} y\cos(y^2)\,dy$$
$$= \frac{\pi}{4} - \frac{1}{2}\int_0^{\sqrt{\pi/2}} \left[\sin(y^2)\right]'\,dy = \frac{\pi}{4} - \frac{1}{2}\left[\sin(y^2)\right]_0^{\sqrt{\pi/2}} = \frac{\pi}{4} - \frac{1}{2},$$

where the first equality follows from Theorem 15.2, and the fifth equality uses

$$y\cos(y^2) = \frac{1}{2}\left[\sin(y^2)\right]'$$

together with Theorem 5.28.[13]

The question is, how does the order $dydx$ compare with $dxdy$? Is it still reasonably doable? In this case we straightaway come up against the integral

$$\int_0^{\sqrt{\pi/2}} y^3 \sin(xy^2)\,dy,$$

which demands the substitution $u = y^2$ in order to get

$$\frac{1}{2}\int_0^{\pi/2} u\sin(xu)\,du,$$

where x is treated as a constant. However, integration by parts next gives

$$\frac{1}{2x^2}\sin\left(\frac{\pi}{2}x\right) - \frac{\pi}{4x}\cos\left(\frac{\pi}{2}x\right),$$

which we would need to integrate with respect to x as the next step:

$$\int_0^1 \left[\frac{1}{2x^2}\sin\left(\frac{\pi}{2}x\right) - \frac{\pi}{4x}\cos\left(\frac{\pi}{2}x\right)\right] dx.$$

This turns out to be highly problematic! ■

[13]This approach is an alternative to applying the Substitution Rule with $u = y^2$.

15.2 – Double Integrals Over General Regions

A first definition of the double integral usually restricts the region of integration to sets in $\mathbb{R}^2$ that are closed and bounded, which is to say any region $R \subseteq \mathbb{R}^2$ that is compact. But there is little to prevent considering other kinds of regions. Note that if R is a bounded set, then the closure of R, $\overline{R} = R \cup \partial R$ (see §13.0), is a compact set.

Definition 15.4. *If R is a bounded region, not necessarily closed, and f is a function that is continuous on the closure $\overline{R}$ of R, then we define*

$$\iint_R f = \iint_{\overline{R}} f.$$

We now set out to improve on Theorem 15.2, so that it applies to other kinds of compact regions other than rectangles.

Theorem 15.5 (Fubini's Theorem). *For continuous functions $g, h : [a,b] \to \mathbb{R}$, where $g < h$ on $[a,b]$, let*

$$R = \{(x,y) : a \leq x \leq b \text{ and } g(x) \leq y \leq h(x)\}.$$

If $f : R \to \mathbb{R}$ is continuous, then

$$\iint_R f = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx. \tag{15.3}$$

For continuous functions $g, h : [c,d] \to \mathbb{R}$, where $g < h$ on $[c,d]$, let

$$R = \{(x,y) : c \leq y \leq d \text{ and } g(y) \leq x \leq h(y)\}.$$

If $f : R \to \mathbb{R}$ is continuous, then

$$\iint_R f = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy. \tag{15.4}$$

There are in fact a whole host of theorems in mathematics, of varying degree of generality, that are referred to as Fubini's Theorem. Thus, even Theorem 15.2 can be fairly called Fubini's Theorem.

Recall that the symbol $\int_a^b dx$ is taken to mean $\int_a^b (1)\,dx$. Indeed if we let $I = [a,b]$ we could even write $\int_I dx$, where

$$\int_I dx = \int_a^b dx = b - a = \mathcal{L}(I),$$

the length of I. In a wholly analogous fashion $\iint_R dA$ is short-hand for $\iint_R (1)\,dA$, and it relates to the area of R, $\mathcal{A}(R)$.

Definition 15.6. *The **area** of a closed, bounded region R in the xy-plane is*

$$\mathcal{A}(R) = \iint_R dA.$$

It is important to bear in mind that this definition gives the area of a region R on a plane that is coordinatized with the customary rectangular coordinate system. It may be surprising to learn that there exist bounded regions in $\mathbb{R}^2$ that do not have an area, which is to say that $\mathcal{A}(R)$ as defined above does not equal any real number (not even 0), but we will not consider these here. A region $R \subseteq \mathbb{R}^2$ that has a defined area is called **Jordan measurable**.

Proposition 15.7. *Suppose that $R \subseteq \mathbb{R}^2$ is a bounded region such that $\mathcal{A}(R) = 0$. If $f : R \to \mathbb{R}$ is a bounded function, then*

$$\iint\limits_R f = 0.$$

Proposition 15.8. *Suppose that $R_1, R_2, R_1 \cap R_2 \subseteq \mathbb{R}^2$ are compact Jordan measurable sets. If $f : R \to \mathbb{R}$ is a continuous function, then*

$$\iint\limits_{R_1 \cup R_2} f = \iint\limits_{R_1} f + \iint\limits_{R_2} f - \iint\limits_{R_1 \cap R_2} f.$$

Thus if $\mathcal{A}(R_1 \cap R_2) = 0$, then

$$\iint\limits_{R_1 \cup R_2} f = \iint\limits_{R_1} f + \iint\limits_{R_2} f.$$

The last statement in Proposition 15.8 follows from the first statement in conjunction with Proposition 15.7, since it is a fact that a function that is continuous on a compact set is necessarily bounded. If R is a bounded set, then $\overline{R} = R \cup \partial R$ is compact, and since $\mathcal{A}(\partial R) = 0$ (assuming the boundary of R is Jordan measurable), by Proposition 15.7 we have

$$\iint\limits_{\partial R} f = 0,$$

and so

$$\iint\limits_{R \cup \partial R} f = \iint\limits_{\overline{R}} f = \iint\limits_{R} f = \iint\limits_{R} f + \iint\limits_{\partial R} f$$

by Definition 15.4. This natural result is in fact the motivation for Definition 15.4.

Throughout this chapter and the next we will assume, without comment, that all regions are Jordan measurable!

Example 15.9. Evaluate $\iint_R f$ for $f(x, y) = x + y$, where $R \subseteq \mathbb{R}^2$ is the region in the first quadrant that is bounded by the line $x = 0$ and the curves $g(x) = x^2$ and $h(x) = 8 - x^2$.

Solution. First we determine where the curves given by g and h by solving $g(x) = h(x)$:

$$g(x) = h(x) \quad \Rightarrow \quad x^2 = 8 - x^2 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2.$$

Since it is given that R is in Quadrant I only $x = 2$ is relevant, and this yields the intersection point $(2, 4)$. Thus R is as shown in Figure 69. In this region it can be seen that, for each $0 \le x \le 2$, we have $x^2 \le y \le 8 - x^2$; thus $\iint_R f$ can be resolved into an iterated integral in

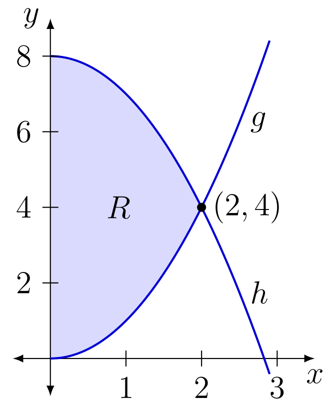


FIGURE 69.

which the inner integral integrates with respect to y from x^2 to $8-x^2$, and the outer integral integrates with respect to x from 0 to 2:

$$\begin{aligned}\iint_R f &= \int_0^2 \int_{x^2}^{8-x^2} (x+y)\,dy\,dx = \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{x^2}^{8-x^2} dx \\ &= \int_0^2 \left[\left(x(8-x^2) + \frac{1}{2}(8-x^2)^2\right) - \left(x^3 + \frac{1}{2}(x^2)^2\right)\right] dx \\ &= \int_0^2 \left(32 + 8x - 8x^2 - 2x^3\right) dx = \frac{152}{3}.\end{aligned}$$

■

Sometimes, as was already seen in Example 15.3, one order of integration is significantly easier than another. Thus, if a given iterated integral appears intractable, it may be worthwhile to reverse the order of integration.

Example 15.10. Evaluate

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1}\,dy\,dx$$

by reversing the order of integration.

Solution. The reader is certainly welcome to attempt evaluation without reversing the order of integration, but it will prove a frustrating enterprise. For each $0 \le x \le 4$ we have $g(x) = \sqrt{x} \le y \le 2 = h(x)$, and since the curves given by g and h intersect at $(4, 2)$, the region of integration R is as in Figure 70.

We wish to reverse the order of integration so that integration with respect to x, rather than y, is done first. To do this observe that, for each $0 \le y \le 2$, the value of x is bound by the y-axis (i.e. $x = 0$) and the curve g where $y = \sqrt{x}$ implies that $x = y^2$; that is, for each $0 \le y \le 2$ we have $0 \le x \le y^2$. Thus,

$$\iint_R f = \int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{x}{y^5+1}\,dx\,dy,$$

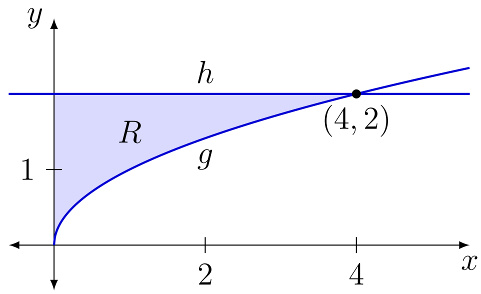


FIGURE 70.

where the integral $\int_0^{y^2} x/(y^5+1)\,dx$ is easy to evaluate since y^5+1 is regarded as a constant and thus can be removed from the integrand! Winding up our propeller beanies, we calculate as follows, making the substitution $u = y^5 + 1$ along the way:

$$\begin{aligned}
\int_0^2 \int_0^{y^2} \frac{x}{y^5+1}\,dx\,dy &= \int_0^2 \left(\frac{1}{y^5+1} \int_0^{y^2} x\,dx \right) dy = \int_0^2 \frac{1}{y^5+1} \left[\frac{x^2}{2} \right]_0^{y^2} dy \\
&= \int_0^2 \frac{y^4}{2(y^5+1)}\,dy = \int_1^{33} \frac{1}{10u}\,du = \left[\frac{1}{10} \ln |u| \right]_1^{33} = \frac{1}{10} \ln(33).
\end{aligned}$$

■

15.3 – Double Integrals in Polar Coordinates

It can happen that a region R in the xy-plane, which can be denoted by $\mathbb{R}^2_{xy}$, is such that the evaluation of $\iint_R f$ by means of Theorem 15.5 is quite arduous or even impossible. In such situations the best strategy is to find a more compliant region S on another plane having a different coordinate system that nicely "transforms" into R. In general this other plane can be referred to as the uv-plane, denoted by $\mathbb{R}^2_{uv}$. Points in $\mathbb{R}^2_{xy}$ have coordinates (x, y), and points in $\mathbb{R}^2_{uv}$ have coordinates (u, v). The "transforming" from $S \subseteq \mathbb{R}^2_{uv}$ to $R \subseteq \mathbb{R}^2_{xy}$ is effected by a function $T : S \to R$ called a **mapping** or **transformation** (we shall freely use either term). The region R is called the "image of S under T" and is frequently denoted by $T(S)$, so that $R = T(S)$.

Much more will be said about transformations in a general setting in §14.7, but for now we'll be specifically interested in the transformation T_{pol} that converts from polar coordinates (r, θ) to rectangular coordinates (x, y), which is given by

$$T_{\text{pol}}(r, \theta) = (r \cos\theta, r \sin\theta) \tag{15.5}$$

and does nothing more than effect the usual conversions $x = r\cos\theta$ and $y = r\sin\theta$ first introduced in section 11.2. In particular T_{pol} takes a point (r, θ) in $\mathbb{R}^2_{r\theta}$ (the $r\theta$-plane) and maps it to a point (x, y) in $\mathbb{R}^2_{xy}$ such that $(x, y) = (r\cos\theta, r\sin\theta)$.

Theorem 15.11. *Let $S \subseteq \mathbb{R}^2_{r\theta}$ be given by*

$$S = \{(r, \theta) : \alpha \le \theta \le \beta \text{ and } 0 \le g(\theta) \le r \le h(\theta)\}$$

for continuous g, h (where $\beta - \alpha \le 2\pi$). If T_{pol} is one-to-one on $\text{Int}(S)$*, $R = T_{\text{pol}}(S)$, and $f : R \to \mathbb{R}$ is continuous, then*

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(r\cos\theta, r\sin\theta) r\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r\, dr d\theta \tag{15.6}$$

The proof of this theorem involves using Theorem 15.30 to obtain

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(r\cos\theta, r\sin\theta) r\, dA$$

(which is shown in section 14.7), and then applying (15.4) in Fubini's Theorem 15.5.

Notation. The double integral

$$\iint\limits_S f(r\cos\theta, r\sin\theta) r\, dA$$

could be represented more compactly by the symbol $\iint_S \varphi$ if we were to define a function φ by $\varphi(r, \theta) = f(r\cos\theta, r\sin\theta) r$.

Example 15.12. Evaluate $\iint_R f$, where R is the region in the xy-plane shown on the right side of Figure 71, and $f(x, y) = 2xy$.

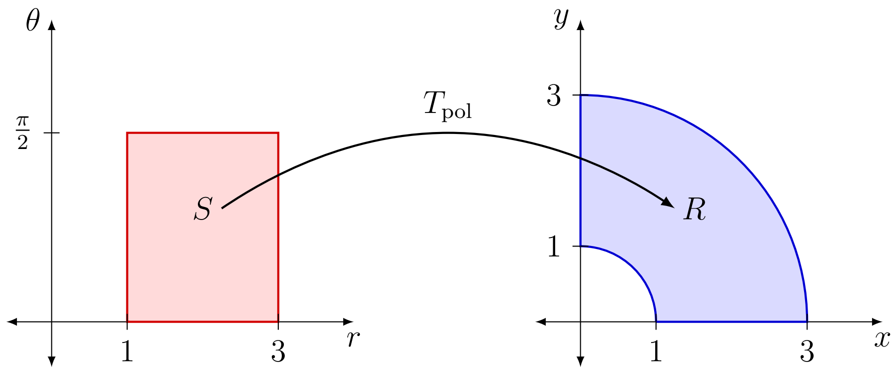


FIGURE 71.

Solution. An examination of R makes it clear that it consists of points having a distance r from the origin between 1 and 3, and making an angle θ with respect to the positive x-axis between 0 and $\pi/2$. Thus, $R = T_{\text{pol}}(S)$, where

$$S = \{(r, \theta) : 1 \le r \le 3,\ 0 \le \theta \le \pi/2\}$$

is the region in the $r\theta$-plane shown on the left side of Figure 71. Since f is continuous on R, we can use Theorem 15.11 with $g(\theta) = 1$ and $h(\theta) = 3$ to obtain

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)r\,drd\theta \\
&= \int_0^{\pi/2}\int_1^3 2(r\cos\theta)(r\sin\theta)\cdot r\,drd\theta \\
&= \int_0^{\pi/2} 2\cos\theta\sin\theta\int_1^3 r^3\,drd\theta = \int_0^{\pi/2} 2\cos\theta\sin\theta\left[\frac{1}{4}r^4\right]_1^3 d\theta \\
&= 40\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = 40\int_0^{\pi/2}\frac{\sin 2\theta}{2}\,d\theta \\
&= -10\,[\cos 2\theta]_0^{\pi/2} = 20,
\end{aligned}$$

where the identity $2\cos\theta\sin\theta = \sin 2\theta$ is used along the way. ■

Example 15.13. Find the area of the region R in the xy-plane bounded by the curve $r = 2(1 - \sin\theta)$.

Solution. The curve is a cardioid when graphed in the xy-plane, as shown on the right side of Figure 72. Certainly in order to obtain a curve that fully encloses a region in $\mathbb{R}^2_{xy}$ it is necessary to take θ from 0 to 2π; and then, for each $\theta \in [0, 2\pi]$, the points that lie in R are a distance r from the origin ranging from 0 to $2 - 2\sin\theta$. From these observations it can be determined that the region S in the $r\theta$-plane for which $T_{\text{pol}}(S) = R$ is given by

$$S = \{(r, \theta) : 0 \le r \le 2 - 2\sin\theta,\ 0 \le \theta \le 2\pi\},$$

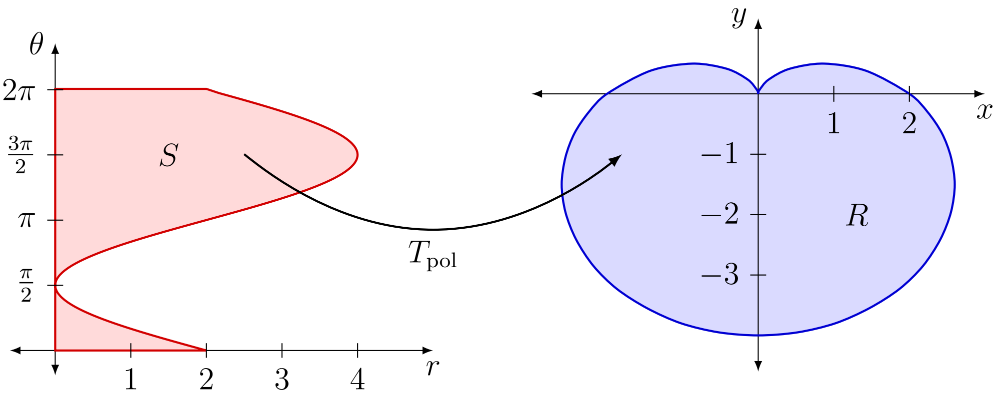


FIGURE 72.

as shown on the left side of Figure 72. Since $f(x, y) = 1$ is continuous on R, we can use Theorem 15.11 with $g(\theta) = 0$ and $h(\theta) = 2 - 2\sin\theta$ to obtain

$$\begin{aligned}
\mathcal{A}(R) &= \iint\limits_R (1)\, dA = \iint\limits_S r\, dA = \int_0^{2\pi} \int_0^{2-2\sin\theta} r\, dr d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_0^{2-2\sin\theta} d\theta = 2\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta \\
&= 2\int_0^{2\pi} \left(1 - 2\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta \\
&= 2\left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 2(3\pi) = 6\pi,
\end{aligned}$$

where the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ is employed along the way. ■

Example 15.14. Evaluate

$$\int_{-4}^{4} \int_0^{\sqrt{16-y^2}} (16 - x^2 - y^2)\, dx\, dy. \tag{15.7}$$

Solution. The region of integration R is shown on the right of Figure 73. It is the image under T_{pol} of the region S on the left of the figure, which is a simple rectangle. Letting ω denote the integral (15.7), we obtain

$$\begin{aligned}
\omega &= \iint\limits_R (16 - x^2 - y^2)\, dA && \text{By Fubini's Theorem} \\
&= \iint\limits_S \left(16 - r^2\cos^2\theta - r^2\sin^2\theta\right) r\, dA && \text{By Theorem 15.11} \\
&= \int_{-\pi/2}^{\pi/2} \int_0^4 (16 - r^2) r\, dr d\theta && \text{By Fubini's Theorem} \\
&= \int_{-\pi/2}^{\pi/2} \left[8r^2 - \frac{1}{4}r^4\right]_0^4 d\theta
\end{aligned}$$

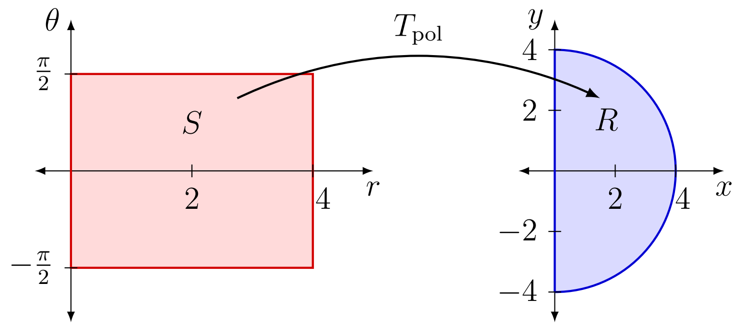


FIGURE 73.

$$= \int_{-\pi/2}^{\pi/2} 64\, d\theta = 64\pi.$$

Note that the identity $\cos^2\theta + \sin^2\theta = 1$ is used during the process. ■

Example 15.15. Let $R' \subseteq \mathbb{R}^2_{xy}$ be the region bounded by the circle $r = 1$, and let $R'' \subseteq \mathbb{R}^2_{xy}$ be the region bounded by the cardioid $r = 1 - \cos\theta$. Find the area of $R = R' \cap R''$.

Solution. The region R is shown at left in Figure 74. We can partition the region into three subregions R_1, R_2 and R_3 as shown at right in Figure 74. Specifically $R_i = T_{\text{pol}}(S_i)$ for $i = 1, 2, 3$, where T is the transformation (15.5) and

$$\begin{aligned}
S_1 &= \{(r,\theta) : 0 \le \theta \le \pi/2 \text{ and } 0 \le r \le 1 - \cos\theta\},\\
S_2 &= \{(r,\theta) : \pi/2 \le \theta \le 3\pi/2 \text{ and } 0 \le r \le 1\},\\
S_3 &= \{(r,\theta) : 3\pi/2 \le \theta \le 2\pi \text{ and } 0 \le r \le 1 - \cos\theta\}.
\end{aligned}$$

We can use Proposition 15.8 to write

$$\mathcal{A}(R) = \iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA + \iint_{R_3} dA.$$

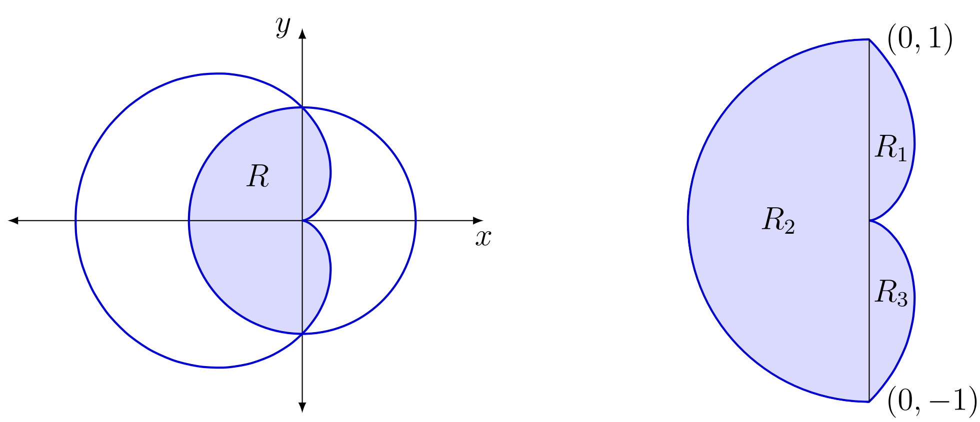


FIGURE 74.

By Theorem 15.11,

$$\iint\limits_{R_1} dA = \iint\limits_{S_1} r\,dA = \int_0^{\pi/2}\int_0^{1-\cos\theta} r\,drd\theta = \int_0^{\pi/2}\frac{1}{2}(1-\cos\theta)^2 d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2}-2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{2}\left[\frac{3\theta}{2}-2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{3\pi}{8}-1.$$

By symmetry it should be clear that

$$\iint\limits_{R_3} dA = \mathcal{A}(R_3) = \mathcal{A}(R_1) = \iint\limits_{R_1} dA = \frac{3\pi}{8}-1.$$

Finally, R_2 is half of a closed circular disk with radius 1, so that

$$\iint\limits_{R_2} dA = \mathcal{A}(R_2) = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

Therefore

$$\mathcal{A}(R) = \left(\frac{3\pi}{8}-1\right)+\frac{\pi}{2}+\left(\frac{3\pi}{8}-1\right) = \frac{5\pi}{4}-2$$

is the area of R. ■

15.4 – Triple Integrals in Rectangular Coordinates

Definition 15.16. *Given a region $D \subseteq \mathbb{R}^3$ and a function $f : D \to \mathbb{R}$, the* ***triple integral of f over*** *D is*

$$\iiint_D f = \lim_{\|P\|\to 0} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \tag{15.8}$$

provided the limit exists. If the limit does exist, f is said to be ***integrable*** *over D.*

Other symbols used to denote the triple integral (15.8) are

$$\iiint_D f(x,y,z)\,dV \quad \text{and} \quad \iiint_D f(x,y,z)\,dx\,dy\,dz,$$

which are useful when the function in the integrand has not been assigned a symbol of its own (such as f). A full elucidation of the whys and wherefores of Definition 15.16 will be included in these notes at a later date, but for now the following result (which is another Fubini theorem) will be used to evaluate triple integrals.

Theorem 15.17. *Let*

$$D = \{(x,y,z) : a \le x \le b, g(x) \le y \le h(x), G(x,y) \le z \le H(x,y)\},$$

where g, h, G, H are continuous functions. If $f : D \to \mathbb{R}$ is continuous, then

$$\iiint_D f = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\,dy\,dx. \tag{15.9}$$

The right-hand side of (15.9) is an iterated integral which is evaluated "from the inside out":

$$\int_a^b \left[\int_{g(x)}^{h(x)} \left(\int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\right) dy\right] dx$$

Thus, integration is first done with respect to z (treating y and x as constant), then with respect to y (treating x as constant), and finally with respect to x. However, this is not the only order of integration that is possible. Indeed, Theorem 15.17 presents only one of *five* possible orders of integration, with the other orders being $dz\,dx\,dy$, $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, and $dydzdx$.

Definition 15.18. *The* ***volume*** *of a closed, bounded region D in xyz-space is*

$$\mathcal{V}(D) = \iiint_D dV.$$

Example 15.19. Find the volume of the region D shown in the left of Figure 75.

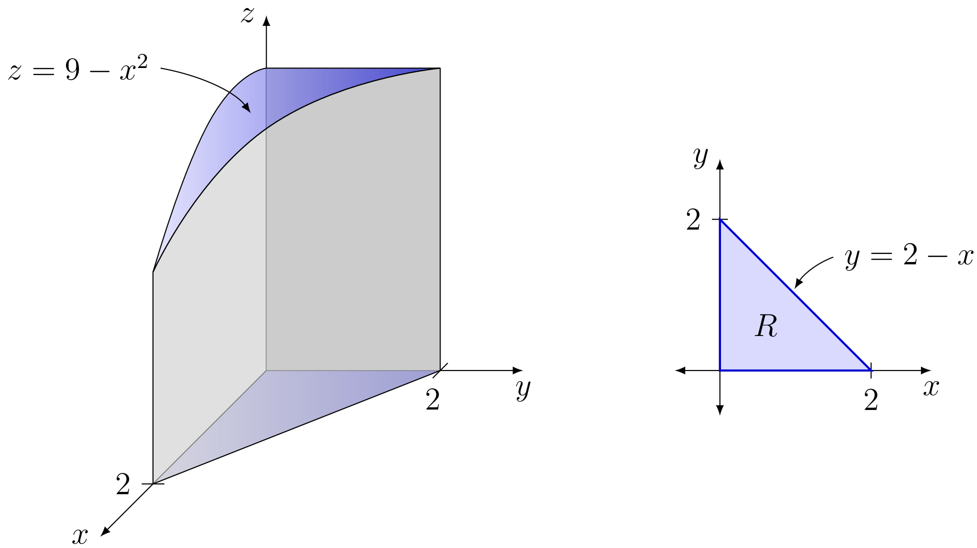


FIGURE 75.

Solution. For any $(x, y, z) \in D$ we have $0 \leq z \leq 9 - x^2$. Thus, if we evaluate $\iiint_D dV$ in the order presented in Theorem 15.17, then we will first evaluate with respect to z using $G(x, y) = 0$ and $H(x, y) = 9 - x^2$ as the limits of integration.

To determine the limits of integration for y and x, project D onto the xy-plane to obtain the region R shown in the right of Figure 75. There it can be seen that if $(x, y) \in R$, then $0 \leq y \leq 2 - x$ for $0 \leq x \leq 2$, and so the limits of integration for y will be $g(x) = 0$ and $h(x) = 2 - x$ in the notation of Theorem 15.17, and the limits of integration for x will be $a = 0$ and $b = 2$. We obtain

$$
\begin{aligned}
\mathcal{V}(D) &= \iiint\limits_D dV && \text{By Definition 15.18} \\
&= \int_0^2 \int_0^{2-x} \int_0^{9-x^2} dz\, dy\, dx && \text{By Theorem 15.17} \\
&= \int_0^2 \int_0^{2-x} (9 - x^2)\, dy\, dx && \text{Integrating with respect to } z \\
&= \int_0^2 \left[9y - x^2 y\right]_0^{2-x} dx && \text{Integrating with respect to } y \\
&= \int_0^2 \left[9(2 - x) - x^2(2 - x)\right] dx \\
&= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{9}{2}x^2 + 18x\right]_0^2 = \frac{50}{3}.
\end{aligned}
$$

It can be instructive to try determining the volume of D by integrating in the orders $dz\, dx\, dy$ and $dy\, dz\, dx$. ■

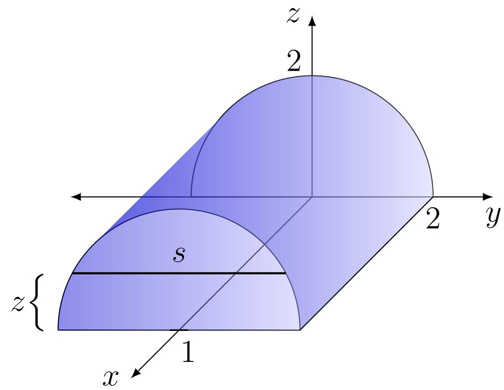


FIGURE 76.

Example 15.20. Rewrite the iterated integral

$$\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz\, dy\, dx$$

in the order $dydzdx$, and then evaluate the resulting integral.

Solution. First we have $0 \le z \le \sqrt{4-y^2}$, where $z = \sqrt{4-y^2}$ graphs on the yz-plane as the upper half of a circle with radius 2 centered at the origin. The entire semicircle is attained since $-2 \le y \le 2$ is given, and from $0 \le x \le 1$ it follows that the semicircle it free to "slide" along the x-axis from $x = 0$ to $x = 1$. In this way a region D is swept out that is the "top" half of a circular cylinder as shown in Figure 76. As can be seen from line segment s in the figure, a point $(x, y, z) \in D$ must have

$$-\sqrt{4-z^2} \le y \le \sqrt{4-z^2}$$

for each $z \in [0, 2]$. Thus we have

$$\begin{aligned} \int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz\, dy\, dx &= \int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx \\ &= \int_0^1 \int_0^2 2\sqrt{4-z^2}\, dz\, dx \end{aligned}$$

With the trigonometric substitution $z = 2\sin\theta$ we can replace dz with $2\cos\theta\, d\theta$, $z = 0$ with $\theta = 0$, and $z = 2$ with $\theta = \pi/2$, so that

$$\begin{aligned} \int_0^1 \int_0^2 2\sqrt{4-z^2}\, dzdx &= \int_0^1 \int_0^{\pi/2} 2\sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\, d\theta\, dx \\ &= \int_0^1 \int_0^{\pi/2} 8\cos^2\theta\, d\theta dx = \int_0^1 \int_0^{\pi/2} 4(1+\cos 2\theta)\, d\theta\, dx \\ &= \int_0^1 [4\theta + 2\sin 2\theta]_0^{\pi/2}\, dx = \int_0^1 2\pi\, dx = 2\pi \end{aligned}$$

Therefore

$$\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz\, dy\, dx = \int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx = 2\pi,$$

by the appropriate extension of Fubini's Theorem. ■

15.5 – Triple Integrals in Curvilinear Coordinates

A **coordinate line** of a coordinate system is a line on which all but one of the coordinate variables are held constant. Thus, in the rectangular coordinate system on the xy-plane, there are two families of coordinate lines: the vertical lines, which is the family of equations $\{x = a : a \in \mathbb{R}\}$; and the horizontal lines, which is the family of equations $\{y = b : b \in \mathbb{R}\}$. A coordinate system is **curvilinear** if at least one of its families of coordinate lines is comprised of lines having *curved* images under the canonical transformation T that maps to rectangular coordinates. The polar coordinate system is an example in $\mathbb{R}^2$, where the coordinate line $r = a$ in $\mathbb{R}^2_{r\theta}$ has an image in $\mathbb{R}^2_{xy}$ that is a circle centered at the origin with radius a under the usual coordinate transformation

$$T_{\text{pol}}(r, \theta) = (r\cos\theta, r\sin\theta),$$

as shown in Figure 77. In $\mathbb{R}^3$ we'll be considering exclusively the cylindrical and spherical coordinate systems.

Definition 15.21. *Let $(x, y, z) \in \mathbb{R}^3_{xyz}$. If an ordered triple (r, θ, z) is such that*

$$(x, y, z) = (r\cos\theta, r\sin\theta, z),$$

*then (r, θ, z) are **cylindrical coordinates** for the point (x, y, z). The **cylindrical coordinate system**, denoted by $\mathbb{R}^3_{r\theta z}$, is the system in which all points in $\mathbb{R}^3$ are given in cylindrical coordinates. The mapping $T_{\text{cyl}} : \mathbb{R}^3_{r\theta z} \to \mathbb{R}^3_{xyz}$ given by*

$$T_{\text{cyl}}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

is the conversion transformation to rectangular coordinates.

Theorem 15.22. *Let $E \subseteq \mathbb{R}^3_{r\theta z}$ be given by*

$$E = \{(r, \theta, z) : \alpha \le \theta \le \beta,\ 0 \le g(\theta) \le r \le h(\theta),\ G(r, \theta) \le z \le H(r, \theta)\}$$

for continuous g, h, G, H (where $\beta - \alpha \le 2\pi$). If T_{cyl} is one-to-one on $\text{Int}(E)$, $D = T_{\text{cyl}}(E)$, and $f : D \to \mathbb{R}$ is continuous, then

$$\iiint\limits_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r,\theta)}^{H(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta.$$

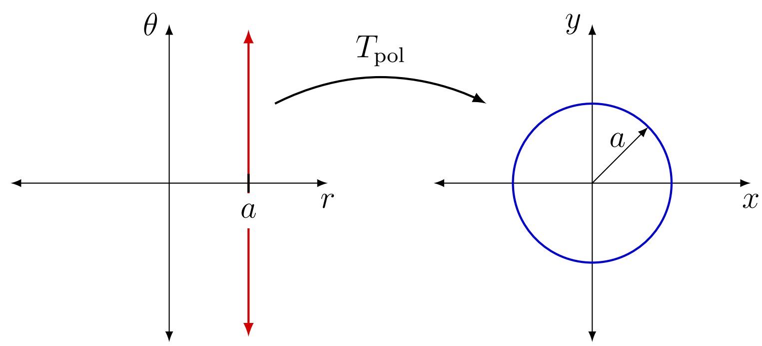


Figure 77.

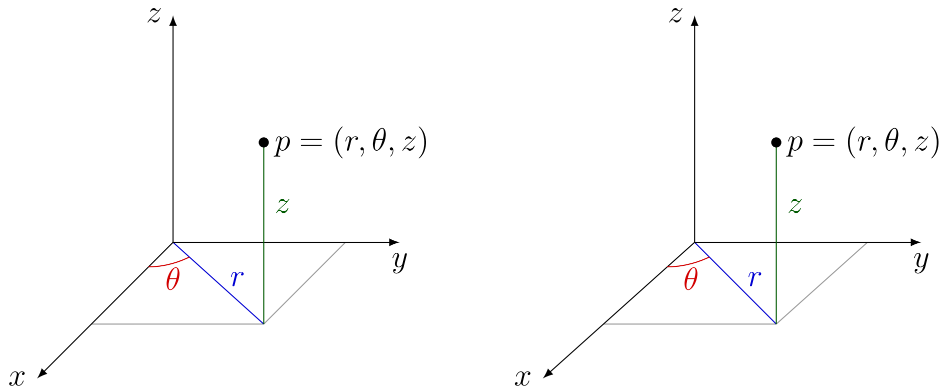


FIGURE 78. Cylindrical coordinates. Usually $r \geq 0$, but it is not required.

The proof of this theorem involves using Theorem 15.32 in §15.6 to obtain

$$\iiint_D f(x, y, z)\, dV = \iiint_E f(r\cos\theta, r\sin\theta, z) r\, dV,$$

and then applying the appropriate version of Theorem 15.17.

Example 15.23. Find the volume of the region D bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$.

Solution. The region D is shown at left in Figure 79. It will be convenient to work in cylindrical coordinates, where $x = r\cos\theta$ and $y = r\sin\theta$ so that the equation of the paraboloid becomes

$$z = x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2,$$

and the equation of the plane remains $z = 25$.

The intersection of the surfaces $z = 25$ and $z = x^2 + y^2$ is the set of points

$$\{(x, y, 25) : x^2 + y^2 = 25\},$$

which is a curve that projects onto the xy-plane as a circle of radius 5 centered at the origin. Thus, the projection of D onto the xy-plane is a region R that is a closed disc with radius 5 centered at the origin, shown at right in Figure 79.

Now, a point in R may have a θ-coordinate value ranging anywhere from $\theta = 0$ to $\theta = 2\pi$; that is, if $(r, \theta) \in R$, then $0 \leq \theta \leq 2\pi$.

If we fix $\theta \in [0, 2\pi]$, then a point $(r, \theta) \in R$ must lie on the line segment joining $o = (0, 0)$ and $a = (5, \theta)$, shown at right in Figure 79. That is, given $\theta \in [0, 2\pi]$, a point $(r, \theta) \in R$ can have r-coordinate value ranging anywhere from $r = 0$ to $r = 5$, which is to say $0 \leq r \leq 5$.

Finally, fixing $\theta \in [0, 2\pi]$ and $r \in [0, 5]$, we consider the limits on z in order for (r, θ, z) to be a point that lies in D. We find that generally z must be such that (r, θ, z) is above the paraboloid $z = r^2$ and below the plane $z = 25$, which is to say $r^2 \leq z \leq 25$.

Thus we find that the region $E \subseteq \mathbb{R}^3_{r\theta z}$ for which $T_{\text{cyl}}(E) = D$ is

$$E = \{(r, \theta, z) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 5,\ r^2 \leq z \leq 25\},$$

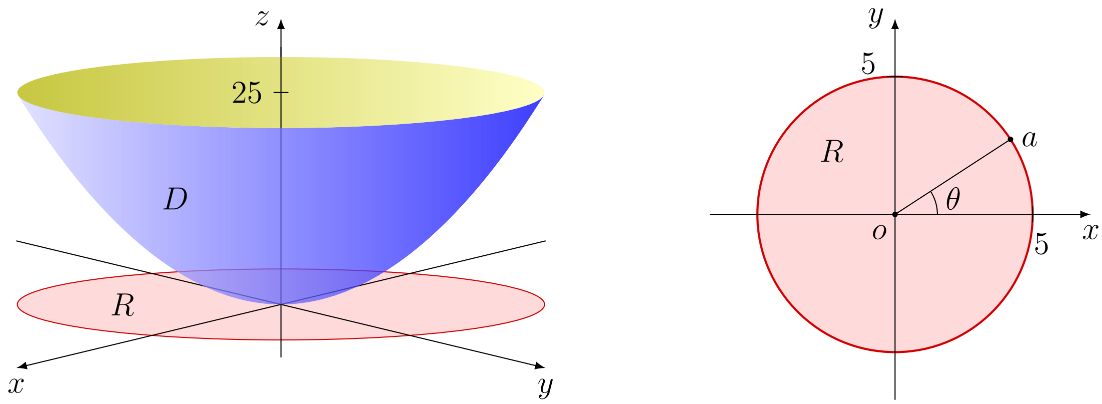


Figure 79.

and so by Definition 15.18 and Theorem 15.22

$$\begin{aligned}\mathcal{V}(D) &= \iiint\limits_D dV = \iiint\limits_E r\,dV = \int_0^{2\pi}\int_0^5\int_{r^2}^{25} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^5 (25r - r^3)\,dr d\theta = \int_0^{2\pi}\left[\frac{25}{2}r^2 - \frac{1}{4}r^4\right]_0^5 d\theta \\ &= \int_0^{2\pi}\frac{625}{4}\,d\theta = \frac{625}{4}\cdot 2\pi = \frac{625}{2}\pi\end{aligned}$$

is the volume of the region D. ■

Definition 15.24. *Let $(x,y,z)\in\mathbb{R}^3_{xyz}$. If an ordered triple (ρ,φ,θ) is such that*

$$(x,y,z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi),$$

*then (ρ,φ,θ) are **spherical coordinates** for the point (x,y,z). The **spherical coordinate system**, denoted by $\mathbb{R}^3_{\rho\varphi\theta}$, is the system in which all points in $\mathbb{R}^3$ are given in spherical coordinates. The mapping $T_{\text{sph}}:\mathbb{R}^3_{\rho\varphi\theta}\to\mathbb{R}^3_{xyz}$ given by*

$$T_{\text{sph}}(\rho,\varphi,\theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \tag{15.10}$$

is the conversion transformation to rectangular coordinates.

Let $p=(x,y,z)$ have spherical coordinates (ρ,φ,θ). Let $p_0=(x,y,0)$ be the projection of p onto the xy-plane. In geometrical terms (as will be shown later), ρ is the distance p is from $o=(0,0,0)$, so that $\rho^2=x^2+y^2+z^2$; φ is the angle the ray $\overrightarrow{op}$ makes with the positive z-axis; and θ is the angle the ray $\overrightarrow{op_0}$ makes with the positive x-axis. See Figure 80.

The next example illustrates how a set of points in $\mathbb{R}^3_{xyz}$ may be more conveniently described in terms of the spherical coordinates ρ, φ, and θ.

Example 15.25. Describe the set of points in $\mathbb{R}^3_{xyz}$ that satisfy $\rho = 4\cos\varphi$.

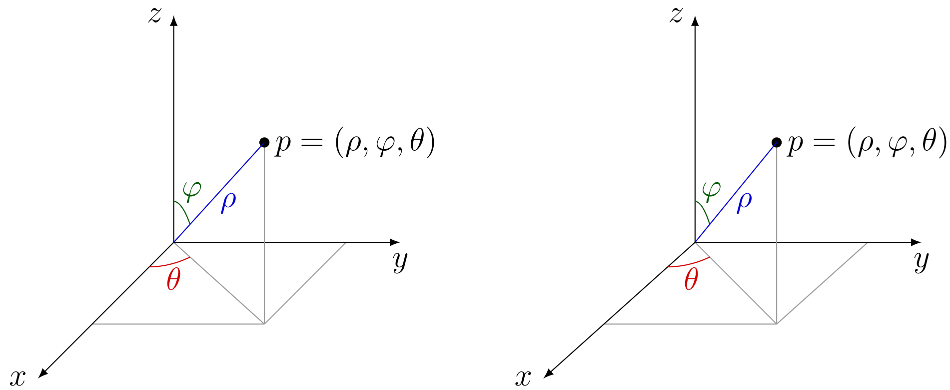


FIGURE 80. Spherical coordinates. Physicists reverse the roles of θ and φ, which conflicts with conventional polar coordinate notation!

Solution. Multiply both sides of $\rho = 4\cos\varphi$ by ρ to obtain $\rho^2 = 4\rho\cos\varphi$. This gives

$$x^2 + y^2 + z^2 = 4z.$$

Thus

$$x^2 + y^2 + (z^2 - 4z + 4) = 4,$$

from which we finally obtain

$$x^2 + y^2 + (z-2)^2 = 4.$$

The set of points in $\mathbb{R}^3_{xyz}$ that satisfy this equation clearly form a sphere centered at $(0,0,2)$ with radius 2. ■

Theorem 15.26. *Let $E \subseteq \mathbb{R}^3_{\rho\varphi\theta}$ be given by*

$$E = \{(\rho, \varphi, \theta) : \alpha \le \theta \le \beta,\ 0 \le g(\theta) \le \varphi \le h(\theta),\ G(\varphi, \theta) \le \rho \le H(\varphi, \theta)\}$$

for continuous g, h, G, H (where $\beta - \alpha \le 2\pi$ and $h(\theta) - g(\theta) \le \pi$). If T_{sph} is one-to-one on $\text{Int}(E)$*, $D = T_{\text{sph}}(E)$, and $f : D \to \mathbb{R}$ is continuous, then*

$$\iiint\limits_D f(x,y,z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(\varphi,\theta)}^{H(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho d\varphi d\theta.$$

This theorem's proof is also a straightforward application of Theorem 15.32 to get

$$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,dV,$$

and then invoking a version of Theorem 15.17.

Example 15.27. Evaluate

$$\iiint\limits_D z\,dV,$$

where D is the region in the first octant that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

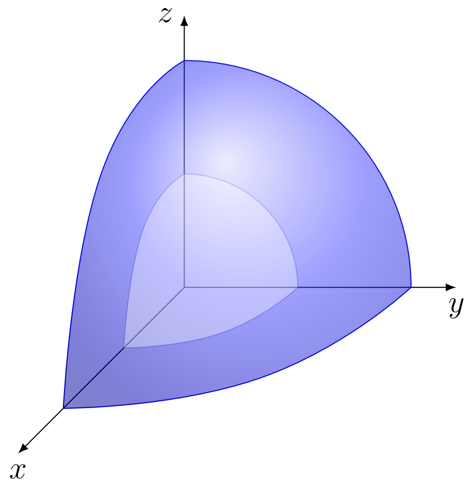


FIGURE 81.

Solution. The region D is shown in Figure 81. In spherical coordinates the sphere $x^2+y^2+z^2=1$, which has center at the origin and radius 1, is given by $\rho=1$; and the sphere $x^2+y^2+z^2=4$, which has center at the origin and radius 2, is given by $\rho=2$.

Since D lies in the first octant, it should be clear that any point in D must have θ-coordinate ranging anywhere from $\theta=0$ to $\theta=\pi/2$; that is, if $(\rho,\varphi,\theta)\in D$, then $0\le\theta\le\pi/2$.

Let $\theta\in[0,\pi/2]$ be fixed. Then if $p=(\rho,\varphi,\theta)\in D$, the φ-coordinate of p can range anywhere from $\varphi=0$ (which places p on the intersection of D with the z-axis) to $\varphi=\pi/2$ (placing p on the intersection of D and the xy-plane). That is, for fixed $0\le\theta\le\pi/2$, we have $0\le\varphi\le\pi/2$.

Let $\theta,\varphi\in[0,\pi/2]$ both be fixed. Then if $p=(\rho,\varphi,\theta)\in D$, the ρ-coordinate of p can range anywhere from $\rho=1$ (placing p on the smaller sphere bounding D) to $\rho=2$ (placing p on the larger sphere bounding D). That is, for fixed $0\le\theta,\varphi\le\pi/2$, we have $1\le\rho\le 2$.

Thus we find that the region $E\subseteq\mathbb{R}^3_{\rho\varphi\theta}$ for which $T_{\text{sph}}(E)=D$ is

$$E=\{(\rho,\varphi,\theta):0\le\theta\le\pi/2,\ 0\le\varphi\le\pi/2,\ 1\le\rho\le 2\},$$

and so by Theorem 15.26

$$\begin{aligned}\iiint\limits_D z\,dV &= \iiint\limits_E \rho\cos\varphi\cdot\rho^2\sin\varphi\,dV\\ &=\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2\rho^3\cos\varphi\sin\varphi\,d\rho d\varphi d\theta=\int_0^{\pi/2}\int_0^{\pi/2}\frac{15}{4}\cos\varphi\sin\varphi\,d\varphi d\theta\\ &=\int_0^{\pi/2}\frac{15}{8}d\theta=\frac{15}{8}\cdot\frac{\pi}{2}=\frac{15\pi}{16}\end{aligned}$$

is the value of the integral. ■

Example 15.28. Find the volume of the region D that lies outside the cone $\varphi=\pi/4$ and inside the sphere $\rho=4\cos\varphi$.

Solution. In $\mathbb{R}^3_{xyz}$ a point that satisfies $\varphi=\pi/4$ is a point that lies on a ray with initial point at $(0,0,0)$ which makes an angle of $\pi/4$ with the positive z-axis, and so the set of all such points

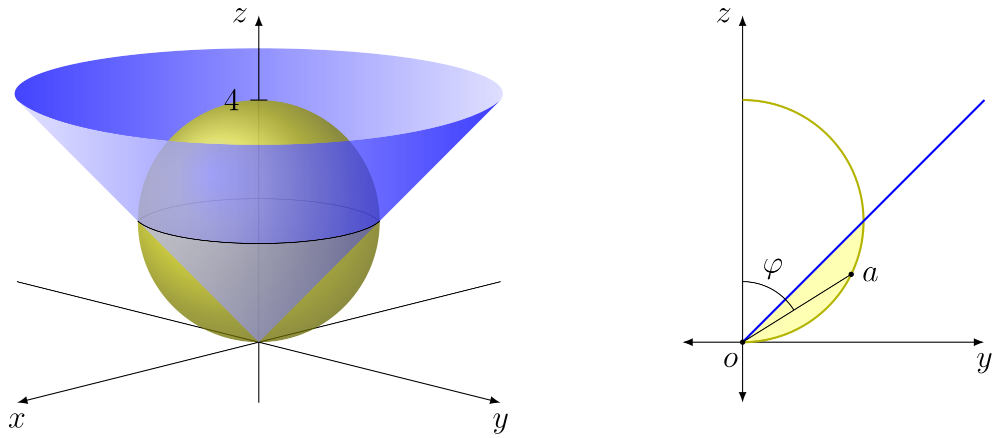


FIGURE 82.

form a cone as shown at left in Figure 82. As for $\rho = 4\cos\varphi$, it was already found in Example 15.25 that this is a sphere centered at $(0,0,2)$ with radius 2, also shown at left in Figure 82. The region $E \subseteq \mathbb{R}^3_{\rho\varphi\theta}$ that the transformation T_{sph} given by (15.10) maps onto D needs to be determined, and this can be done by examining the geometry of the region $D \subseteq \mathbb{R}^3_{xyz}$.

Clearly a point in D can have a θ coordinate ranging anywhere from 0 to 2π, and any cross-section of D at a given value of $\theta \in [0, 2\pi]$ will look the same. Thus, we can examine the cross-section of D that lies in, say, the yz-plane (where $\theta = \pi/2$), as shown at right in Figure 82. A point p in this cross-section can have a φ coordinate ranging from $\pi/4$ to $\pi/2$, and at a given value of $\varphi \in [\pi/4, \pi/2]$ the ρ coordinate of the point can be any value from 0 to $4\cos\varphi$ as indicated by the segment $[o, a]$ at right in Figure 82. By Definition 15.18 and Theorem 15.26

$$
\begin{aligned}
\mathcal{V}(D) &= \iiint_D dV = \iiint_E \rho^2 \sin\varphi \, dV = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \left[\frac{1}{3}\rho^3 \sin\varphi\right]_0^{4\cos\varphi} d\varphi d\theta = \frac{64}{3} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \cos^3\varphi \sin\varphi \, d\varphi \, d\theta \\
&= \frac{64}{3} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{d}{d\varphi}\left(-\frac{1}{4}\cos^4\varphi\right) d\varphi d\theta = \frac{64}{3} \int_0^{2\pi} \left[-\frac{1}{4}\cos^4\varphi\right]_{\pi/4}^{\pi/2} d\theta \\
&= \frac{16}{3}\left(\frac{1}{\sqrt{2}}\right)^4 \int_0^{2\pi} d\theta = \frac{4}{3} \cdot 2\pi = \frac{8\pi}{3}
\end{aligned}
$$

is the volume of the region D. ■

15.6 – Multiple Integral Change of Variables

Where we used to write $x = g(u, v)$ and $y = h(u, v)$, we shall now write $x = x(u, v)$ and $y = y(u, v)$. Thus, x is a function that receives a point $(u, v) \in \mathbb{R}^2_{uv}$ and returns a real number x, and y is a function that receives a point $(u, v) \in \mathbb{R}^2_{uv}$ and returns a real number y. In this way a point $(x, y) \in \mathbb{R}^2_{xy}$ is obtained. This is how the transformation T in the definition that follows operates.

Definition 15.29. *Let $S \subseteq \mathbb{R}^2_{uv}$ be a region. If $T : S \to \mathbb{R}^2_{xy}$ is a transformation given by $T(u, v) = (x(u, v), y(u, v))$, where $x, y : \mathbb{R}^2_{uv} \to \mathbb{R}$ are differentiable functions on S, then the* ***Jacobian*** *of T is the function $J_T : S \to \mathbb{R}$ given by*

$$J_T(u, v) = \begin{vmatrix} x_u(u, v) & x_v(u, v) \\ y_u(u, v) & y_v(u, v) \end{vmatrix} = x_u(u, v)y_v(u, v) - x_v(u, v)y_u(u, v).$$

Thus $J_T = x_u y_v - x_v y_u$. Another symbol used for J_T is

$$\frac{\partial(x, y)}{\partial(u, v)},$$

but this will not be used here.

Theorem 15.30 (**Double Integral Change of Variables**). *Let $S \subseteq \mathbb{R}^2_{uv}$ be a closed, bounded region, $T : S \to \mathbb{R}^2_{xy}$ a mapping given by $T(u, v) = (x(u, v), y(u, v))$, and $R = T(S)$. If T is one-to-one on $\operatorname{Int}(S)$, x and y have continuous first partials on $\operatorname{Int}(S)$, and $f : R \to \mathbb{R}$ is continuous, then*

$$\iint_R f(x, y)\, dA = \iint_S (f \circ T)(u, v)\big|J_T(u, v)\big|\, dA.$$

Moving up a dimension, we make the following definition.

Definition 15.31. *Let $E \subseteq \mathbb{R}^3_{uvw}$ be a region. If $T : E \to \mathbb{R}^3_{xyz}$ is a mapping given by*

$$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)),$$

where $x, y, z : \mathbb{R}^3_{uvw} \to \mathbb{R}$ are differentiable functions on E, then the ***Jacobian*** *of T is the function $J_T : S \to \mathbb{R}$ given by*

$$J_T = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

It's understood that the arguments of all the functions featured in the equation for J_T in this definition are (u, v, w). Another symbol used instead of J_T is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)},$$

which has some descriptive advantages but shall nonetheless be shunned here.

Theorem 15.32 (Triple Integral Change of Variables). *Let $E \subseteq \mathbb{R}^3_{uvw}$ be a closed, bounded region, $T : E \to \mathbb{R}^3_{xyz}$ a mapping given by*

$$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)),$$

and $D = T(E)$. If T is one-to-one on $\operatorname{Int}(E)$, x, y, and z are of class $\mathcal{C}'$ on $\operatorname{Int}(E)$, and $f : D \to \mathbb{R}$ is continuous, then

$$\iiint\limits_D f(x, y, z)\, dV = \iiint\limits_E (f \circ T)(u, v, w) \big| J_T(u, v, w) \big| \, dV.$$

16
Vector Calculus

16.1 – Vector Fields

A region $R \subseteq \mathbb{R}^2$ is called **vertically simple** if there exist continuous functions f_1 and f_2 such that

$$R = \{(x, y) : a \leq x \leq b \text{ and } f_1(x) \leq y \leq f_2(x)\}.$$

Similarly, R is **horizontally simple** if there exist continuous functions g_1 and g_2 such that

$$R = \{(x, y) : c \leq y \leq d \text{ and } g_1(y) \leq x \leq g_2(y)\}.$$

Later in this chapter there will be frequent consideration of a region R in $\mathbb{R}^2$ that is enclosed by a planar curve C, in which case it is taken as understood that C is also part of R; that is, $C \subseteq R$, and so R is a closed set.

Before defining a vector field, we summarize some terminology first introduced in Chapter 13. Recall that a function of several independent variables is said to be continuously differentiable on an open set U if it has continuous first partial derivatives on U (i.e. the first partials exist and are continuous at every point in U). A real-valued function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be differentiable (resp. continuously differentiable) on an arbitrary set $S \subseteq D$ if there exists an open set U such that $S \subseteq U \subseteq D$ and f is differentiable (resp. continuously differentiable) on U. For f to be continuous on S, of course, simply means f is continuous at each point in S regardless of whether S happens to be an open set or not.

Definition 16.1. *Let R be a region in $\mathbb{R}^2$, and suppose $f, g : R \to \mathbb{R}$. A **vector field** in R is a function $\mathbf{F} : R \to \mathbb{R}^2$ given by*

$$\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle,$$

and we say $\mathbf{F}$ is continuous, differentiable, or continuously differentiable on R if both f and g are continuous, differentiable, or continuously differentiable on R, respectively.

*More generally, if $f_1, \ldots, f_n : R \subseteq \mathbb{R}^n \to \mathbb{R}$, where $n \geq 2$, then a **vector field** in R is a function $\mathbf{F} : R \to \mathbb{R}^n$ given by*

$$\mathbf{F}(\mathbf{x}) = \langle f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\rangle. \tag{16.1}$$

F *is said to be continuous, differentiable, or continuously differentiable on* R *if* $f_1, \ldots, f_n$ *are all continuous, differentiable, or continuously differentiable on* R*, respectively.*

The function $\mathbf{F}$ given by (16.1) can also be denoted by $\langle f_1, \ldots, f_n \rangle$, so that $\mathbf{F} = \langle f_1, \ldots, f_n \rangle$. The natural way to think of $\mathbf{F}$ is as a function that assigns a vector

$$\langle f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}) \rangle$$

to each point $\mathbf{x} = \langle x_1, \ldots, x_n \rangle \in R$.

Suppose C is a smooth curve in $\mathbb{R}^n$ and $\mathbf{x} \in C$. We say a vector field $\mathbf{F}$ is **orthogonal** to C at $\mathbf{x}$ if the vector $\mathbf{F}(\mathbf{x})$ (i.e. the vector that $\mathbf{F}$ assigns to $\mathbf{x}$) is orthogonal to the tangent line to C at $\mathbf{x}$; that is $\mathbf{F}(\mathbf{x})$ is a normal vector for C at $\mathbf{x}$. Similarly, $\mathbf{F}$ is **parallel** to C at $\mathbf{x}$ if $\mathbf{F}(\mathbf{x})$ is parallel to the tangent line to C at $\mathbf{x}$; that is $\mathbf{F}(\mathbf{x})$ is a tangent vector for C at $\mathbf{x}$.

Example 16.2. Determine whether the vector field

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{4}, \frac{x}{4} \right\rangle$$

is orthogonal to or parallel to the points on the curve $C = \{(x, y) : x^2 + y^2 = 19.36\}$.

Solution. One approach to take is to parameterize C, which is a circle centered at $(0, 0)$ with radius $22/5$, by

$$\mathbf{r}(t) = \frac{22}{5}\langle \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi.$$

Then an arbitrary point (x, y) on C corresponding to parameter value t is given by

$$(x(t), y(t)) = \left(\frac{22\cos t}{5}, \frac{22\sin t}{5} \right)$$

for some $t \in [0, 2\pi]$, which is to say $x(t) = 4.4\cos t$ and $y(t) = 4.4\sin t$. Thus

$$\mathbf{F}(x(t), y(t)) = \frac{1}{4}\langle -y(t), x(t) \rangle = \frac{11}{10}\langle -\sin t, \cos t \rangle,$$

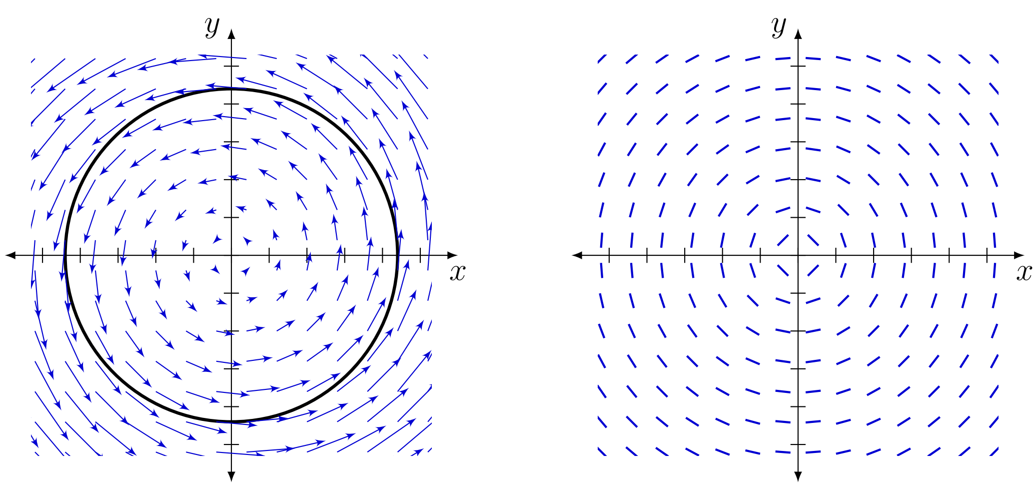


FIGURE 83. At left, the vector field $\mathbf{F}(x, y) = \langle -y/4, x/4 \rangle$ along with the curve C. At right, the direction field for $\mathbf{F}$.

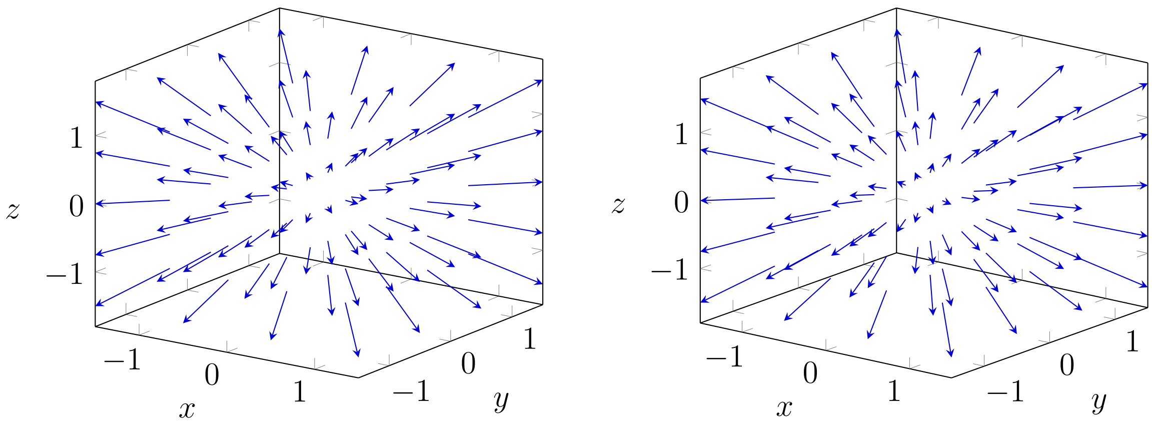


FIGURE 84. The vector field $\mathbf{F}(x, y, z) = \frac{1}{2}\langle x, y, z\rangle$.

while a tangent vector to C at $(x(t), y(t))$ is given by

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \frac{22}{5}\langle -\sin t, \cos t\rangle.$$

It's now seen that

$$\mathbf{F}(x(t), y(t)) = \frac{1}{4}\mathbf{r}'(t);$$

that is, $\mathbf{F}(x(t), y(t))$ is a nonzero scalar multiple of $\mathbf{r}'(t)$, which shows that $\mathbf{F}(x(t), y(t))$ is parallel to $\mathbf{r}'(t)$. Therefore $\mathbf{F}(x(t), y(t))$ is parallel to the tangent line to C at $(x(t), y(t))$, and since $(x(t), y(t)) \in C$ is arbitrary we can conclude that $\mathbf{F}$ is parallel to all points on C. See Figure 83. ■

In Figure 83 we see an example of a vector field in $\mathbb{R}^2$. A vector field in $\mathbb{R}^3$ can be harder to visualize, but the stereoscopic image in Figure 84 gives depth to the vector field

$$\mathbf{F}(x, y, z) = \frac{1}{2}\langle x, y, z\rangle.$$

We have actually encountered vector fields already. Given a differentiable scalar-valued function $\varphi : R \subseteq \mathbb{R}^2 \to \mathbb{R}$, say, there is the gradient of φ,

$$\nabla\varphi(x, y) = \langle \varphi_x(x, y), \varphi_y(x, y)\rangle\,,$$

written as $\nabla\varphi = \langle \varphi_x, \varphi_y\rangle$.

Definition 16.3. *Let φ be a differentiable function on an open region $R \subseteq \mathbb{R}^n$. The vector field $\mathbf{F} = \nabla\varphi$ is a **gradient field**, and φ is a **potential function** for $\mathbf{F}$.*

Returning to the plane, let φ be a function with continuous first partials on an open region $R \subseteq \mathbb{R}^2$, and let $\mathbf{F}$ be a vector field on R for which φ is a potential function. Suppose C is the level curve $\varphi(x, y) = c$ of φ, where c is some constant and $C \subseteq R$. (The level curves of φ are called **equipotential curves**.) If $(x, y) \in C$ and $\mathbf{F}(x, y) = \nabla\varphi(x, y) \neq \mathbf{0}$, then by Proposition 13.46 the vector $\mathbf{F}(x, y)$ is orthogonal to C at (x, y).[14]

[14]Of course, if $\mathbf{F}(x, y) = \mathbf{0}$ then it is trivially orthogonal to all vectors and curves.

16.2 – Line Integrals

Let R be a region in $\mathbb{R}^2$, let f be a real-valued function $f(x, y) = z$ with domain R, and let C be a smooth curve in R parameterized in terms of arc length s by

$$\mathbf{r}(s) = \langle x(s), y(s), 0\rangle, \quad s \in [a, b].$$

To give the developments that follow a geometrical motivation, assume that $f(x, y) \geq 0$ for all $(x, y) \in R$, so that f generates a surface S given by

$$S = \{(x, y, f(x, y)) : (x, y) \in R\}$$

in $\mathbb{R}^3$ that lies above R in the xy-plane, and in particular f traces out a curve C_f given by

$$\boldsymbol{\rho}(s) = \big\langle x(s), y(s), f(x(s), y(s))\big\rangle, \quad s \in [a, b],$$

that lies in S directly above C. What would be the "area" of the surface Σ in $\mathbb{R}^3$ that lies "between" the curves C below and C_f above? To be more precise, for each $a \leq s \leq b$ let $[\mathbf{r}(s), \boldsymbol{\rho}(s)]$ be the line segment in $\mathbb{R}^3$ with endpoints $\mathbf{r}(s) \in C$ and $\boldsymbol{\rho}(s) \in C_f$. Then the surface Σ is the union of all the points on all of these line segments:

$$\Sigma = \bigcup_{s\in[a,b]} [\mathbf{r}(s), \boldsymbol{\rho}(s)]$$

We return to the question: what is the area $\mathcal{A}(\Sigma)$ of Σ? More to the point, what would be a natural *definition* for the area of Σ? We start by subdividing C into n smaller curves by forming a partition P of $[a, b]$:

$$a = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = b$$

For each $k = 1, 2, \ldots, n$, let

$$C_k = \{\langle x(s), y(s)\rangle : s \in [s_{k-1}, s_k]\},$$

so C_k is the piece of C we get when we restrict ourselves to $s_{k-1} \leq s \leq s_k$, and since our parameter is arc length we know that the length of C_k is $\Delta s_k = s_k - s_{k-1}$. Now, for each k, choose a point (any point) s_k^* from the kth subinterval $[s_{k-1}, s_k]$, which corresponds to a point $(x(s_k^*), y(s_k^*))$ on C_k. Our subdivision of C now gives rise to a subdivision of Σ into n "panels," with the kth panel being the piece of Σ that lies above C_k. The height of the kth panel can be taken to be approximately $f(x(s_k^*), y(s_k^*))$, which then enables us to approximate the area of the kth panel as

$$(\text{height})(\text{length}) = f(x(s_k^*), y(s_k^*))\Delta s_k.$$

Then, the total area of S can be approximated by a Riemann sum:

$$\mathcal{A}(\Sigma) \approx \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*))\Delta s_k.$$

Now, let

$$\|P\| = \max\{\Delta s_k : 1 \leq k \leq n\},$$

and note that if $\|P\|$ converges to 0 then of necessity n must converge to ∞. We define the area of Σ as follows:

$$\mathcal{A}(\Sigma) = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*))\Delta s_k.$$

The limit, if it exists, is called the line integral of f over C, and we write

$$\int_C f(x(s), y(s))\, ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*))\Delta s_k.$$

Many references are quick to point out that a line integral is more accurately referred to as a curve integral. The general definition in $\mathbb{R}^n$ is as follows.

Definition 16.4. *Let $C \subseteq \mathbb{R}^n$ be a piecewise smooth curve parameterized in terms of arc length s by*

$$\mathbf{r}(s) = \langle x_1(s), \ldots, x_n(s)\rangle, \quad s \in [a, b], \tag{16.2}$$

and let $f : C \to \mathbb{R}$. The ***line integral of*** f ***over*** C *is*

$$\int_C f = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\mathbf{r}(s_k^*))\Delta s_k,$$

provided the limit exists. If the limit exists then f is said to be ***integrable*** *on C.*

As with the definite integral defined in Chapter 5 the interval $[a, b]$ can be replaced by an infinite interval such as $[a, \infty)$ or $(-\infty, b]$, but we shan't go into that here. Alternate symbols for the line integral of f over C are, in order of increasing long-windedness,

$$\int_C f\, ds, \quad \int_C f(\mathbf{r}(s))\, ds, \quad \text{and} \quad \int_C f(x_1(s), \ldots, x_n(s))\, ds.$$

It's a fact, not to be shown here, that if a piecewise smooth curve C has *finite* length, and $f : C \to \mathbb{R}$ is continuous, then f will be integrable on C. Also, notice that if $f(\mathbf{r}(s)) = 1$ for all $s \in [a, b]$ then

$$\int_C f = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta s_k = \lim_{\|P\|\to 0} \mathcal{L}(C) = \mathcal{L}(C),$$

where $\mathcal{L}(C)$ is our symbol for the length of C.

While Definition 16.4 is nice conceptually, it offers no easy means of evaluating line integrals. But do not despair! A quick manipulation yields

$$\int_C f = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (f \circ \mathbf{r})(s_k^*)\Delta s_k = \int_a^b (f \circ \mathbf{r})(s)\, ds$$

by Theorem 5.10; for after all, $f \circ \mathbf{r}$ is nothing more than a real-valued function defined on an interval $[a, b]$. In short,

$$\int_C f = \int_a^b f(\mathbf{r}(s))\, ds. \tag{16.3}$$

This equation may seem to be wholly natural, but the catch is that it is only valid if C is parameterized in terms of arc length. What if it is not?

Suppose C is a smooth curve that has a parametrization in terms of arc length given by (16.2), and moreover arc length s itself is given as a function of some other arbitrary parameter t such that $s = s(t)$ for $\alpha \leq t \leq \beta$. Then $\boldsymbol{\rho}(t) = \mathbf{r}(s(t))$, $t \in [\alpha, \beta]$, is a parametrization of C in terms of t. We assume that $s(t)$ increases monotonically as t increases from α to β, so that $s(\alpha) = a$, $s(\beta) = b$, and $s'(t) \geq 0$ for all $\alpha \leq t \leq \beta$. (We define $s'(\alpha)$ and $s'(\beta)$ by one-sided limits as is done in the definition of a smooth curve in §12.6.) Now, recalling that $\|\mathbf{r}'(s)\| = 1$ for all s (see §12.8), we obtain

$$\boldsymbol{\rho}'(t) = \mathbf{r}'(s(t))s'(t) \quad \Rightarrow \quad \|\boldsymbol{\rho}'(t)\| = \|\mathbf{r}'(s(t))\||s'(t)| \quad \Rightarrow \quad \|\boldsymbol{\rho}'(t)\| = |s'(t)|,$$

and thus $s'(t) = \|\boldsymbol{\rho}'(t)\|$. We now employ an argument that amounts to little more than an application of the Substitution Rule for Definite Integrals as given in §5.5:

$$\begin{aligned}
\int_a^b f(\mathbf{r}(s))\,ds &= \int_{s(\alpha)}^{s(\beta)} (f \circ \mathbf{r})(s)\,ds && \text{Since } s(\alpha) = a \text{ and } s(\beta) = b \\
&= \int_\alpha^\beta (f \circ \mathbf{r})(s(t))s'(t)\,dt && \text{Substitution Rule with } s = s(t) \\
&= \int_\alpha^\beta f(\mathbf{r}(s(t)))s'(t)\,dt && \text{By definition of } f \circ \mathbf{r} \\
&= \int_\alpha^\beta f(\boldsymbol{\rho}(t))\|\boldsymbol{\rho}'(t)\|\,dt && \boldsymbol{\rho}(t) = \mathbf{r}(s(t)) \text{ and } s'(t) = \|\boldsymbol{\rho}'(t)\|
\end{aligned}$$

It might be helpful to follow the steps in the reverse direction. Combining the result above with (16.3), we obtain

$$\int_C f = \int_\alpha^\beta f(\boldsymbol{\rho}(t))\|\boldsymbol{\rho}'(t)\|\,dt.$$

The same result is obtained even if we assume that $s(t)$ is monotonically *decreasing*, so that $s(\alpha) = b$, $s(\beta) = a$, and $s'(t) \leq 0$ for all $\alpha \leq t \leq \beta$ (and it is a worthwhile exercise to verify this). We have now obtained a formula that enables us to evaluate a line integral as a conventional definite integral no matter how C is parameterized, at least as long as the parametrization is smooth.

Suppose C is a piecewise smooth curve with parametrization (16.2). Then we may find a partition of $[a, b]$,

$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b,$$

such that, for each $1 \leq i \leq n$, the curve C_i given by $\mathbf{r}_i(s) = \mathbf{r}(s)$ for $s \in [s_{i-1}, s_i]$ is smooth. We may then apply Theorem 5.19 to obtain

$$\int_C f = \int_a^b f(\mathbf{r}(s))\,ds = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} f(\mathbf{r}_i(s))\,ds.$$

Now, for each i let $\boldsymbol{\rho}_i(t) = \mathbf{r}_i(s(t))$, $t \in [t_{i-1}, t_i]$, be some other parametrization of C_i, where

$$\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta.$$

By the same arguments as above we find that

$$\int_{C_i} f = \int_{s_{i-1}}^{s_i} f(\mathbf{r}_i(s))\,ds = \int_{t_{i-1}}^{t_i} f(\boldsymbol{\rho}_i(t))\|\boldsymbol{\rho}_i'(t)\|\,dt$$

for each i, and thus by Theorem 5.19

$$\int_C f = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} f(\mathbf{r}_i(s))\,ds = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f(\boldsymbol{\rho}_i(t))\|\boldsymbol{\rho}_i'(t)\|\,dt = \int_\alpha^\beta f(\boldsymbol{\rho}(t))\|\boldsymbol{\rho}'(t)\|\,dt$$

once again. We have now proved the following theorem, applicable in $\mathbb{R}^n$ for $n \geq 2$.

Theorem 16.5. *If C is a piecewise smooth curve given by*

$$\mathbf{r}(t) = \langle x_1(t), \ldots, x_n(t)\rangle, \quad t \in [a, b], \tag{16.4}$$

and $f : C \to \mathbb{R}$ is continuous, then

$$\int_C f = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt.$$

An immediate consequence of the proof of this theorem is the following, which makes clear that the value of a line integral over a smooth curve is independent of the smooth vector function chosen to parameterize the curve.

Corollary 16.6. *If $\boldsymbol{\rho}_1(t)$, $t \in [\alpha_1, \beta_1]$, and $\boldsymbol{\rho}_2(t)$, $t \in [\alpha_2, \beta_2]$, are parametrizations of a smooth curve C, then*

$$\int_{\alpha_1}^{\beta_1} f(\boldsymbol{\rho}_1(t))\|\boldsymbol{\rho}_1'(t)\|dt = \int_{\alpha_2}^{\beta_2} f(\boldsymbol{\rho}_2(t))\|\boldsymbol{\rho}_2'(t)\|dt$$

Proof. Let $\mathbf{r}(s)$, $s \in [a, b]$, be a parametrization of C in terms of arc length. Then

$$\int_{\alpha_1}^{\beta_1} f(\boldsymbol{\rho}_1(t))\|\boldsymbol{\rho}_1'(t)\|dt = \int_C f(\mathbf{r}(s))\,ds = \int_{\alpha_2}^{\beta_2} f(\boldsymbol{\rho}_2(t))\|\boldsymbol{\rho}_2'(t)\|dt,$$

which finishes the proof. ∎

If C_1 is an oriented curve from $\mathbf{x}_1$ to $\mathbf{y}_1$, and C_2 is an oriented curve from $\mathbf{x}_2$ to $\mathbf{y}_2$, then the **concatenation** of C_1 and C_2, written $C_1 + C_2$, may be regarded as the oriented curve C that starts at $\mathbf{x}_1$, follows C_1 to $\mathbf{y}_1$, and then jumps to $\mathbf{x}_2$ and follows C_2 to $\mathbf{y}_2$. Oftentimes $\mathbf{y}_1 = \mathbf{x}_2$ (i.e. C_1 ends where C_2 begins), but it is not necessary.

Let C_1 and C_2 be curves parameterized by $\mathbf{r}_1(t)$, $t \in [a, b]$, and $\mathbf{r}_2(t)$, $t \in [b, c]$, respectively. Suppose $\mathbf{r}_1(b) = \mathbf{r}_2(b)$. Then $C_1 + C_2$ is a curve that may be parameterized by

$$\mathbf{r}(t) = \begin{cases} \mathbf{r}_1(t), & a \leq t \leq b \\ \mathbf{r}_2(t), & b \leq t \leq c \end{cases}$$

Moreover, if C_1 and C_2 are each smooth curves, then $C = C_1 + C_2$ is piecewise smooth, and by Theorem 16.5 along with Theorem 5.19 we have

$$\int_C f = \int_a^c f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt = \int_a^b f(\mathbf{r}_1(t))\|\mathbf{r}_1'(t)\|dt + \int_b^c f(\mathbf{r}_2(t))\|\mathbf{r}_2'(t)\|dt = \int_{C_1} f + \int_{C_2} f.$$

That is,

$$\int_{C_1+C_2} f = \int_{C_1} f + \int_{C_2} f. \tag{16.5}$$

As another example we may have C_1 parameterized by $\mathbf{r}_1(t)$, $t \in [a,b]$, and C_2 by $\mathbf{r}_2(t)$, $t \in [a,b]$, with $\mathbf{r}_1(b) \neq \mathbf{r}_2(a)$; that is, C_1 does not end at the same point where C_2 begins. We could still, if desired, parameterize $C_1 + C_2$ with a single piecewise defined function as follows

$$\mathbf{r}(t) = \begin{cases} \mathbf{r}_1(t), & a \leq t < b \\ \mathbf{r}_2(t-b+a), & b \leq t \leq 2b-a \end{cases}$$

For instance, if C_1 is given by $\mathbf{r}_1(t) = \langle \cos t, \sin t \rangle$, $t \in [0, 2\pi]$, and C_2 by $\mathbf{r}_2(t) = \langle 3\cos t, 3\sin t \rangle$, $t \in [0, 2\pi]$, then the concatenation $C_1 + C_2$ may be parameterized by

$$\mathbf{r}(t) = \begin{cases} \langle \cos t, \sin t \rangle, & 0 \leq t < 2\pi \\ \langle 3\cos t, 3\sin t \rangle, & 2\pi \leq t \leq 4\pi \end{cases}$$

In general, given curves $C_1, \ldots, C_k$ in $\mathbb{R}^n$, it is not desirable to determine a single (piecewise defined) function $\mathbf{r}$ that parameterizes the concatenation $C = C_1 + \cdots + C_k$, regardless of whether C is continuous or not. Indeed, if $C_i \cap C_j = \varnothing$ whenever $i \neq j$, so that the curves $C_1, \ldots, C_k$ are mutually disjoint, there is little to be gained by insisting that travel along $C_1 + \cdots + C_k$ be taken in any particular order such as $C_1, C_2, \ldots, C_k$ or $C_k, C_{k-1}, \ldots, C_1$. Each of the curves $C_1, \ldots, C_k$ can be parameterized by functions $\mathbf{r}_1, \ldots, \mathbf{r}_k$ having distinct or identical domains, depending on whatever is most convenient. The only question is, how should a line integral over $C_1 + \cdots + C_n$ be defined?

The equation (16.5) obtained above in the case of a piecewise smooth curve $C_1 + C_2$ that has smooth "pieces" C_1 and C_2 that are linked end-to-end (which is to say $C_1 + C_2$ is a continuous curve) motivates the following definition concerning the line integral over an arbitrary concatenation of piecewise smooth curves.

Definition 16.7. *Given any finite collection of piecewise smooth curves $C_1, C_2, \ldots, C_k$ in $\mathbb{R}^n$, we define*

$$\int_{C_1+C_2+\cdots+C_k} f = \int_{C_1} f + \int_{C_2} f + \cdots + \int_{C_k} f. \tag{16.6}$$

Example 16.8. Let $\mathbf{p} = \langle -1, 0 \rangle$, $\mathbf{q} = \langle 0, 1 \rangle$ and $\mathbf{r} = \langle 1, 0 \rangle$. Evaluate $\int_C f$, where

$$f(x,y) = 2x - 3y$$

and $C = [\mathbf{p}, \mathbf{q}] + [\mathbf{q}, \mathbf{r}]$.

Solution. By (16.6) we have $\int_C f = \int_{[\mathbf{p},\mathbf{q}]} f + \int_{[\mathbf{q},\mathbf{r}]} f$. Parameterize $[\mathbf{p}, \mathbf{q}]$ by

$$\mathbf{r}_1(t) = (1-t)\mathbf{p} + t\mathbf{q} = \langle -1, 0 \rangle + t\langle 1, 1 \rangle = \langle t-1, t \rangle, \quad t \in [0,1],$$

and parameterize $[\mathbf{q}, \mathbf{r}]$ by

$$\mathbf{r}_2(t) = (1-t)\mathbf{q} + t\mathbf{r} = \langle 0, 1 \rangle + t\langle 1, -1 \rangle = \langle t, 1-t \rangle, \quad t \in [0,1].$$

Now we obtain

$$\begin{aligned}
\int_C f &= \int_{[\mathbf{p},\mathbf{q}]} f + \int_{[\mathbf{q},\mathbf{r}]} f = \int_0^1 f(\mathbf{r}_1(t))\|\mathbf{r}_1'(t)\|dt + \int_0^1 f(\mathbf{r}_2(t))\|\mathbf{r}_2'(t)\|dt \\
&= \sqrt{2}\int_0^1 f(t-1,t)dt + \sqrt{2}\int_0^1 f(t,1-t)dt \\
&= \sqrt{2}\int_0^1 \big[2(t-1)-3t\big]dt + \sqrt{2}\int_0^1 \big[2t-3(1-t)\big]dt \\
&= -\sqrt{2}\int_0^1 (t+2)dt + \sqrt{2}\int_0^1 (5t-3)dt \\
&= -\sqrt{2}\left[\frac{1}{2}t^2+2t\right]_0^1 + \sqrt{2}\left[\frac{5}{2}t^2-3t\right]_0^1 = -3\sqrt{2},
\end{aligned}$$

by Theorem 16.5. ■

In what follows recall that for any given vector function $\mathbf{r}$ the unit tangent vector function $\mathbf{T}$ is given by $\mathbf{T} = \mathbf{r}'/\|\mathbf{r}'\|$.

Definition 16.9. *Let $\mathbf{F}$ be a continuous vector field on a region $R \subseteq \mathbb{R}^n$, and let C be a smooth oriented curve in R having parametrization with respect to arc length $\mathbf{r}(s)$, $s \in [a,b]$, consistent with the orientation. The* ***line integral of $\mathbf{F}$ over** C is*

$$\int_C \mathbf{F}\cdot\mathbf{T} = \int_C \mathbf{F}(\mathbf{r}(s))\cdot\mathbf{T}(s)\,ds. \tag{16.7}$$

If $R \subseteq \mathbb{R}^3$, then $\int_C \mathbf{F}\cdot\mathbf{T}$ is the ***circulation of $\mathbf{F}$ on** C.*

Another notation for the line integral of $\mathbf{F}$ over C as defined by (16.7) is $\int_C \mathbf{F}\cdot d\mathbf{r}$, and yet another notation will be introduced at the end of §16.3. Recalling from Proposition 13.23 that $\|\mathbf{r}'(s)\| = 1$ whenever s is arc length, we have

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s),$$

and so

$$\int_C \mathbf{F}\cdot\mathbf{T} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(s))\cdot\mathbf{r}'(s)\,ds. \tag{16.8}$$

The question now arises: how is $\int_C \mathbf{F}\cdot\mathbf{T}$ to be evaluated if C is not parameterized in terms of arc length, but rather in terms of an arbitrary parameter t? To find out, let $\boldsymbol{\rho}(t)$, $t \in [\alpha,\beta]$, be any smooth parametrization of C. Then there exists some function s such that $\boldsymbol{\rho}(t) = \mathbf{r}(s(t))$ for $\alpha \le t \le \beta$, and as on page 400 we find that $s'(t) = \|\boldsymbol{\rho}'(t)\|$ if s is assumed to be a monotone increasing function, so that $a = s(\alpha)$ and $b = s(\beta)$. Then, since $\boldsymbol{\rho}'(t) = \mathbf{r}'(s(t))s'(t)$, we obtain

$$\mathbf{r}'(s(t)) = \frac{\boldsymbol{\rho}'(t)}{s'(t)} = \frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|}. \tag{16.9}$$

We now argue as in the proof of Theorem 16.5, picking up where (16.8) left off:

$$\int_a^b \mathbf{F}(\mathbf{r}(s))\cdot\mathbf{r}'(s)\,ds = \int_{s(\alpha)}^{s(\beta)} (\mathbf{F}\circ\mathbf{r})(s)\cdot\mathbf{r}'(s)\,ds \qquad \text{Since } a = s(\alpha),\ b = s(\beta)$$

$$
\begin{aligned}
&= \int_{s(\alpha)}^{s(\beta)} \big((\mathbf{F}\circ\mathbf{r})\cdot\mathbf{r}'\big)(s)\,ds \\
&= \int_{\alpha}^{\beta} \big((\mathbf{F}\circ\mathbf{r})\cdot\mathbf{r}'\big)(s(t))s'(t)\,dt && \text{Substitution: } s=s(t) \\
&= \int_{\alpha}^{\beta} \mathbf{F}(\boldsymbol{\rho}(t))\cdot\mathbf{r}'(s(t))\|\boldsymbol{\rho}'(t)\|\,dt && \boldsymbol{\rho}(t)=\mathbf{r}(s(t)),\ s'(t)=\|\boldsymbol{\rho}'(t)\| \\
&= \int_{\alpha}^{\beta} \mathbf{F}(\boldsymbol{\rho}(t))\cdot\boldsymbol{\rho}'(t)\,dt && \text{By equation (16.9)}
\end{aligned}
$$

We have now proved the following.

Proposition 16.10. *Let* $\mathbf{F}$ *be continuous vector field on a region in* $\mathbb{R}^n$ *containing a smooth curve* C *parameterized by* $\mathbf{r}(t)$, $t\in[a,b]$. *Then*

$$
\int_C \mathbf{F}\cdot\mathbf{T} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \tag{16.10}
$$

If $\mathbf{F}$ is a force field (say a magnetic or gravitational field), then the **work** W done in moving an object along C over time t in accordance with the position function $\mathbf{r}$ is given by (16.10).

Example 16.11. Find the work required to move an object from point $(2,0,0)$ to $(2,0,1)$ along the helical curve C given by

$$
\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t/2\pi\rangle, \quad 0\le t\le 2\pi,
$$

in the force field $\mathbf{F}(x,y,z)=\langle -y,x,z\rangle$.

Solution. Putting Proposition 16.10 to work, we have

$$
\begin{aligned}
W &= \int_C \mathbf{F}\cdot\mathbf{T} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_0^{2\pi} \mathbf{F}\Big(2\cos t, 2\sin t, \frac{t}{2\pi}\Big)\cdot\Big\langle -2\sin t, 2\cos t, \frac{1}{2\pi}\Big\rangle\,dt \\
&= \int_0^{2\pi} \Big\langle -2\sin t, 2\cos t, \frac{t}{2\pi}\Big\rangle\cdot\Big\langle -2\sin t, 2\cos t, \frac{1}{2\pi}\Big\rangle\,dt \\
&= \int_0^{2\pi} \Big(4\sin^2 t + 4\cos^2 t + \frac{t}{4\pi^2}\Big)dt = \int_0^{2\pi}\Big(4+\frac{t}{4\pi^2}\Big)\,dt \\
&= \Big[4t + \frac{t^2}{8\pi^2}\Big]_0^{2\pi} = 8\pi + \frac{1}{2}.
\end{aligned}
$$

If force is measured in newtons N and distance in meters m, then the work is approximately 25.63 J. ■

The following proposition informs us that the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ does not change if we change our choice of parametrization for C, at least so long as we do not change the orientation of C.

Proposition 16.12. *Let* $\mathbf{F}$ *be a continuous vector field on a region in* $\mathbb{R}^n$ *containing a smooth curve* C *parameterized by* $\boldsymbol{\rho}_1(t)$, $t \in [\alpha_1, \beta_1]$. *If* $\boldsymbol{\rho}_2(t)$, $t \in [\alpha_2, \beta_2]$ *is another parametrization for* C *consistent with the orientation induced by* $\boldsymbol{\rho}_1$, *then*

$$\int_C \mathbf{F} \cdot d\boldsymbol{\rho}_1 = \int_C \mathbf{F} \cdot d\boldsymbol{\rho}_2.$$

Thus, the value of the line integral of $\mathbf{F}$ *over* C *is invariant under orientation-preserving reparametrization.*

Proof. Let $\boldsymbol{\rho}_1(t)$, $t \in [\alpha_1, \beta_1]$, and $\boldsymbol{\rho}_2(t)$, $t \in [\alpha_2, \beta_2]$, be two parametrizations for C that induce the same orientation, and let $\mathbf{r}(s)$, $s \in [a, b]$, be a parametrization for C with respect to arc length that is consistent with this orientation. By Proposition 16.10

$$\begin{aligned}\int_C \mathbf{F} \cdot d\boldsymbol{\rho}_1 &= \int_{\alpha_1}^{\beta_1} \mathbf{F}(\boldsymbol{\rho}_1(t)) \cdot \boldsymbol{\rho}_1'(t)\, dt = \int_a^b \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{r}'(s)\, ds \\ &= \int_{\alpha_2}^{\beta_2} \mathbf{F}(\boldsymbol{\rho}_2(t)) \cdot \boldsymbol{\rho}_2'(t)\, dt = \int_C \mathbf{F} \cdot d\boldsymbol{\rho}_2,\end{aligned}$$

where in particular the steps that justify the second and third equalities can be found in the proof of Proposition 16.10. ∎

Given an oriented curve C, let $-C$ denote the same curve, *but with opposite orientation.* Thus, if C has parametrization $\mathbf{r}(t)$, $t \in [a, b]$, then a suitable parametrization for $-C$ would be, say, $\boldsymbol{\rho}(\tau) = \mathbf{r}(a + b - \tau)$, $a \le \tau \le b$. (It can be seen that $\mathbf{r}(t)$ starts at $\mathbf{r}(a)$ and stops at $\mathbf{r}(b)$, while $\boldsymbol{\rho}(\tau)$ starts at $\mathbf{r}(b)$ and stops at $\mathbf{r}(a)$.) Given curves C_1 and C_2 we naturally define

$$C_1 - C_2 = C_1 + (-C_2),$$

so that by Definition 16.7

$$\int_{C_1 - C_2} f = \int_{C_1 + (-C_2)} f = \int_{C_1} f + \int_{-C_2} f.$$

Proposition 16.13. *Let* C *be an oriented curve. If* $\mathbf{r}(t)$, $t \in [a, b]$, *is a parametrization for* C *consistent with its orientation, and* $\boldsymbol{\rho}(\tau)$, $\tau \in [\alpha, \beta]$ *is a parametrization for* $-C$ *consistent with its orientation, then*

$$\int_{-C} \mathbf{F} \cdot d\boldsymbol{\rho} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Proof. Let $\mathbf{r}(t)$, $t \in [a, b]$, be a parametrization for C consistent with its orientation. Define a parametrization for $-C$ by

$$\bar{\mathbf{r}}(\tau) = \mathbf{r}(a + b - \tau), \quad \tau \in [a, b],$$

and note that this parametrization is consistent with the orientation of $-C$. By 16.10

$$\int_{-C} \mathbf{F} \cdot d\bar{\mathbf{r}} = \int_a^b \mathbf{F}(\bar{\mathbf{r}}(\tau)) \cdot \bar{\mathbf{r}}'(\tau)\, d\tau = -\int_a^b \mathbf{F}(\mathbf{r}(a + b - \tau)) \cdot \mathbf{r}'(a + b - \tau)\, d\tau,$$

where $\bar{\mathbf{r}}'(\tau) = -\mathbf{r}'(a+b-\tau)$ by the Chain Rule. Making the substitution $t = a+b-\tau$, so that $d\tau$ is replaced by $(-1)dt$, we obtain

$$\begin{aligned}\int_{-C} \mathbf{F}\cdot d\bar{\mathbf{r}} &= -\int_b^a \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)(-1)\,dt = \int_b^a \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= -\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = -\int_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

Now, if $\boldsymbol{\rho}(\tau)$, $\tau\in[\alpha,\beta]$, is an arbitrary parametrization for $-C$ consistent with its orientation, then by Proposition 16.12

$$\int_{-C} \mathbf{F}\cdot d\boldsymbol{\rho} = \int_{-C} \mathbf{F}\cdot d\bar{\mathbf{r}},$$

and therefore

$$\int_{-C} \mathbf{F}\cdot d\boldsymbol{\rho} = -\int_C \mathbf{F}\cdot d\mathbf{r}$$

obtains, as was to be shown. ■

Given any curve C in $\mathbb{R}^2$ parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $t\in[a,b]$, the **outward unit normal vector**, $\mathbf{n}$, is defined by $\mathbf{n} = \mathbf{T}\times\mathbf{k}$. Thus, for any $a\le t\le b$,

$$\mathbf{n}(t) = \mathbf{T}(t)\times\mathbf{k}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\times\langle 0,0,1\rangle = \frac{\langle x'(t), y'(t), 0\rangle\times\langle 0,0,1\rangle}{\|\mathbf{r}'(t)\|} = \frac{\langle y'(t), -x'(t), 0\rangle}{\|\mathbf{r}'(t)\|},$$

and so as a vector in $\mathbb{R}^2$ we have

$$\mathbf{n}(t) = \frac{\langle y'(t), -x'(t)\rangle}{\|\mathbf{r}'(t)\|}. \tag{16.11}$$

Definition 16.14. *Let $\mathbf{F}$ be a continuous vector field on a region $R\subseteq\mathbb{R}^2$, and let C be a smooth curve in R parameterized by $\mathbf{r}(s) = \langle x(s), y(s)\rangle$ for $s\in[a,b]$, where s is arc length. If $\mathbf{n} = \mathbf{T}\times\mathbf{k}$, then the **flux of $\mathbf{F}$ across C** is*

$$\int_C \mathbf{F}\cdot\mathbf{n} = \int_C \mathbf{F}(\mathbf{r}(s))\cdot\mathbf{n}(s)\,ds.$$

As with $\int_C \mathbf{F}\cdot\mathbf{T}$, we would like to develop a formula that allows for easy computation of $\int_C \mathbf{F}\cdot\mathbf{n}$ regardless of how C is parameterized. Let

$$\mathbf{r}(s) = \langle x(s), y(s)\rangle,\quad s\in[a,b],$$

and

$$\boldsymbol{\rho}(t) = \langle \bar{x}(t), \bar{y}(t)\rangle,\quad t\in[\alpha,\beta],$$

be parametrizations with respect to arc length s and an arbitrary parametrization t, respectively. Observing that $\|\mathbf{r}'(s)\| = 1$ for all $a\le s\le b$, define

$$\mathbf{n}(s) = \frac{\langle y'(s), -x'(s)\rangle}{\|\mathbf{r}'(s)\|} = \langle y'(s), -x'(s)\rangle \quad\text{and}\quad \bar{\mathbf{n}}(t) = \frac{\langle \bar{y}'(t), -\bar{x}'(t)\rangle}{\|\boldsymbol{\rho}'(t)\|}.$$

As before, there is a function $s(t)$ that maps from the parameter t to the parameter s such that $\boldsymbol{\rho}(t) = \mathbf{r}(s(t))$ for all $\alpha\le t\le\beta$. From this we obtain

$$\boldsymbol{\rho}'(t) = (\mathbf{r}\circ s)'(t) = \mathbf{r}'(s(t))s'(t),$$

or

$$\langle \bar{x}'(t), \bar{y}'(t)\rangle = \big\langle x'(s(t))s'(t), y'(s(t))s'(t)\big\rangle,$$

and thus

$$\langle \bar{y}'(t), -\bar{x}'(t)\rangle = \big\langle y'(s(t))s'(t), -x'(s(t))s'(t)\big\rangle. \tag{16.12}$$

for $\alpha \le t \le \beta$. Recall that $\|\boldsymbol{\rho}'(t)\| = s'(t)$. Then, if the left side of (16.12) is divided by $\|\boldsymbol{\rho}'(t)\|$ and the right side is divided by $s'(t)$, we get

$$\bar{\mathbf{n}}(t) = \frac{\langle \bar{y}'(t), -\bar{x}'(t)\rangle}{\|\boldsymbol{\rho}'(t)\|} = \big\langle y'(s(t)), -x'(s(t))\big\rangle = \mathbf{n}(s(t)) \tag{16.13}$$

and therefore

$$\mathbf{n}(s(t))\|\boldsymbol{\rho}'(t)\| = \langle \bar{y}'(t), -\bar{x}'(t)\rangle. \tag{16.14}$$

We shall need this enthralling factoid as we once more perform a series of manipulations that makes use of the Substitution Rule. Letting $\mathbf{F} = \langle f, g\rangle$ and recalling equation (16.3), we have

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{n} &= \int_a^b \mathbf{F}(\mathbf{r}(s))\cdot\mathbf{n}(s)\,ds = \int_{s(\alpha)}^{s(\beta)} \big((\mathbf{F}\circ\mathbf{r})\cdot\mathbf{n}\big)(s)\,ds \\
&= \int_\alpha^\beta \big((\mathbf{F}\circ\mathbf{r})\cdot\mathbf{n}\big)(s(t))s'(t)\,dt = \int_\alpha^\beta \mathbf{F}(\mathbf{r}(s(t)))\cdot\mathbf{n}(s(t))\|\boldsymbol{\rho}'(t)\|dt \\
&= \int_\alpha^\beta \mathbf{F}(\boldsymbol{\rho}(t))\cdot\langle \bar{y}'(t), -\bar{x}'(t)\rangle\,dt = \int_\alpha^\beta \langle f(\boldsymbol{\rho}(t)), g(\boldsymbol{\rho}(t))\rangle\cdot\langle \bar{y}'(t), -\bar{x}'(t)\rangle\,dt \\
&= \int_\alpha^\beta \big[f(\boldsymbol{\rho}(t))\bar{y}'(t) - g(\boldsymbol{\rho}(t))\bar{x}'(t)\big]\,dt
\end{aligned}$$

What has been proved is the following.

Proposition 16.15. *Let $\mathbf{F} = \langle f, g\rangle$ be continuous vector field on a region in $\mathbb{R}^2$ containing a smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $t \in [a, b]$. Then*

$$\int_C \mathbf{F}\cdot\mathbf{n} = \int_a^b \big[f(\mathbf{r}(t))y'(t) - g(\mathbf{r}(t))x'(t)\big]dt.$$

Example 16.16. Compute the flux for the vector field $\mathbf{F}(x, y) = \langle y - x, x\rangle$ across C given by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.

Solution. Here we have $x(t) = 2\cos t$ and $y(t) = 2\sin t$, so $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$, and by Proposition 16.15 we obtain

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{n} &= \int_0^{2\pi} \big[f(\mathbf{r}(t))y'(t) - g(\mathbf{r}(t))x'(t)\big]dt \\
&= \int_0^{2\pi} \big[f(2\cos t, 2\sin t)(2\cos t) - g(2\cos t, 2\sin t)(-2\sin t)\big]dt \\
&= \int_0^{2\pi} \big[(2\sin t - 2\cos t)(2\cos t) - (2\cos t)(-2\sin t)\big]dt \\
&= 4\int_0^{2\pi} 2\cos t\sin t\,dt - 4\int_0^{2\pi} \cos^2 t\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \sin(2t)\,dt - 4\int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt \\
&= 4\left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} - 2\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\
&= 4\cdot 0 - 2\cdot 2\pi = -4\pi.
\end{aligned}
$$

From a physical standpoint, then, there is a net flux of 4π *into* the region enclosed by C. ■

16.3 – Fundamental Theorem of Line Integrals

Recall that a function φ is called a potential function for a vector field $\mathbf{F}$ if $\nabla\varphi = \mathbf{F}$. It is a fact that not every vector field has a potential function, and those that do are of particular interest in mathematics and the sciences.

Definition 16.17. *A vector field* $\mathbf{F}$ *is* ***conservative*** *on a region* $R \subseteq \mathbb{R}^n$ *if there exists an open set* U *containing* R *and a function* $\varphi : U \to \mathbb{R}$ *such that* $\nabla\varphi = \mathbf{F}$ *on* U.

Because our definition of partial derivative is valid only at interior points of a function's domain, the function $\nabla\varphi$ would of necessity have to be defined on an open set U containing R. How can we tell whether a vector field is conservative? For vector fields defined on a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ we have the following.

Proposition 16.18. *Let* $\mathbf{F} = \langle f, g\rangle$ *be a vector field on a connected and simply connected region* $R \subseteq \mathbb{R}^2$, *where* f *and* g *have continuous first partial derivatives on an open set* $V \supseteq R$. *Then* $\mathbf{F}$ *is conservative on* R *if and only if* $f_y = g_x$ *on an open set* $W \supseteq R$.

Let $\mathbf{F} = \langle f, g, h\rangle$ *be a vector field on a connected and simply connected region* $D \subseteq \mathbb{R}^3$, *where* f, h *and* g *have continuous first partial derivatives on an open set* $V \supseteq D$. *Then* $\mathbf{F}$ *is conservative on* D *if and only if* $f_y = g_x$, $f_z = h_x$, *and* $g_z = h_y$ *on an open set* $W \supseteq D$.

Proof. Suppose $\mathbf{F} = \langle f, g\rangle : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a conservative vector field, so there exists an open set $U \supseteq R$ and a function $\varphi : U \to \mathbb{R}$ such that $\mathbf{F} = \nabla\varphi$ on U. Let $W = U \cap V$, so W is an open set such that $R \subseteq W \subseteq U, V$. Then $f = \varphi_x$ and $g = \varphi_y$ on the open set W, and since f and g have continuous first partials on W it follows that φ_x and φ_y have continuous second partials on W. Now, by Clairaut's Theorem (see section 13.4) we obtain

$$f_y = (\varphi_x)_y = \varphi_{xy} = \varphi_{yx} = (\varphi_y)_x = g_x$$

on W (and therefore $f_y = g_x$ on R).

The proof is much the same for the three-dimensional case. As for the proof of the converse (i.e. $f_y = g_x$ on R implies that $\mathbf{F}$ is conservative on R), see section 16.4. ∎

Example 16.19. Show that the vector field

$$\mathbf{F}(x, y, z) = \left\langle \frac{1}{y}, -\frac{x}{y^2}, 2z - 1\right\rangle$$

is conservative on $R = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\}$, and then determine a potential function φ.

Solution. Here R is not a connected region, but if we define

$$R_1 = \{(x, y, z) \in \mathbb{R}^3 : y > 0\} \quad \text{and} \quad R_2 = \{(x, y, z) \in \mathbb{R}^3 : y < 0\},$$

then we see that R_1 and R_2 are connected and simply connected disjoint open regions such that $R = R_1 \cup R_2$.

We begin by carrying out the analysis in R_1, where $y > 0$. Letting $\mathbf{x} = \langle x, y, z\rangle$, we have $\mathbf{F} = \langle f, g, h\rangle$ with

$$f(\mathbf{x}) = \frac{1}{y}, \quad g(\mathbf{x}) = -\frac{x}{y^2}, \quad \text{and} \quad h(\mathbf{x}) = 2z - 1.$$

Now,

$$f_y(\mathbf{x}) = -\frac{1}{y^2} = g_x(\mathbf{x}), \quad f_z(\mathbf{x}) = 0 = h_x(\mathbf{x}), \quad \text{and} \quad g_z(\mathbf{x}) = 0 = h_y(\mathbf{x}),$$

which immediately implies that $\mathbf{F}$ is conservative on R_1 by appealing to Proposition 16.18 with D, V, and W all set equal to R_1. The same analysis leads us to conclude that $\mathbf{F}$ is conservative on R_2 as well, and therefore $\mathbf{F}$ is conservative on the entire open set R.

Now we set about finding a potential function φ. From $\nabla\varphi = \mathbf{F}$ we have

$$\varphi_x(\mathbf{x}) = \frac{1}{y}, \quad \varphi_y(\mathbf{x}) = -\frac{x}{y^2} \quad \text{and} \quad \varphi_z(\mathbf{x}) = 2z - 1.$$

In particular $\varphi_x(\mathbf{x}) = 1/y$ implies that

$$\varphi(\mathbf{x}) = \int \varphi_x(\mathbf{x})\,dx = \int \frac{1}{y}\,dx = \frac{x}{y} + c(y, z), \tag{16.15}$$

where $c(y, z)$ represents a function of y and z, which is to say that it is constant with respect to x. (Note that differentiating $x/y + c(y, z)$ with respect to x does indeed return us to the integrand $1/y$.)

Differentiating (16.15) with respect to y gives $\varphi_y(\mathbf{x}) = -x/y^2 + c_y(y, z)$, which, when compared to $\varphi_y(\mathbf{x}) = -x/y^2$, informs us that $c_y(y, z) = 0$ and therefore $c(y, z) = c(z)$. That is, the function c must not be a function of y.

At this point we have $\varphi(\mathbf{x}) = x/y + c(z)$. This implies that $\varphi_z(\mathbf{x}) = c'(z)$, which, when compared to $\varphi_z(\mathbf{x}) = 2z - 1$ above, gives $c'(z) = 2z - 1$. So

$$c(z) = \int c'(z)\,dz = \int (2z - 1)\,dz = z^2 - z + c$$

for arbitrary constant c. Choosing c to be zero, we obtain

$$\varphi(\mathbf{x}) = \frac{x}{y} + z^2 - z$$

at last. ■

The following is the foremost result of this section, and one that is analogous to the Fundamental Theorem of Calculus given in Chapter 5.

Theorem 16.20 (**Fundamental Theorem of Line Integrals**). *Let $\mathbf{F}$ be a continuous vector field on an open region $U \subseteq \mathbb{R}^n$. If $\varphi : U \to \mathbb{R}$ is such that $\mathbf{F} = \nabla\varphi$, then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{b}) - \varphi(\mathbf{a})$$

for any piecewise-smooth curve $C \subseteq U$ from $\mathbf{a}$ to $\mathbf{b}$.

Proof. Suppose $\mathbf{F} = \nabla\varphi$ on U, and let $C \subseteq U$ be a smooth curve given by $\mathbf{r}(t)$, $t \in [a,b]$, with $\mathbf{r}(a) = \mathbf{a}$ and $\mathbf{r}(b) = \mathbf{b}$. Using Chain Rule 1 in section 13.5,

$$(\varphi \circ \mathbf{r})'(t) = \nabla\varphi(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Now we employ Proposition 16.10 to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b (\varphi \circ \mathbf{r})'(t)\,dt,$$

and since

$$\int_a^b (\varphi \circ \mathbf{r})'(t)\,dt = (\varphi \circ \mathbf{r})(b) - (\varphi \circ \mathbf{r})(a),$$

by the Fundamental Theorem of Calculus, it follows that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a)) = \varphi(\mathbf{b}) - \varphi(\mathbf{a}).$$

If C is a piecewise-smooth curve, then there exist smooth curves $C_1, \ldots, C_k$ such that $C = C_1 + \cdots + C_k$, where each C_i is parameterized by $\mathbf{r}_i(t)$, $t \in [a_i, b_i]$, such that

$$\mathbf{b}_i = \mathbf{r}_i(b_i) = \mathbf{r}_{i+1}(a_{i+1}) = \mathbf{a}_{i+1}, \tag{16.16}$$

and in particular $\mathbf{r}_1(a_1) = \mathbf{a}$ and $\mathbf{r}_k(b_k) = \mathbf{b}$. Now, using Definition 16.7 and letting $\mathbf{r}$ denote the parametrization of C itself, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+\cdots+C_k} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \cdots + \int_{C_k} \mathbf{F} \cdot d\mathbf{r}_k \\ &= \int_{a_1}^{b_1} \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\,dt + \cdots + \int_{a_k}^{b_k} \mathbf{F}(\mathbf{r}_k(t)) \cdot \mathbf{r}_k'(t)\,dt \\ &= \int_{a_1}^{b_1} (\varphi \circ \mathbf{r}_1)'(t)\,dt + \cdots + \int_{a_k}^{b_k} (\varphi \circ \mathbf{r}_k)'(t)\,dt\end{aligned}$$

The last expression, by the Fundamental Theorem of Calculus, becomes

$$[\varphi(\mathbf{r}_1(b_1)) - \varphi(\mathbf{r}_1(a_1))] + [\varphi(\mathbf{r}_2(b_2)) - \varphi(\mathbf{r}_2(a_2))] + \cdots + [\varphi(\mathbf{r}_k(b_k)) - \varphi(\mathbf{r}_k(a_k))],$$

which yields

$$\begin{aligned}&[\varphi(\mathbf{b}_1) - \varphi(\mathbf{a})] + [\varphi(\mathbf{b}_2) - \varphi(\mathbf{a}_2)] + [\varphi(\mathbf{b}_3) - \varphi(\mathbf{a}_3)] + [\varphi(\mathbf{b}_4) - \varphi(\mathbf{a}_4)] + \cdots \\ &\qquad \cdots + [\varphi(\mathbf{b}_{k-2}) - \varphi(\mathbf{a}_{k-2})] + [\varphi(\mathbf{b}_{k-1}) - \varphi(\mathbf{a}_{k-1})] + [\varphi(\mathbf{b}) - \varphi(\mathbf{a}_k)].\end{aligned}$$

Recalling (16.16), this leads to

$$\begin{aligned}&[\varphi(\mathbf{a}_2) - \varphi(\mathbf{a})] + [\varphi(\mathbf{a}_3) - \varphi(\mathbf{a}_2)] + [\varphi(\mathbf{a}_4) - \varphi(\mathbf{a}_3)] + [\varphi(\mathbf{a}_5) - \varphi(\mathbf{a}_4)] + \cdots \\ &\qquad \cdots + [\varphi(\mathbf{a}_{k-1}) - \varphi(\mathbf{a}_{k-2})] + [\varphi(\mathbf{a}_k) - \varphi(\mathbf{a}_{k-1})] + [\varphi(\mathbf{b}) - \varphi(\mathbf{a}_k)],\end{aligned}$$

which collapses to give $-\varphi(\mathbf{a}) + \varphi(\mathbf{b})$ and therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{b}) - \varphi(\mathbf{a})$$

results once more. ■

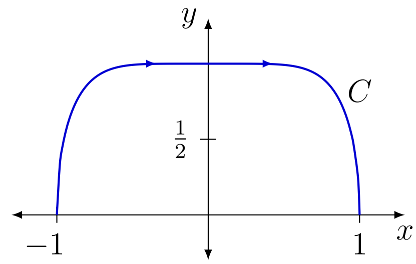


FIGURE 85.

Thus, if $\mathbf{F}$ is a continuous vector field that is conservative on an open connected region U, then there exists a function φ such that $\mathbf{F}(\mathbf{x}) = \nabla\varphi(\mathbf{x})$ for all $\mathbf{x} \in U$, in which case given any piecewise-smooth curve $C \subseteq U$ parameterized by $\mathbf{r}(t)$, $a \le t \le b$, we find that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a)).$$

So the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ does not hinge on the particular path that is taken from the point $\mathbf{r}(a)$ to the point $\mathbf{r}(b)$, which is to say the line integral has the property known as **independence of path**. This is an important property, because oftentimes $\mathbf{F}$ is known to be conservative but a potential function φ has not been determined, in which case the evaluation of $\int_C \mathbf{F} \cdot d\mathbf{r}$ may still be greatly eased by passing from C to some more convenient path C' from $\mathbf{r}(a)$ to $\mathbf{r}(b)$.

The converse of the Fundamental Theorem of Line Integrals is not true in general unless U happens to also be a connected region. This is given as a separate proposition.

Proposition 16.21. *Let $\mathbf{F}$ be a continuous vector field on an open connected region $U \subseteq \mathbb{R}^n$. If $\varphi : U \to \mathbb{R}$ is such that*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{b}) - \varphi(\mathbf{a})$$

for any piecewise-smooth curve $C \subseteq U$ from $\mathbf{a}$ to $\mathbf{b}$, then $\mathbf{F} = \nabla\varphi$.

The proof of this proposition will be supplied in time, but for now let's turn to an example.

Example 16.22. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y^2, 2xy\rangle$ and C is the curve given by

$$\mathbf{r}(t) = \left\langle t, \sqrt{1 - t^6}\right\rangle, \quad -1 \le t \le 1.$$

Solution. The curve C begin at the point $(-1, 0)$ and ends at $(1, 0)$, as shown in Figure 85. It would not be the easiest curve to work with. However, $\mathbf{F} = \langle f, g\rangle$ with $f(x, y) = y^2$ and $g(x, y) = 2xy$, and since $f_y(x, y) = 2y = g_x(x, y)$ it follows from Proposition 16.18 that $\mathbf{F}$ is conservative. So, by the Fundamental Theorem of Line Integrals the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path taken from $(-1, 0)$ to $(1, 0)$. Therefore we can evaluate the line integral along the path C' given by $\boldsymbol{\rho}(t) = \langle t, 0\rangle$, $-1 \le t \le 1$, which is the line segment connecting these two points. Now,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\boldsymbol{\rho} = \int_{-1}^{1} \mathbf{F}(\boldsymbol{\rho}(t)) \cdot \boldsymbol{\rho}'(t)\, dt = \int_{-1}^{1} \mathbf{F}(t, 0) \cdot \langle 1, 0\rangle\, dt$$

$$= \int_{-1}^{1} \langle 0,0\rangle \cdot \langle 1,0\rangle \, dt = \int_{-1}^{1} (0)\, dt = 0.$$

It's worthwhile also evaluating the line integral along the original path C which, while not as nice, is still quite feasible. ■

As we have seen in the previous section, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the function $\mathbf{r}$ that is used to parameterize C, as long as orientation is preserved. (We say $\int_C \mathbf{F}\cdot d\mathbf{r}$ is invariant under orientation-preserving reparametrization.) Thus the $\mathbf{r}$ and $\mathbf{T}$ in the equivalent symbols $\int_C \mathbf{F}\cdot d\mathbf{r}$ and $\int_C \mathbf{F}\cdot\mathbf{T}$ become something of an encumbrance in situations when having a specific parametrization for C is unnecessary. This leads us to so-called differential form notation, which has wide utility in higher mathematics, but for our purposes will serve primarily as a means of denoting line integrals without making explicit mention of parametrizations. Given a vector field $\mathbf{F} = \langle f, g\rangle$ on a region R containing a curve C parameterized by $\mathbf{r}$, we define

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f\,dx + g\,dy \tag{16.17}$$

Here $f\,dx + g\,dy$, called a "differential one-form," is considered to be a single object, so usually it is not encapsulated in parentheses. An easy way to remember (16.17) is to think of $d\mathbf{r}$ as $\langle dx, dy\rangle$, so

$$\mathbf{F}\cdot d\mathbf{r} = \langle f, g\rangle \cdot \langle dx, dy\rangle = f\,dx + g\,dy.$$

In the special case when $f = 0$ or $g = 0$ we define

$$\int_C 0\,dx + g\,dy = \int_C g\,dy \quad\text{and}\quad \int_C f\,dx + 0\,dy = \int_C f\,dx. \tag{16.18}$$

Of particular interest for us is the result of Proposition 16.13, which previously required explicit mention of parametrizations $\mathbf{r}$ and $\boldsymbol{\rho}$ for C and $-C$, respectively, but in our new notation is written simply as

$$\int_{-C} f\,dx + g\,dy = -\int_C f\,dx + g\,dy.$$

Next we define

$$\int_C \mathbf{F}\cdot\mathbf{n} = \int_C f\,dy - g\,dx, \tag{16.19}$$

which can be remembered by thinking of $\mathbf{n}$ as being $\langle dy, -dx\rangle$ and carrying out the dot product. (Consider: if $d\mathbf{r} = \langle dx, dy\rangle$ is thought of as a vector tangent to C, then $\langle dy, -dx\rangle$ would be a vector normal to C.)

In the statement of the following proposition the symbol $\oint_C$ is used to indicate that the line integral is over a curve C that is *closed*.

Proposition 16.23. *Let $U \subseteq \mathbb{R}^n$ be an open region. Then $\mathbf{F}$ is conservative on U if and only if*

$$\oint_C f\,dx + g\,dy = 0$$

on all simple closed piecewise-smooth curves $C \subseteq U$.

16.4 – Green's Theorem

Throughout the remainder of this chapter the symbols

$$\ointctrclockwise_C \quad \text{and} \quad \ointclockwise_C$$

will denote a line integral over a closed curve C in the xy-plane that has a positive (counter-clockwise) orientation and negative (clockwise) orientation, respectively, as viewed from $z > 0$ in the standard right-handed $\mathbb{R}^3_{xyz}$ system. Proposition 16.13 implies that

$$\ointctrclockwise_C f\,dx + g\,dy = -\ointclockwise_C f\,dx + g\,dy,$$

which should be remembered during the developments to come.

Lemma 16.24. *Given any continuous vector field $\mathbf{F} = \langle f, g\rangle : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ and piecewise-smooth curve $C \subseteq R$,*

$$\int_C f\,dx + g\,dy = \int_C f\,dx + \int_C g\,dy \quad \textit{and} \quad \int_C f\,dy - g\,dx = \int_C f\,dy - \int_C g\,dx.$$

Proof. Letting $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $t \in [a, b]$, be a parametrization for C, we obtain

$$\begin{aligned}
\int_C f\,dx + \int_C g\,dy &= \left(\int_C f\,dx + 0\,dy\right) + \left(\int_C 0\,dx + g\,dy\right) && \text{Equation (16.18)}\\
&= \int_C \langle f, 0\rangle \cdot d\mathbf{r} + \int_C \langle 0, g\rangle \cdot d\mathbf{r} && \text{Equation (16.17)}\\
&= \int_a^b \langle f(\mathbf{r}(t)), 0\rangle \cdot \mathbf{r}'(t)\,dt + \int_a^b \langle 0, g(\mathbf{r}(t))\rangle \cdot \mathbf{r}'(t)\,dt && \text{Prop. 16.10}\\
&= \int_a^b f(\mathbf{r}(t))x'(t)\,dt + \int_a^b g(\mathbf{r}(t))y'(t)\,dt && \text{Dot product}\\
&= \int_a^b [f(\mathbf{r}(t))x'(t) + g(\mathbf{r}(t))y'(t)]\,dt && \text{Section 5.2}\\
&= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt && \text{Dot product}\\
&= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f\,dx + g\,dy, && \text{Prop. 16.10 and (16.17)}
\end{aligned}$$

which verifies the first equation in the lemma. Verification of the second equation is done similarly. ■

We now come to the first form of Green's Theorem, known as the circulation form. It is important to remember that we define the region $R \subseteq \mathbb{R}^2$ bounded by a closed curve C to include C itself, and thus R is a closed set.

Theorem 16.25 (**Green's Theorem—Circulation Form**). *Let $C \subseteq \mathbb{R}^2$ be a simple closed piecewise-smooth curve that bounds a region R, so that $C = \partial R$. If f and g have continuous first partial derivatives on an open region containing R, then*

$$\oint_{\partial R} f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA. \tag{16.20}$$

Note the use of the symbol $\oint$ indicating that ∂R must have a positive (counterclockwise) orientation. Also, some books state Green's Theorem a little differently, giving conditions on R instead of on C. Specifically, instead of stating that C must be simple, closed, and piecewise-smooth, one could equivalently require that R be a connected and simply connected region with piecewise-smooth boundary.

If f, g, and C are as in Green's Theorem, we define $\mathbf{F} = \langle f, g\rangle$, and we give C a positively oriented piecewise-smooth parametrization $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $t \in [a, b]$, then (16.20) may be written

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA,$$

where we use Leibniz notation at right for variety's sake.

Proof. We will prove Theorem 16.25 for the special case when R is both a vertically and horizontally simple region, and $C = \partial R$ is smooth. Thus

$$R = \{(x, y) : a \le x \le b, \varphi_1(x) \le y \le \varphi_2(x)\} = \{(x, y) : c \le y \le d, \psi_1(y) \le x \le \psi_2(y)\},$$

and so if C_1 is the curve given by $\mathbf{r}_1(t) = \langle t, \varphi_1(t)\rangle$, $t \in [a, b]$, and C_2 is the curve given by $\mathbf{r}_2(t) = \langle t, \varphi_2(t)\rangle$, $t \in [a, b]$, then $C = C_1 + (-C_2) = C_1 - C_2$ as shown at left in Figure 86. Now, by Definition 16.7 and Proposition 16.13,

$$\int_C f\,dx = \int_{C_1} f\,dx + \int_{-C_2} f\,dx = \int_{C_1} f\,dx - \int_{C_2} f\,dx,$$

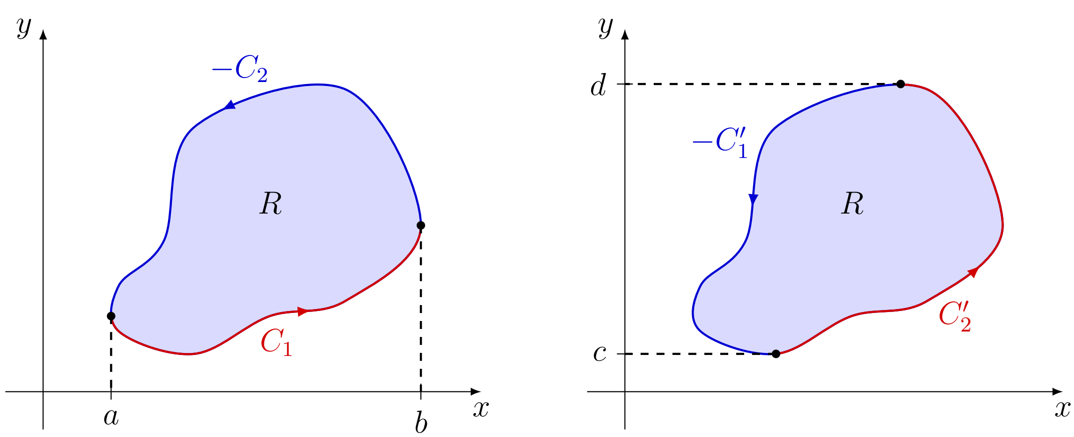


Figure 86.

and so by Proposition 16.10

$$\begin{aligned}\int_C f\,dx &= \int_a^b \langle f(\mathbf{r}_1(t)), 0\rangle \cdot \mathbf{r}_1'(t)\,dt - \int_a^b \langle f(\mathbf{r}_2(t)), 0\rangle \cdot \mathbf{r}_2'(t)\,dt \\ &= \int_a^b f(t,\varphi_1(t))\,dt - \int_a^b f(t,\varphi_2(t))\,dt \\ &= \int_a^b [f(t,\varphi_1(t)) - f(t,\varphi_2(t))]\,dt \qquad (16.21)\end{aligned}$$

where we use the fact that $\mathbf{r}_i'(t) = \langle 1, \varphi_i'(t)\rangle$ for each i. On the other hand Fubini's Theorem and the Fundamental Theorem of Calculus yield

$$\iint\limits_R f_y\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f_y(x,y)\,dy\,dx = \int_a^b [f(x,\varphi_2(x)) - f(x,\varphi_1(x))]\,dx,$$

and a comparison of this result with (16.21) makes clear that

$$\iint\limits_R f_y\,dA = -\int_C f\,dx. \qquad (16.22)$$

Next, let C_1' be given by

$$\boldsymbol{\rho}_1(t) = \langle \psi_1(t), t\rangle, \quad t \in [c,d],$$

and C_2' by

$$\boldsymbol{\rho}_2(t) = \langle \psi_2(t), t\rangle, \quad t \in [c,d],$$

so that $C = -C_1' + C_2'$ as shown at right in Figure 86. By Definition 16.7 and Proposition 16.13,

$$\int_C g\,dy = \int_{-C_1'} g\,dy + \int_{C_2'} g\,dy = -\int_{C_1'} g\,dy + \int_{C_2'} g\,dy,$$

and so by Proposition 16.10

$$\begin{aligned}\int_C g\,dy &= -\int_c^d \langle 0, g(\boldsymbol{\rho}_1(t))\rangle \cdot \boldsymbol{\rho}_1'(t)\,dt + \int_c^d \langle 0, g(\boldsymbol{\rho}_2(t))\rangle \cdot \boldsymbol{\rho}_2'(t)\,dt \\ &= \int_c^d g(\psi_2(t),t)\,dt - \int_a^b g(\psi_1(t),t)\,dt \\ &= \int_c^d [g(\psi_2(t),t) - g(\psi_1(t),t)]\,dt \qquad (16.23)\end{aligned}$$

where we use the fact that $\boldsymbol{\rho}_i'(t) = \langle \psi_i'(t), 1\rangle$ for each i. On the other hand

$$\iint\limits_R g_x\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} g_x(x,y)\,dx\,dy = \int_c^d [g(\psi_2(y),y) - g(\psi_1(y),y)]\,dy,$$

and a comparison of this result with (16.23) shows that

$$\iint\limits_R g_x\,dA = \int_C g\,dy. \qquad (16.24)$$

Finally, equations (16.22) and (16.24) taken together give

$$\int_C f\,dx + g\,dy = \int_C f\,dx + \int_C g\,dy = -\iint\limits_R f_y\,dA + \iint\limits_R g_x\,dA = \iint\limits_R (g_x - f_y)\,dA,$$

where the first equality is justified by Lemma 16.24. ■

The first application of the theorem will be to finish the proof of Proposition 16.18. The argument in $\mathbb{R}^2$ should be sufficient to convey the general approach.

Proof of Proposition 16.18 Concluded. Suppose $f_y = g_x$ on an open set W containing $R \subseteq \mathbb{R}^2$, so that $g_x - f_y = 0$. We can assume without loss of generality that W is contained in V and is itself both connected and simply connected. Hence f and g have continuous first partials on W.

Now, let $C \subseteq W$ be a simple closed piecewise-smooth curve that bounds a region S. Since $S \subseteq W$ it's seen immediately that f and g have continuous first partials on an open region containing S, and so Green's Theorem can be invoked to obtain

$$\oint_C f\,dx + g\,dy = \iint\limits_S (g_x - f_y)\,dA = \iint\limits_S (0)\,dA = 0.$$

By Proposition 16.23, $\mathbf{F}$ is conservative on W and therefore is conservative on R. ■

Example 16.26. Use Green's Theorem to evaluate the line integral

$$\oint_C (y^2 - x^2 y)\,dx + xy^2\,dy,$$

where C consists of the arc on the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(\sqrt{2}, \sqrt{2})$, the line segment from $(\sqrt{2}, \sqrt{2})$ to $(0,0)$, and the line segment from $(0,0)$ to $(2,0)$.

Solution. Let R be the region bounded by C, so R is a circular sector as shown in Figure 87, and $C = \partial R$. Clearly the transformation $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ maps the region

$$S = \{(r,\theta) : 0 \le r \le 2 \text{ and } 0 \le \theta \le \pi/4\}$$

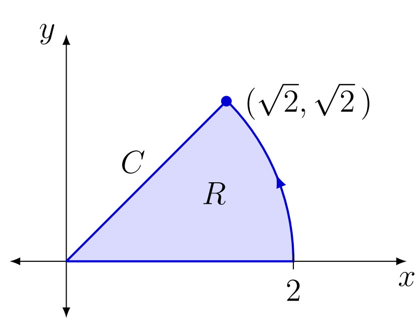


FIGURE 87.

onto R, so Theorem 15.11 will be of use here:

$$\begin{aligned}
\oint_C (y^2 - x^2 y)\,dx + xy^2\,dy &= \iint_R \left[y^2 - (2y - x^2)\right] dA \\
&= \iint_S \left(r^2 \sin^2\theta - 2r\sin\theta + r^2\cos^2\theta\right) r\,dA \\
&= \int_0^{\pi/4} \int_0^2 \left(r^3 - 2r^2\sin\theta\right) dr\,d\theta \\
&= \int_0^{\pi/4} \left[\frac{r^4}{4} - \frac{2r^3}{3}\sin\theta\right]_0^2 d\theta = \int_0^{\pi/4} \left(4 - \frac{16}{3}\sin\theta\right) d\theta \\
&= \left[4\theta + \frac{16}{3}\cos\theta\right]_0^{\pi/4} = \pi + \frac{16}{3\sqrt{2}} - \frac{16}{3}.
\end{aligned}$$

■

Let f and g have continuous first partials on an open region containing $R \subseteq \mathbb{R}^2$. Suppose R is not connected and simply-connected, but there exist measurable sets (see §14.2) R_1 and R_2 such that $R = R_1 \cup R_2$ and $\mathcal{A}(R_1 \cap R_2) = 0$. Then by Proposition 15.8 we can write

$$\iint_R (g_x - f_y)\,dA = \iint_{R_1} (g_x - f_y)\,dA + \iint_{R_2} (g_x - f_y)\,dA. \tag{16.25}$$

Now, assuming that R_1, R_2 are each both connected and simply-connected, ∂R_1, ∂R_2 are each simple closed piecewise-smooth curves, and $\mathbf{r}_1$, $\mathbf{r}_2$ are each parametrizations for ∂R_1, ∂R_2 with positive orientation, then by Green's Theorem equation (16.25) becomes

$$\iint_R (g_x - f_y)\,dA = \oint_{\partial R_1} \mathbf{F} \cdot d\mathbf{r}_1 + \oint_{\partial R_2} \mathbf{F} \cdot d\mathbf{r}_2. \tag{16.26}$$

It may be that the right side of (16.26) is much easier to evaluate than the left side.

Proposition 16.27. *Let C_1, C_2 be closed simple piecewise-smooth curves, with C_1 lying in the interior of the region bounded by C_2. If C_1 has negative and C_2 positive orientation, R is the region bounded by C_1 and C_2, and g, f have continuous first partial derivatives on an open region containing R, then*

$$\iint_R (g_x - f_y)\,dA = \oint_{C_1} f\,dx + g\,dy + \oint_{C_2} f\,dx + g\,dy.$$

Proof. Let p_1, p_2 be points on C_1 and q_1, q_2 points on C_2. Now, define the curve $A = A_1 + A_2 + A_3 + A_4$, where A_1 is a simple, piecewise-smooth path from p_1 to q_1, A_2 is the path on C_2 from q_1 to q_2 consistent with the orientation of C_2, A_3 is a simply, piecewise-smooth path from q_2 to p_2, and A_4 is the path on C_1 from p_2 to p_1 consistent with the orientation of C_1. Next, define the curve $B = B_1 + B_2 + B_3 + B_4$, where B_1 is $-A_1$ from q_1 to p_1, B_2 is the path on C_1 from p_1 to p_2 consistent with the orientation of C_1, B_3 is $-A_3$ from p_2 to q_2, and B_4 is the path on C_2 from q_2 to q_1 consistent with the orientation of C_2. See Figure 88. Curve A is the boundary of a connected and simply-connected region R_1, while B is the boundary of a connected and simply-connected region R_2; and since $R = R_1 \cup R_2$ and $\mathcal{A}(R_1 \cap R_2) = 0$, by (16.25) we obtain

$$\iint\limits_{R} (g_x - f_y)\, dA = \iint\limits_{R_1} (g_x - f_y)\, dA + \iint\limits_{R_2} (g_x - f_y)\, dA. \tag{16.27}$$

Now, by construction $A = \partial R_1$ and $B = \partial R_2$ are each closed, simple, piecewise-smooth, and positively oriented curves, so by Green's Theorem (16.27) becomes

$$\iint\limits_{R} (g_x - f_y)\, dA = \oint_A f\, dx + g\, dy + \oint_B f\, dx + g\, dy. \tag{16.28}$$

Suppressing integrands in the interests of brevity, we have

$$\oint_A + \oint_B = \int_{A_1} + \int_{A_2} + \int_{A_3} + \int_{A_4} + \int_{B_1} + \int_{B_2} + \int_{B_3} + \int_{B_4}, \tag{16.29}$$

and since

$$\int_{B_1} = \int_{-A_1} = -\int_{A_1} \quad \text{and} \quad \int_{B_3} = \int_{-A_3} = -\int_{A_3},$$

equation (16.29) becomes

$$\begin{aligned} \oint_A + \oint_B &= \int_{A_2} + \int_{A_4} + \int_{B_2} + \int_{B_4} = \left(\int_{B_2} + \int_{A_4}\right) + \left(\int_{A_2} + \int_{B_4}\right) \\ &= \int_{B_2 + A_4} + \int_{A_2 + B_4} = \oint_{C_1} + \oint_{C_2} \end{aligned}$$

This result, combined with (16.28), yields

$$\iint\limits_{R} (g_x - f_y)\, dA = \oint_{C_1} f\, dx + g\, dy + \oint_{C_2} f\, dx + g\, dy,$$

which finishes the proof. ∎

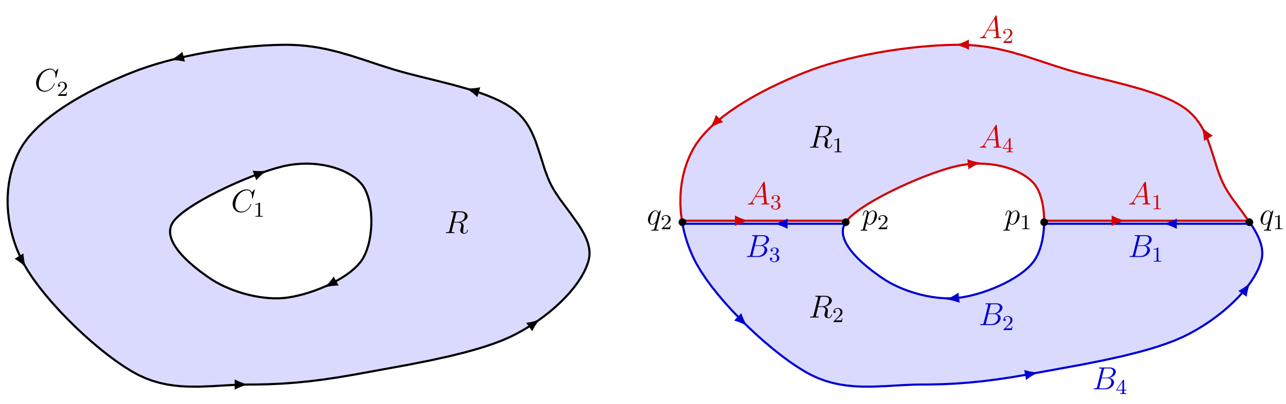


FIGURE 88. At right, the curves A and B.

In the classical notation, if $\mathbf{F} = \langle f, g \rangle$, and parametrizations for C_1 and C_2 are given by functions $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, then Proposition 16.27 gives

$$\iint_R (g_x - f_y)\, dA = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}_2.$$

Compare this result to (16.26), for the differences are subtle. Of course, explicit parametrizations are not always practical for every curve, as the next example illustrates.

Example 16.28. Given the vector field

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle,$$

show that if $C \subseteq \mathbb{R}^2$ is a simple closed piecewise-smooth curve that encloses the origin, then

$$\oint_C f\, dx + g\, dy = 2\pi.$$

Solution. Let C be an arbitrary simple closed piecewise-smooth curve that encloses $(0, 0)$ and has positive orientation. As shown at left in Figure 89, C may be a curve that would be quite difficult to parameterize in any fashion, much less in a fashion that would admit easy analysis. The way to proceed is to introduce a second curve C' that also encloses the origin, but which lies in the interior of the region R bounded by C, has negative orientation, and is easily parameterized. Since $(0, 0) \in \operatorname{Int}(R)$, there exists some $\epsilon > 0$ such that $\overline{B}_\epsilon(0, 0) \subseteq R$, and so we can let C' be the circle centered at $(0, 0)$ with radius ϵ.

Let R' be the region bounded by C' and C; that is, $R' = R \setminus B_\epsilon(0, 0)$ as shown at right in Figure 89. Since

$$f(x, y) = -\frac{y}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x}{x^2 + y^2}$$

have continuous first partials on $\mathbb{R}^2 \setminus (0, 0)$, which contains R', by Proposition 16.27

$$\begin{aligned} \oint_{C'} f\, dx + g\, dy + \oint_C f\, dx + g\, dy &= \iint_{R'} (g_x - f_y)\, dA \\ &= \iint_{R'} \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] dA \end{aligned}$$

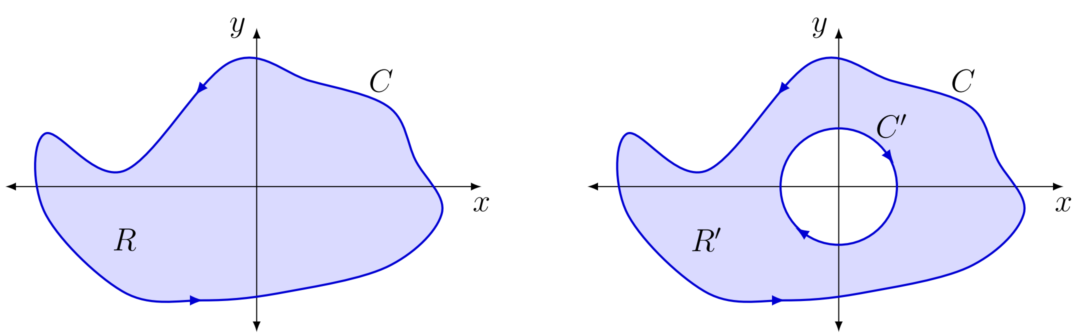


Figure 89.

$$= \iint_{R'} (0)\, dA = 0,$$

and thus

$$\oint_C f\, dx + g\, dy = -\oint_{C'} f\, dx + g\, dy. \tag{16.30}$$

Now, the circle C' with negative orientation is easily parameterized by $\mathbf{r}(t) = \epsilon\langle \sin t, \cos t\rangle$ for $t \in [0, 2\pi]$. With this parametrization we obtain

$$\begin{aligned}
\oint_{C'} f\, dx + g\, dy = \oint_{C'} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \frac{\epsilon}{(\epsilon \sin t)^2 + (\epsilon \cos t)^2} \langle -\cos t, \sin t\rangle \cdot \langle \epsilon \cos t, -\epsilon \sin t\rangle\, dt \\
&= \int_0^{2\pi} \langle -\cos t, \sin t\rangle \cdot \langle \cos t, -\sin t\rangle\, dt = \int_0^{2\pi} (-\cos^2 t - \sin^2 t)\, dt \\
&= \int_0^{2\pi} (-1)\, dt = -2\pi,
\end{aligned}$$

and so by equation (16.30) we conclude that

$$\oint_C f\, dx + g\, dy = 2\pi,$$

as was to be shown. ■

It's worth reflecting on the example above, because the result is really quite astonishing: no matter how bizarre the curve C is, we are assured that the value of the line integral of $\mathbf{F}$ over C is the same. And not only that, we are able to determine that the value is 2π!

Theorem 16.29 (Green's Theorem—Flux Form). *Let $C \subseteq \mathbb{R}^2$ be a simple closed piecewise-smooth curve that bounds a region R, so that $C = \partial R$. If f and g have continuous first partial derivatives on an open region containing R, then*

$$\oint_{\partial R} f\, dy - g\, dx = \iint_R (f_x + g_y)\, dA.$$

This form of Green's Theorem can easily be proven by replacing g and f with f and $-g$, respectively, in the circulation form. Recalling the classical notation for the line integral and writing the double integral in Leibniz notation, the theorem tells us that

$$\oint_{\partial R} \mathbf{F} \cdot \mathbf{n} = \oint_{\partial R} f\, dy - g\, dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA,$$

where it is understood that $\mathbf{F} = \langle f, g\rangle$ and $\mathbf{n}$ is the outward unit normal vector function for C.

Example 16.30. Use Green's Theorem to evaluate the line integral

$$\oint_C f\, dy - g\, dx,$$

where $\langle f, g\rangle = \langle 0, xy\rangle$ and C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$.

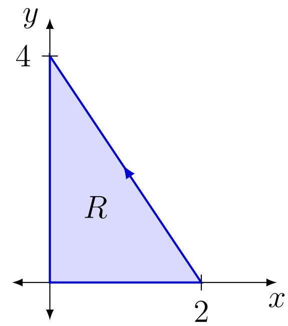


FIGURE 90.

Solution. The form of the line integral calls for the flux form of Green's Theorem. Letting R be the triangular region bounded by C, we obtain

$$\oint_C f\,dy - g\,dx = \iint_R [\partial_x(0) + \partial_y(xy)]\,dA = \iint_R x\,dA = \int_0^2 \int_0^{-2x+4} x\,dy\,dx$$
$$= \int_0^2 [xy]_0^{-2x+4}\,dx = \int_0^2 (-2x^2+4x)\,dx = \left[-\frac{2}{3}x^3 + 2x^2\right]_0^2 = \frac{8}{3}.$$

See Figure 90. ■

Wholly analogous to Proposition 16.27 (and proved by the same argument) is the following.

Proposition 16.31. *Let C_1, C_2 be closed simple piecewise-smooth curves, with C_1 lying in the interior of the region bounded by C_2. If C_1 has negative and C_2 positive orientation, R is the region bounded by C_1 and C_2, and g, f have continuous first partial derivatives on an open region containing R, then*

$$\iint_R (f_x + g_y)\,dA = \oint_{C_1} f\,dy - g\,dx + \oint_{C_2} f\,dy - g\,dx.$$

The usefulness of this result can be exhibited right away.

Example 16.32. Compute the flux of

$$\mathbf{F}(x,y) = \langle xy^2, x^2y \rangle$$

across the boundary of the region $R = \{(x,y) : 1 \le x^2 + y^2 \le 4\}$.

Solution. The region R, as Figure 91 shows, is an annulus with a boundary ∂R that consists of two simple closed piecewise-smooth curves, C_1 and C_2, that are completely separate from one another, with C_1 being in the interior of the region bounded by C_2. Let R_1 and R_2 be the regions bounded by C_1 and C_2, respectively. The flux of $\mathbf{F} = \langle f, g \rangle$ across C_1 *into* R equals the flux of $\mathbf{F}$ *out of* R_1, and thus the flux of $\mathbf{F}$ across C_1 *out of* R is the *negative* of the flux of $\mathbf{F}$ *out of* R_1. As a result the (outward) flux of $\mathbf{F}$ across $\partial R = C_1 \cup C_2$ as a whole is given by

$$\text{Flux of } \mathbf{F} \text{ across } \partial R = (\text{Flux of } \mathbf{F} \text{ across } C_2) - (\text{Flux of } \mathbf{F} \text{ across } C_1)$$

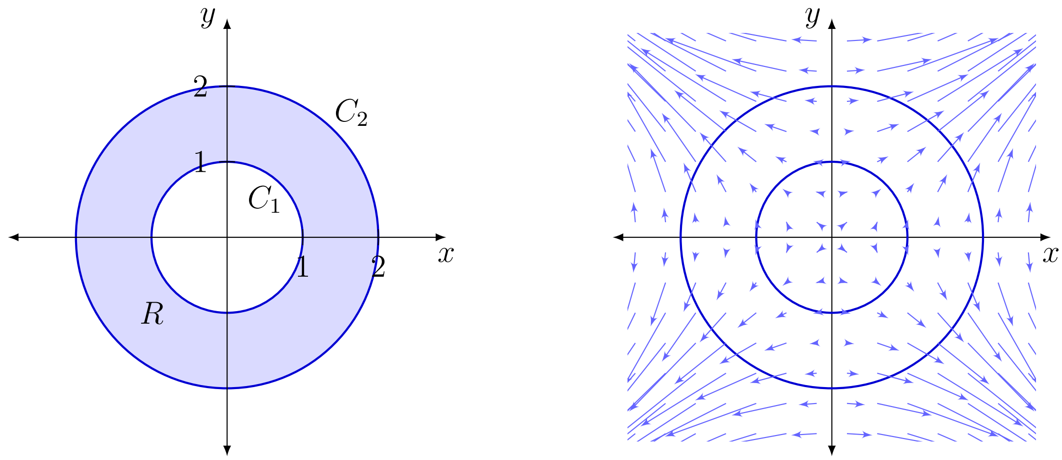


FIGURE 91. At right, the vector field $\mathbf{F}$ at 1/4-scale.

$$
\begin{aligned}
&= \ointctrclockwise_{C_2} f\,dy - g\,dx - \ointctrclockwise_{C_1} f\,dy - g\,dx \\
&= \ointctrclockwise_{C_2} f\,dy - g\,dx + \ointclockwise_{C_1} f\,dy - g\,dx \\
&= \iint\limits_R (f_x + g_y)\,dA,
\end{aligned}
$$

where the last equality is a consequence of Proposition 16.31.

If $T : \mathbb{R}^2_{r\theta} \to \mathbb{R}^2_{xy}$ is the usual conversion transformation from polar to rectangular coordinates, then it can be seen that $R = T(S)$ for

$$S = \{(r, \theta) : 0 \le \theta \le 2\pi \text{ and } 0 \le r \le 1\},$$

and so Theorem 15.11 gives

$$
\begin{aligned}
\iint\limits_R (f_x + g_y)\,dA = \iint\limits_R (x^2 + y^2)\,dA &= \iint\limits_S (r^2 \sin^2\theta + r^2\cos^2\theta) r\,dA \\
&= \iint\limits_S r^3\,dA = \int_0^{2\pi} \int_1^2 r^3\,dr d\theta = \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_1^2 d\theta \\
&= \int_0^{2\pi} \frac{15}{4}\,d\theta = \frac{15}{2}\pi.
\end{aligned}
$$

Therefore the (outward) flux of $\mathbf{F}$ across ∂R is $15\pi/2$. ■

16.5 – Divergence and Curl

The **gradient operator** in $\mathbb{R}^n$ is notationally represented as

$$\nabla = \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle, \tag{16.31}$$

and is defined by

$$\nabla f = \langle \partial_{x_1} f, \dots, \partial_{x_n} f \rangle = \langle f_{x_1}, \dots, f_{x_n} \rangle$$

for a function f with domain a subset of $\mathbb{R}^n$. Thus, if $\mathbf{x} \in \mathbb{R}^n$ is a point where the partial derivatives f_{x_i} of f all exist, then

$$\nabla f(\mathbf{x}) = \langle f_{x_1}(\mathbf{x}), \dots, f_{x_n}(\mathbf{x}) \rangle.$$

We see that ∇f is the familiar "gradient of f" from Chapter 13.

It must be stressed that the expression on the right-hand side of (16.31) is not a vector, but rather a convenient symbol that facilitates the proper use of the operator ∇. For instance, if $\mathbf{F} = \langle f_1, \dots, f_n \rangle$ is a vector field with domain a subset of $\mathbb{R}^n$, then we may define the function

$$\nabla \cdot \mathbf{F} = \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle \cdot \langle f_1, \dots, f_n \rangle = \sum_{i=1}^{n} \partial_{x_i} f_i. \tag{16.32}$$

Similarly, if $\mathbf{F} = \langle f, g, h \rangle$ is a vector field with domain a subset of $\mathbb{R}^3$, then we may define the function

$$\nabla \times \mathbf{F} = \langle \partial_x, \partial_y, \partial_z \rangle \times \langle f, g, h \rangle = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle. \tag{16.33}$$

The equalities in (16.32) and (16.33) are established *by definition*, with the middle expressions serving as convenient symbols to help connect the functions $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ to their intended meanings. When the dot product is applied to the middle expression in (16.32) as if $\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$ were a vector, the expression at right follows immediately. As for (16.33), we need only treat the middle expression as a cross product of two vectors to obtain

$$\begin{aligned}
\nabla \times \mathbf{F} = \langle \partial_x, \partial_y, \partial_z \rangle \times \langle f, g, h \rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} \\
&= \begin{vmatrix} \partial_y & \partial_z \\ g & h \end{vmatrix} \mathbf{i} - \begin{vmatrix} \partial_x & \partial_z \\ f & h \end{vmatrix} \mathbf{j} + \begin{vmatrix} \partial_x & \partial_y \\ f & g \end{vmatrix} \mathbf{k} \\
&= (\partial_y h - \partial_z g)\mathbf{i} - (\partial_x h - \partial_z f)\mathbf{j} + (\partial_x g - \partial_y f)\mathbf{k} \\
&= (h_y - g_z)\mathbf{i} + (f_z - h_x)\mathbf{j} + (g_x - f_y)\mathbf{k},
\end{aligned}$$

which readily delivers the expression at right in (16.33).

The setting throughout the remainder of this section will be strictly limited to $\mathbb{R}^3$, the three-dimensional space that is typical in physical applications.

Definition 16.33. *Let $\mathbf{F} = \langle f, g, h \rangle$ be a differentiable vector field on a region $R \subseteq \mathbb{R}^3$. The **divergence** of $\mathbf{F}$ on R is*

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = f_x + g_y + h_z,$$

*and the **curl** of* $\mathbf{F}$ *on* R *is*

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle.$$

If $\operatorname{div}\mathbf{F} = 0$*, then* $\mathbf{F}$ *is **source-free**; and if* $\operatorname{curl}\mathbf{F} = \mathbf{0}$*, then* $\mathbf{F}$ *is **irrotational**.*

Thus, if $\mathbf{x} \in R$, we have

$$(\operatorname{div}\mathbf{F})(\mathbf{x}) = (\nabla \cdot \mathbf{F})(\mathbf{x}) = f_x(\mathbf{x}) + g_y(\mathbf{x}) + h_z(\mathbf{x})$$

and

$$(\operatorname{curl}\mathbf{F})(\mathbf{x}) = (\nabla \times \mathbf{F})(\mathbf{x}) = \big\langle h_y(\mathbf{x}) - g_z(\mathbf{x}), f_z(\mathbf{x}) - h_x(\mathbf{x}), g_x(\mathbf{x}) - f_y(\mathbf{x}) \big\rangle$$

Example 16.34. Find the divergence of $\mathbf{F}(x, y, z) = \langle yz \sin x, xz \cos y, xy \cos z \rangle$.

Solution. We have

$$\begin{aligned}
(\operatorname{div}\mathbf{F})(x, y, z) &= \nabla \cdot \mathbf{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle yz \sin x, xz \cos y, xy \cos z \rangle \\
&= \partial_x(yz \sin x) + \partial_y(xz \cos y) + \partial_z(xy \cos z) \\
&= yz \cos x - xz \sin y - xy \sin z,
\end{aligned}$$

by Definition 16.33. ■

Example 16.35. Find the curl of $\mathbf{F}(x, y, z) = \langle 0, z^2 - y^2, yz \rangle$.

Solution. We have

$$\begin{aligned}
(\operatorname{curl}\mathbf{F})(x, y, z) = (\nabla \times \mathbf{F})(x, y, z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & z^2 - y^2 & yz \end{vmatrix} \\
&= \begin{vmatrix} \partial_y & \partial_z \\ z^2 - y^2 & yz \end{vmatrix} \mathbf{i} - \begin{vmatrix} \partial_x & \partial_z \\ 0 & yz \end{vmatrix} \mathbf{j} + \begin{vmatrix} \partial_x & \partial_y \\ 0 & z^2 - y^2 \end{vmatrix} \mathbf{k} \\
&= \big[\partial_y(yz) - \partial_z(z^2 - y^2)\big]\mathbf{i} - \big[\partial_x(yz) - \partial_z(0)\big]\mathbf{j} + \big[\partial_x(z^2 - y^2) - \partial_y(0)\big]\mathbf{k} \\
&= -z\mathbf{i} = \langle -z, 0, 0 \rangle,
\end{aligned}$$

by Definition 16.33. ■

16.6 – Parametrized Surfaces

In Chapter 12 the concept of a parameterized curve C was introduced, which is a curve that is generated by a vector function $\mathbf{r} : I \subseteq \mathbb{R} \to \mathbb{R}^n$, $n \geq 2$, given as $\mathbf{r}(t) = \langle x_1(t), \dots, x_n(t) \rangle$ for $t \in I$. Thus as a point set $C = \mathbf{r}(I)$, which is to say C is the trace, or range, of $\mathbf{r}$. Now we consider parameterized surfaces.

Definition 16.36. *A* ***parameterized surface*** Σ *is a surface that is generated by a vector function* $\mathbf{r} : R \subseteq \mathbb{R}^2 \to \mathbb{R}^n$*,* $n \geq 3$*, given as*

$$\mathbf{r}(u, v) = \langle x_1(u, v), x_2(u, v), \dots, x_n(u, v) \rangle$$

for $(u, v) \in R$*. Thus as a point set* $\Sigma = \mathbf{r}(R) \subseteq \mathbb{R}^n$*.*

The symbol Σ, and not S, will consistently be used to designate a surface throughout the remainder of this chapter, because S will consistently be used to denote surface area. Also, for the remainder of this chapter we will assume that any surface is in $\mathbb{R}^3$ unless otherwise specified.

Example 16.37. Consider an ellipsoid in $\mathbb{R}^3$ centered at the origin, given by the general equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{16.34}$$

We can parameterize this surface Σ using the fact that any point lying on it can be specified with two bits of information: first, an angle $0 \leq u \leq \pi$ with respect to the positive x-axis which determines a point $\mathbf{x}_u$ on the ellipse C that is the xy-trace of Σ as shown at left in Figure 92, and then an angle $0 \leq v \leq 2\pi$ which determines a point $\mathbf{x}_{uv}$ on the ellipse C_u lying at the intersection of Σ with the plane containing $\mathbf{x}_u$ that is parallel to the yz-plane, as shown at right in Figure 92.

Since C can be parameterized by

$$\boldsymbol{\rho}(t) = \langle a \cos t, b \sin t, 0 \rangle$$

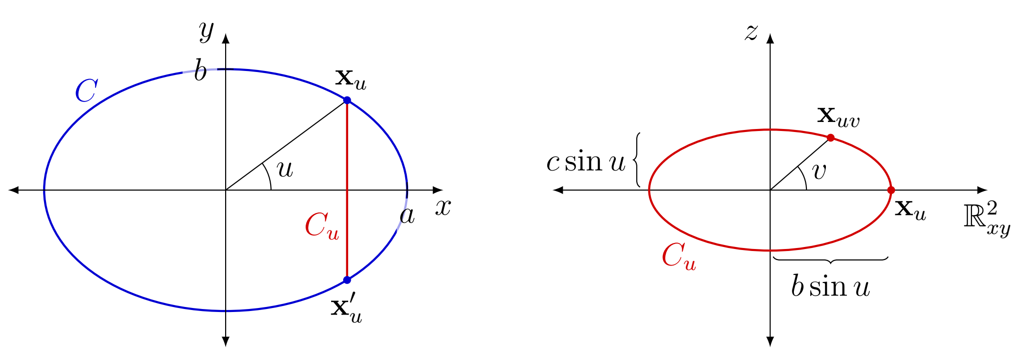


Figure 92. Left: the ellipse C, with segment $[\mathbf{x}_u, \mathbf{x}'_u]$ depicting the cross-sectional ellipse C_u viewed edge-on from above. Right: the cross-sectional ellipse C_u.

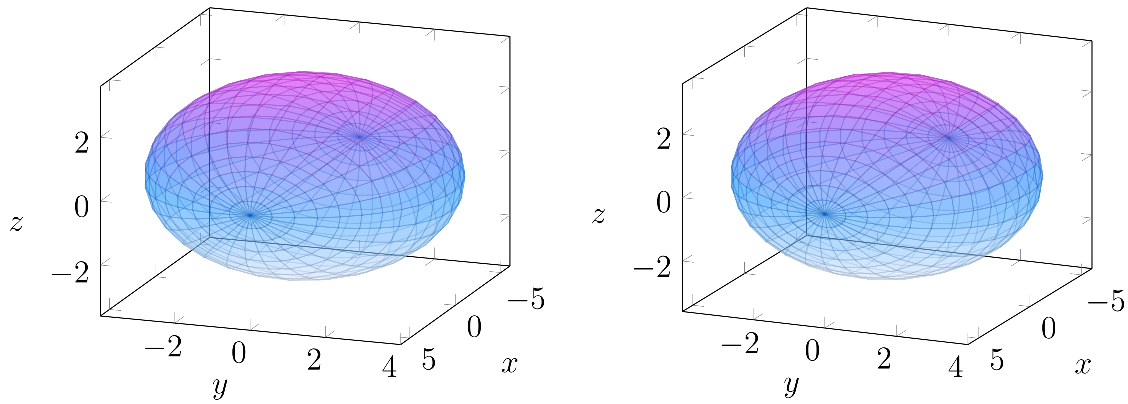


FIGURE 93. Ellipsoid given by (16.36), with "poles" at front and back.

for $t \in [0, 2\pi]$, we find that $\mathbf{x}_u = \boldsymbol{\rho}(u) = \langle a\cos u, b\sin u, 0\rangle$. Now, using (16.34) and the fact that $x = a\cos u$ on C_u, we find an equation for C_u:

$$\frac{(a\cos u)^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad\Rightarrow\quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \cos^2 u \quad\Rightarrow\quad \frac{y^2}{(b\sin u)^2} + \frac{z^2}{(c\sin u)^2} = 1.$$

From this we find a suitable parametrization for C_u to be

$$\boldsymbol{\rho}_u(t) = \langle a\cos u, b\sin u\cos t, c\sin u\sin t\rangle$$

for $t \in [0, 2\pi]$, and so $\mathbf{x}_{uv} = \boldsymbol{\rho}_u(v) = \langle a\cos u, b\sin u\cos v, c\sin u\sin v\rangle$. This is all we need to construct a parametrization for Σ itself:

$$\mathbf{r}(u, v) = \langle a\cos u, b\sin u\cos v, c\sin u\sin v\rangle, \tag{16.35}$$

for $u \in [0, \pi]$, $v \in [0, 2\pi]$. If we let $a = 6$, $b = 4$, and $c = 3$, we obtain an ellipsoid with parametrization

$$\mathbf{r}(u, v) = \langle 6\cos u, 4\sin u\cos v, 3\sin u\sin v\rangle, \tag{16.36}$$

as shown in Figure 93.

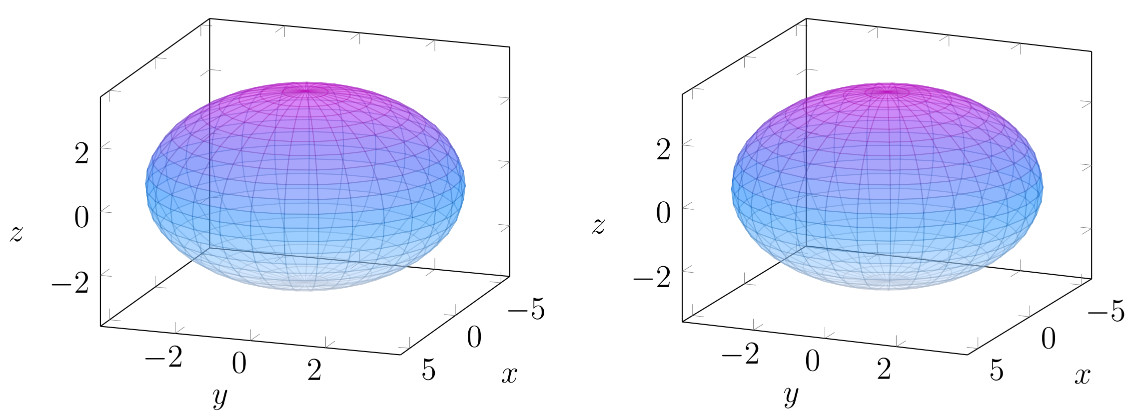


FIGURE 94. Ellipsoid given by (16.37), with "poles" at top and bottom.

As with curves there is always more than one possible parametrization for a given surface Σ. An alternate parametrization for the ellipsoid with $a = 6$, $b = 4$, and $c = 3$ can be obtained from (16.36) by permuting the roles of the variables x, y, and z in our coordinate system. In particular we could relabel axes, letting the x, y, and z axes become the z, x, and y axes, respectively, and replacing a, b, and c with c, b, and a. This would alter (16.36) to give the parametrization

$$\tilde{\mathbf{r}}(u, v) = \langle 6 \sin u \cos v, 4 \sin u \sin v, 3 \cos u \rangle. \tag{16.37}$$

The "new" coordinate system is in fact still a right-handed xyz-coordinate system, so by adjusting our vantage point accordingly we can view the graph of $\tilde{\mathbf{r}}$ with the coordinate axes in the same positions as before. See Figure 94, noting in particular the change in the lines of latitude and longitude, as well as the shift in position of the poles. ∎

Example 16.38. We now consider the surface Σ in Figure 95, called a torus. In Figure 96(a) is shown the xy-trace of Σ, which consists of two concentric circles centered at the origin, one with radius 4 and the other with radius 6. In Figure 96(b) is shown the xz-trace, and indeed any plane in $\mathbb{R}^3$ given by an equation of the form $y = cx$, where c is a constant, will intersect with Σ at two circles with radius 1. The task is to construct a parametrization for Σ.

A point on Σ can be specified with two parameters u and v. The first parameter, $0 \le u \le 2\pi$, can be taken to be an angle with respect to the positive x-axis (just like θ in the cylindrical coordinate system), which determines a point $\mathbf{x}_u$ on the circle C in the xy-plane with center at the origin and radius 5 as shown in Figure 96(c). If we parameterize C by

$$\boldsymbol{\rho}(t) = \langle 5 \cos t, 5 \sin t, 0 \rangle, \quad t \in [0, 2\pi],$$

then $\mathbf{x}_u = \boldsymbol{\rho}(u) = \langle 5 \cos u, 5 \sin u, 0 \rangle$. The points on Σ located at a given value of u will lie on a "longitudinal circle" (i.e. a circle perpendicular to the xy-plane) C_u of Σ having center at $\mathbf{x}_u$ and radius 1, as shown in Figure 96(d). The same figure depicts one viable way that the second parameter $0 \le v \le 2\pi$ may determine a particular point $\mathbf{x}_{uv}$ on C_u, which happens also to be a particular point on Σ itself. The challenge is to find a parametrization for C_u for any fixed value of $u \in [0, 2\pi]$.

Start with the circle C_0 in the xz-plane with center at the origin and radius 1. A parametrization for C_0 is given by

$$\boldsymbol{\rho}_0(t) = \langle \cos t, 0, \sin t \rangle, \quad t \in [0, 2\pi]$$

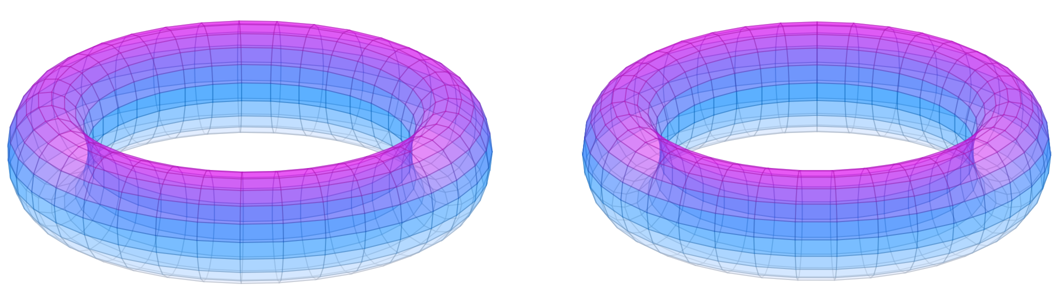
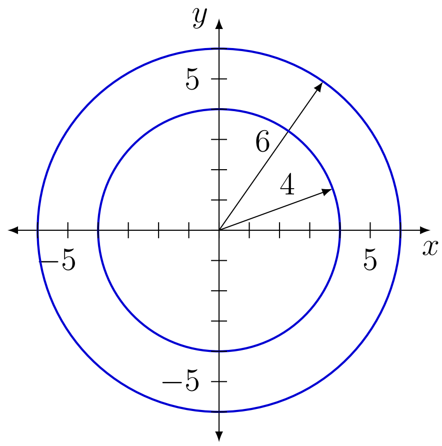
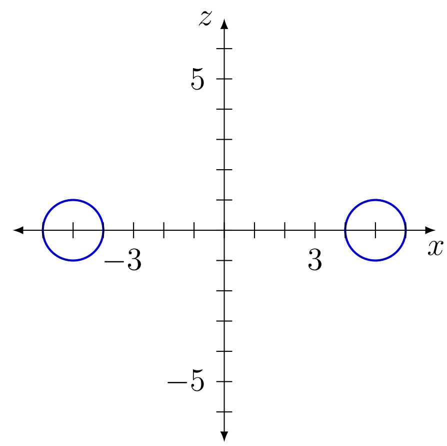


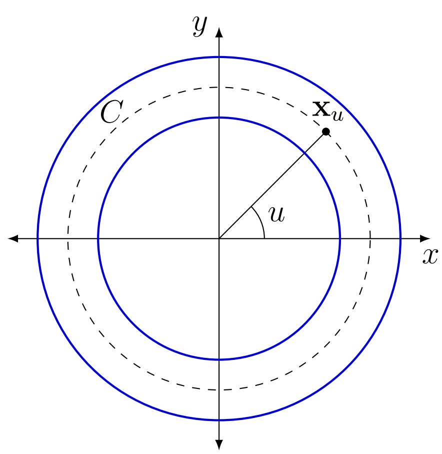
FIGURE 95. Stereoscopic image of a torus.



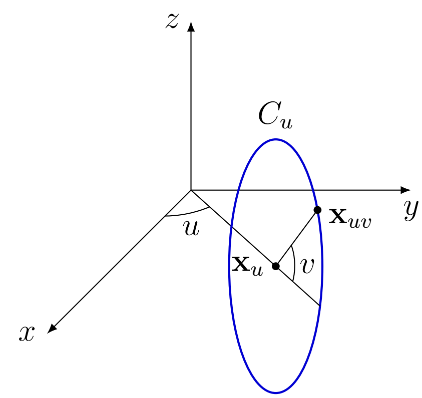
(a) The xy-trace of Σ.



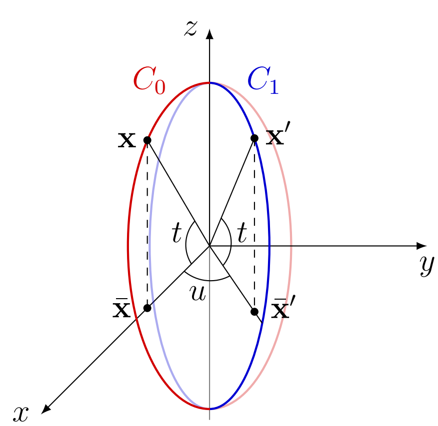
(b) The xz-trace of Σ.



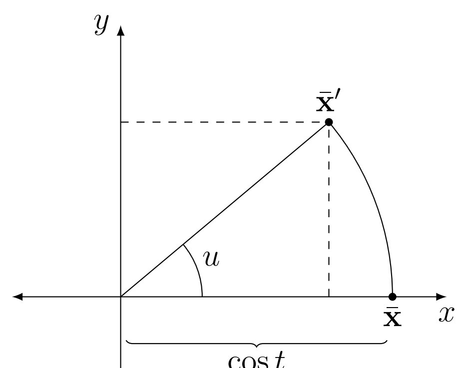
(c) The parameter u.



(d) The parameter v.



(e) The unit circles C_0 and C_1.



(f) Coordinate transformation.

FIGURE 96.

Now, if we rotate C_0 about the z-axis by angle u, we obtain a new circle C_1 parameterized by

$$\boldsymbol{\rho}_1(t) = \langle \cos u \cos t, \sin u \cos t, \sin t \rangle, \quad t \in [0, 2\pi]. \tag{16.38}$$

To see this consult Figure 96(e), which shows C_0 and C_1, a point $\mathbf{x} \in C_0$ and the corresponding point $\mathbf{x}' \in C_1$ obtained after rotating C_0, and the projections of these points onto the xy-plane, denoted in the figure by $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}}'$. If $\mathbf{x} = \langle \cos t, 0, \sin t \rangle$ for some $0 \le t \le 2\pi$, then $\bar{\mathbf{x}} = \langle \cos t, 0, 0 \rangle$, and from Figure 96(f) it can be seen that $\bar{\mathbf{x}}' = \langle \cos u \cos t, \sin u \sin t \rangle$ and thus

$$\mathbf{x}' = \langle \cos u \cos t, \sin u \cos t, \sin t \rangle,$$

(notice the z components of $\mathbf{x}$ and $\mathbf{x}'$ are the same since rotation is about the z-axis).

Next, we translate C_1 horizontally by $5 \cos u$ and vertically by $5 \sin u$ on the xy-plane to bring its center to $\mathbf{x}_u$, which gives us C_u. Thus C_u has parametrization

$$\boldsymbol{\rho}_u(t) = \boldsymbol{\rho}_1(t) + \langle 5 \cos u, 5 \sin u, 0 \rangle;$$

that is,

$$\boldsymbol{\rho}_u(t) = \langle 5 \cos u + \cos u \cos t, 5 \sin u + \sin u \cos t, \sin t \rangle, \quad t \in [0, 2\pi]$$

We specify a point $\mathbf{x}_{uv}$ on C_u, then, by setting $t = v$ for some $0 \le v \le 2\pi$, giving

$$\mathbf{x}_{uv} = \langle 5 \cos u + \cos u \cos v, 5 \sin u + \sin u \cos v, \sin v \rangle$$

and thereby determining a point on Σ. Hence

$$\mathbf{r}(u, v) = \langle 5 \cos u + \cos u \cos v, 5 \sin u + \sin u \cos v, \sin v \rangle, \quad u, v \in [0, 2\pi]$$

is a parametrization for Σ. ■

Notice that, from a mathematical standpoint, a vector function

$$\mathbf{r}(u, v) = \langle x_1(u, v), \ldots, x_n(u, v) \rangle, \quad (u, v) \in R,$$

is nothing more or less than a vector field in $\mathbb{R}^n$, and so Definition 16.1 already makes clear what it means for $\mathbf{r}$ to be continuous, differentiable, or continuously differentiable on any arbitrary set region $S \subseteq R$. What follows is a definition of what it means for $\mathbf{r}$ to be smooth on R which will be sufficient to suit our needs for the remainder of the chapter; but it is a stricter definition than the one developed in differential geometry courses with more sophisticated machinery.

Definition 16.39. *A continuous function* $\mathbf{r} : R \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ *given by*

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \tag{16.39}$$

is ***smooth*** *on* R *if all of the following are true:*

1. R *is a connected and simply connected set.*
2. $\mathbf{r}$ *is continuously differentiable on* $\mathrm{Int}(R)$.
3. $\mathbf{r}$ *is one-to-one on* $\mathrm{Int}(R)$.
4. $(\mathbf{r}_u \times \mathbf{r}_v)(u, v) \neq \mathbf{0}$ *for all* $(u, v) \in \mathrm{Int}(R)$

Definition 16.40. *A continuous function* $\mathbf{r} : R \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ *given by* (16.39) *that is one-to-one on* $\mathrm{Int}(R)$ *is* ***piecewise-smooth*** *on* R *if either one of the following is true:*

1. *R is connected and simply connected, but can be partitioned into sets $R_1, \dots, R_n$ such that* $\mathbf{r}$ *is smooth on each R_k.*
2. *R is a disjoint union of connected and simply connected sets $R_1, \dots, R_n$ such that* $\mathbf{r}$ *is smooth on each R_k.*

In the common situation when $R = [a, b] \times [c, d]$ (a closed rectangle), it is reasonable to define $\mathbf{r}$ to be piecewise-smooth if there exists a partition

$$\{a = u_0 < u_1 < \cdots < u_{m-1} < u_m = b\ ;\ c = v_0 < v_1 < \cdots < v_{n-1} < v_n = d\}$$

of R such that $\mathbf{r}$ is smooth on $R_{ij} = [u_{i-1}, u_i] \times [v_{j-1}, v_j]$ for each $1 \leq i \leq m$, $1 \leq j \leq n$.

For our purposes a surface Σ will be said to be **smooth** if it admits a smooth parametrization in the sense of Definition 16.39, and **piecewise-smooth** if it admits a piecewise-smooth parametrization in the sense of Definition 16.40. Whenever a surface is said to be smooth or piecewise-smooth it is assumed that any parametrization given for the surface is itself smooth or piecewise-smooth.

Example 16.41. Let Σ_1 be the rectangular region with vertices $(2, 0, 0)$, $(2, 1, 0)$, $(0, 1, 0)$, $(0, 0, 0)$, let Σ_2 be the rectangular region with vertices $(2, 0, 0)$, $(0, 0, 0)$, $(0, 0, 3)$, $(2, 0, 3)$, and let Σ_3 be the rectangular region with vertices $(2, 0, 3)$, $(2, 1, 0)$, $(0, 1, 0)$, $(0, 0, 3)$. Consider the surface $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, shown in Figure 97. The curve C with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 3)$ shown in the figure lies on Σ, and can be parameterized by function $\boldsymbol{\rho} : [0, 3] \to \mathbb{R}^3$ given by

$$\boldsymbol{\rho}(v) = \begin{cases} \langle 0, v, 0 \rangle, & v \in [0, 1] \\ \langle 0, 2 - v, 3v - 3 \rangle, & v \in [1, 2] \\ \langle 0, 0, 9 - 3v \rangle, & v \in [2, 3] \end{cases}$$

For each $u \in [0, 2]$ the function $\boldsymbol{\rho}_u : [0, 3] \to \mathbb{R}^3$ given by $\boldsymbol{\rho}_u(v) = \boldsymbol{\rho}(v) + \langle u, 0, 0 \rangle$ generates another triangle that lies on Σ, and the union of all these triangles generates Σ itself. Thus, we

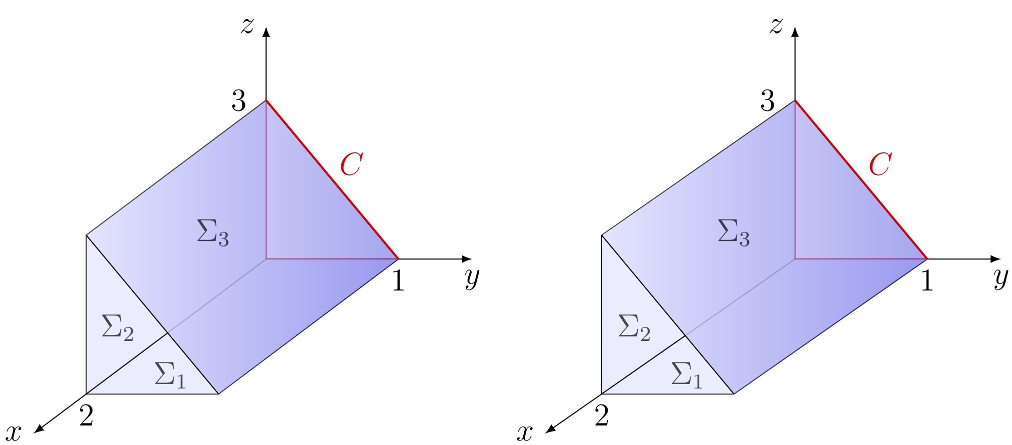


FIGURE 97. The piecewise-smooth surface Σ.

may parameterize Σ by the function $\mathbf{r} : [0,2] \times [0,3] \to \mathbb{R}^3$ given by

$$\mathbf{r}(u,v) = \begin{cases} \langle u, v, 0 \rangle, & (u,v) \in [0,2] \times [0,1] \\ \langle u, 2-v, 3v-3 \rangle, & (u,v) \in [0,2] \times [1,2] \\ \langle u, 0, 9-3v \rangle, & (u,v) \in [0,2] \times [2,3] \end{cases}$$

The surface Σ is not smooth since $\mathbf{r}$ is not continuously differentiable on $(0,2) \times (0,3)$, which is the interior of the domain of $\mathbf{r}$. For instance, the first partial derivatives of $\mathbf{r}$ do not exist along the line segment $[(0,1),(2,1)]$ in the uv-plane connecting the points $(0,1)$ and $(2,1)$. However, Σ is piecewise-smooth according to Definition 16.40(1), since the rectangle $[0,2] \times [0,3]$ can be partitioned into subsets $[0,2] \times [0,1]$, $[0,2] \times [1,2]$, and $[0,2] \times [2,3]$, and $\mathbf{r}$ restricted to each of these subsets *is* smooth.

Another way to parameterize Σ is to use three separate parametrizations for each of the rectangles Σ_1, Σ_2, and Σ_3. We have

$$\begin{aligned} &\Sigma_1 : \quad \mathbf{r}_1(u,v) = \langle u, v, 0 \rangle, && (u,v) \in [0,2] \times [0,1] \\ &\Sigma_2 : \quad \mathbf{r}_2(u,v) = \langle u, 0, v \rangle, && (u,v) \in [0,2] \times [0,3] \\ &\Sigma_3 : \quad \mathbf{r}_3(u,v) = \langle u, v, -3v+3 \rangle, && (u,v) \in [0,2] \times [0,1] \end{aligned}$$

We can regard the domains of the functions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ as being subsets of different "copies" of the uv-plane. Function $\mathbf{r}_1$, as indicated by its subscript, has domain in copy 1 of the uv-plane, $\mathbf{r}_2$ has domain in copy 2, and $\mathbf{r}_3$ has domain in copy 3. Formally, the subscript i in the symbol $\mathbf{r}_i$ is taken to be a third independent variable that designates which copy of the uv-plane the domain of $\mathbf{r}_i$ belongs in. Thus the domains R_1, R_2, R_3 of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are considered to be disjoint, and if we let $R = R_1 \cup R_2 \cup R_3$, then the three functions taken together create a function

$$\mathbf{r} : R \times \{1,2,3\} \to \mathbb{R}^3$$

that is a parametrization of Σ:

$$\mathbf{r}(u,v,i) = \mathbf{r}_i(u,v).$$

We see, however, that Σ is now piecewise-smooth in the sense of Definition 16.40(2), which goes to show there is no substantive difference between the two ways a surface may be piecewise-smooth according to Definition 16.40.

In practice we will be less formal, which is to say we will make no attempt to put separate functions such as $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ together to obtain a single parametrization $\mathbf{r}$ for a surface. Each $\mathbf{r}_i$ gives us a piece of the surface, and the pieces "add up" to the whole surface in question. ∎

16.7 – Surface Integrals

Definition 16.42. *Let f be a scalar-valued function that is continuous on a smooth surface Σ parameterized by $\mathbf{r}(u,v)$ for $(u,v) \in R$. The* ***surface integral of*** *f* ***over*** *Σ is*

$$\iint_{\Sigma} f = \iint_{R} f(\mathbf{r}(u,v)) \|(\mathbf{r}_u \times \mathbf{r}_v)(u,v)\| \, dA.$$

Alternative symbols for the surface integral of f over Σ are

$$\iint_{\Sigma} f \, dS, \quad \iint_{\Sigma} f(\mathbf{r}(u,v)) \, dS, \quad \text{and} \quad \iint_{\Sigma} f(x_1(u,v), \ldots, x_n(u,v)) \, dS,$$

with the idea being that dS represents an "elemental surface area" in a manner consonant with the usual non-rigorous hocus-pocus of Leibnizian cultists. Extending this notion we have the following definition.

Definition 16.43. *If Σ is a smooth surface parameterized by $\mathbf{r}(u,v)$, $(u,v) \in R$, then the* ***surface area*** *of Σ is*

$$\mathcal{A}(\Sigma) = \iint_{\Sigma} dS,$$

provided that the double integral exists. If Σ is not smooth, but Σ' given by $\mathbf{r}(u,v)$, $(u,v) \in R' \subseteq R$, is smooth, where $\mathcal{A}(R \setminus R') = 0$, then we define $\mathcal{A}(\Sigma) = \mathcal{A}(\Sigma')$ provided that $\mathcal{A}(\Sigma')$ is defined.

Thus $\mathcal{A}(\Sigma)$ is just $\iint_{\Sigma} f \, dS$ for the constant function $f(x_1, \ldots, x_n) = 1$. All other "definitions" of area given earlier in this book are actually propositions that could be proven using Definition 16.43.

Example 16.44. Find the lateral surface area of the circular cone with height h and base radius r.

Solution. The lateral surface area would be the area of only the "side" of the cone, which is to say the area of the base (which is a circular disc of radius r) is not included. Let Σ be the lateral surface of the circular cone with height h and base radius r. A parametrization for Σ is

$$\mathbf{r}(u,v) = \left\langle \frac{rv}{h} \cos u, \frac{rv}{h} \sin u, v \right\rangle, \quad (u,v) \in [0, 2\pi] \times [0, h].$$

Let $R = [0, 2\pi] \times [0, h]$. Notice that Σ is not smooth owing to the cone's apex at $(0,0,0)$, so we pass to a new surface Σ' parameterized by $\mathbf{r}(u,v)$ for $(u,v) \in [0, 2\pi] \times (0, h] := R' \subseteq R$, which is simply the cone with its apex removed. Since $R \setminus R'$ is just a line segment in the uv-plane it is clear that $\mathcal{A}(R \setminus R') = 0$, and so $\mathcal{A}(\Sigma) = \mathcal{A}(\Sigma')$ by Definition 16.43. Now,

$$(\mathbf{r}_u \times \mathbf{r}_v)(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\dfrac{rv}{h} \sin u & \dfrac{rv}{h} \cos u & 0 \\ \dfrac{r}{h} \cos u & \dfrac{r}{h} \sin u & 1 \end{vmatrix} = \left\langle \frac{rv}{h} \cos u, \frac{rv}{h} \sin u, -\frac{r^2 v}{h^2} \right\rangle,$$

so that

$$\|(\mathbf{r}_u \times \mathbf{r}_v)(u,v)\| = \sqrt{\frac{r^2v^2}{h^2}\cos^2 u + \frac{r^2v^2}{h^2}\sin^2 u + \left(\frac{r^2v}{h^2}\right)^2} = \frac{rv}{h}\sqrt{1+\frac{r^2}{h^2}}$$

and we're ready to find the surface area of Σ'. Observing that $\|\mathbf{r}_u \times \mathbf{r}_v\|$ is a continuous function on the compact set R (and hence a bounded function), we have

$$\begin{aligned}
\mathcal{A}(\Sigma') &= \iint_{\Sigma'} dS = \iint_{R'} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA && \text{(Definitions 16.43 \& 16.42)}\\
&= \iint_{\overline{R'}} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA = \iint_{R} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA && \text{(Definition 15.4)}\\
&= \int_0^{2/\pi}\int_0^h \frac{rv}{h}\sqrt{1+r^2/h^2}\, dv du && \text{(Theorem 15.5)}\\
&= \int_0^{2\pi} \frac{rh}{2}\sqrt{1+r^2/h^2}\, du\\
&= \pi rh\sqrt{1+r^2/h^2}.
\end{aligned}$$

Therefore

$$\mathcal{A}(\Sigma) = \pi r\sqrt{r^2+h^2}$$

is the sought after area. ■

Example 16.45. Recall from §6.3 the concept of a surface of revolution: given a differentiable function f such that $f(x) > 0$ for all $x \in [a,b]$, we revolve the curve C given by $y = f(x)$, $a \le x \le b$, fully about the x-axis to generate a surface Σ in $\mathbb{R}^3$. The question is, what is a parametrization for Σ?

Let $u \in [a,b]$. A point on Σ at $x = u$ can lie anywhere on a circle C_u in $\mathbb{R}^3$ that is parallel to the $yz-plane$, has center at $(u,0,0)$, and radius $f(u)$. We can parameterize C_u by

$$\boldsymbol{\rho}(t) = \langle u, f(u)\cos t, f(u)\sin t\rangle, \quad t \in [0,2\pi],$$

and so determine a specific point on C_u, and hence on Σ itself, by specifying a value $t = v$ for some $0 \le v \le 2\pi$. Thus a parametrization for Σ is

$$\mathbf{r}(u,v) = \langle u, f(u)\cos v, f(u)\sin v\rangle, \quad (u,v) \in [a,b]\times[0,2\pi].$$

With this parametrization we are in a position to determine the area of Σ. We have

$$\mathbf{r}_u(u,v) = \langle 1, f'(u)\cos v, f'(u)\sin v\rangle \quad \text{and} \quad \mathbf{r}_v(u,v) = \langle 0, -f(u)\sin v, f(u)\cos v\rangle,$$

so that

$$(\mathbf{r}_u \times \mathbf{r}_v)(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u)\cos v & f'(u)\sin v \\ 0 & -f'(u)\sin v & f(u)\cos v \end{vmatrix} = \langle f'(u)f(u), -f(u)\cos v, -f(u)\sin v\rangle,$$

and hence

$$\|(\mathbf{r}_u \times \mathbf{r}_v)(u,v)\| = \sqrt{[f'(u)f(u)]^2 + f^2(u)\cos^2 v + f^2(u)\sin^2 v} = f(u)\sqrt{1+[f'(u)]^2}.$$

By Definition 16.43 we obtain

$$\begin{aligned}\mathcal{A}(\Sigma) &= \iint\limits_{\Sigma} dS = \iint\limits_{R} f(u)\sqrt{1+[f'(u)]^2}\,dA \\ &= \int_a^b \int_0^{2\pi} f(u)\sqrt{1+[f'(u)]^2}\,dvdu = \int_a^b 2\pi f(u)\sqrt{1+[f'(u)]^2}\,du\end{aligned}$$

for the area of a surface of revolution. ■

Definition 16.46. *A smooth surface $\Sigma \subseteq \mathbb{R}^3$ parameterized by*

$$\mathbf{r}(u,v) = \big\langle x(u,v), y(u,v), z(u,v)\big\rangle, \quad (u,v) \in R,$$

*is **orientable** if the function $\hat{\mathbf{n}} : \mathrm{Int}(R) \to \mathbb{R}^3$ given by*

$$\hat{\mathbf{n}}(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}(u,v) \tag{16.40}$$

has the following properties:

1. $\hat{\mathbf{n}}$ *is continuous on* $\mathrm{Int}(R)$*, with a continuous extension* $\hat{\mathbf{n}}_c$ *to* R*.*
2. $\hat{\mathbf{n}}_c(u_1,v_1) = \hat{\mathbf{n}}_c(u_2,v_2)$ *for any* $(u_1,v_1),(u_2,v_2) \in R$ *such that* $\mathbf{r}(u_1,v_1) = \mathbf{r}(u_2,v_2)$*.*

Henceforth we will denote $\hat{\mathbf{n}}_c$ by $\hat{\mathbf{n}}$, which is to say we assume any $\hat{\mathbf{n}}$ is in fact a continuous extension of the function defined by (16.40). We call $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$ the two possible **orientations** for Σ that consist of unit vectors. Once an orientation $\mathbf{n} = \pm\hat{\mathbf{n}}$ has been chosen for Σ we say that the surface is **oriented**. More formally an oriented surface may be regarded as a pairing $(\Sigma, \mathbf{n})$ of a surface Σ with an orientation $\mathbf{n}$.

The function $\mathbf{n}$ is thought of as assigning a unit vector $\mathbf{n}(u,v)$ to each point $\mathbf{r}(u,v)$ on the surface Σ, and it can be shown that $\mathbf{n}(u,v)$ is *orthogonal* to Σ at $\mathbf{r}(u,v)$; that is, $\mathbf{n}(u,v)$ is a normal vector for the tangent plane to Σ at $\mathbf{r}(u,v)$. (The smoothness of Σ at the point $\mathbf{r}(u,v)$ is what ensures that Σ in fact has a tangent plane at $\mathbf{r}(u,v)$.) So $\mathbf{n}$ defines a vector field on Σ consisting entirely of unit vectors that are orthogonal to Σ.

On a local scale, the continuity of $\mathbf{n}$ on R implies that for any $\mathbf{x} \in \Sigma$, there exists a sufficiently small $\epsilon > 0$ such that the unit vectors assigned by $\mathbf{n}$ to the points in $B_\epsilon(\mathbf{x}) \cap \Sigma$ point approximately in the same direction. This means that, globally, the vectors that $\mathbf{n}$ assigns to Σ all issue from the same "side" of the surface. Generally speaking, an orientable surface is a surface for which it is possible to identify two sides. If the two sides were painted different colors, there is no point on the surface where the two colors will necessarily mix.

A surface that is not orientable is called **nonorientable**. The most famous example of a surface that is not orientable is a Möbius band. Shown in Figure 98 is a Möbius band that has parametrization

$$\mathbf{r}(u,v) = \left\langle \cos(u) + \frac{v}{2}\cos(u)\cos\!\left(\frac{u}{2}\right),\ \sin(u) + \frac{v}{2}\sin(u)\cos\!\left(\frac{u}{2}\right),\ \frac{v}{2}\sin\!\left(\frac{u}{2}\right)\right\rangle$$

for $(u,v) \in [0,2\pi] \times [-1/4, 1/4]$.

Let Σ be a piecewise-smooth surface, so that it consists of smaller surfaces $\Sigma_1, \ldots, \Sigma_k$ which are each smooth by themselves. Let $\mathbf{r}(u,v)$, $(u,v) \in R$, be a parametrization for Σ, and define $R' \subseteq R$ by

$$R' = \{(u,v) \in R : \Sigma \text{ is not smooth at } \mathbf{r}(u,v)\}.$$

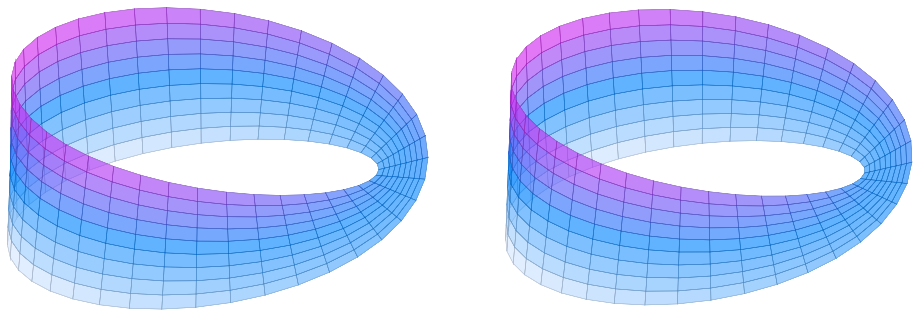


FIGURE 98. Stereoscopic image of a Möbius band, a surface with only one side.

Now, for each $1 \le i \le k$ suppose Σ_i is orientable, and is given orientation $\mathbf{n}_i$. If the functions $\mathbf{n}_1, \ldots, \mathbf{n}_k$ can be chosen so that they assign vectors to Σ that all issue from the same "side" of Σ, then we say Σ is orientable.[15] One possible orientation is the function $\mathbf{n} : R' \to \mathbb{R}^3$ given by $\mathbf{n}(u, v) = \mathbf{n}_i(u, v)$ if $\mathbf{r}(u, v) \in \Sigma_i$.

Definition 16.47. *Let $\mathbf{F} = \langle f, g, h \rangle$ be a continuous vector field on an open region $D \subseteq \mathbb{R}^3$, and let $\Sigma \subseteq D$ be a smooth oriented surface parameterized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, $(u, v) \in R$. The **flux of $\mathbf{F}$ across** Σ is*

$$\iint\limits_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{R} \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{n}(u, v) \|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|\, dA,$$

where $\mathbf{n}$ is either $\hat{\mathbf{n}}$ as given in (16.40), or $-\hat{\mathbf{n}}$, depending on the chosen orientation of Σ.

The integral given in Definition 16.47 is often called simply a **flux integral**, with $\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S}$ being an alternative symbol so that

$$\iint\limits_{\Sigma} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{R} (\mathbf{F} \circ \mathbf{r}) \cdot \mathbf{n} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

by definition. The value of a flux integral is independent of the parametrization $\mathbf{r}$ chosen for Σ, but will change in sign depending on whether the orientation $\mathbf{n}$ is chosen to be $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$. If we choose $\mathbf{n} = \hat{\mathbf{n}}$, then by Definition 16.42

$$\begin{aligned}
\iint\limits_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint\limits_{R} \mathbf{F}(\mathbf{r}(u, v)) \cdot \hat{\mathbf{n}}(u, v) \|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|\, dA \\
&= \iint\limits_{R} \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{(\mathbf{r}_u \times \mathbf{r}_v)(u, v)}{\|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|} \|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|\, dA \\
&= \iint\limits_{R} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)(u, v)\, dA \\
&= \iint\limits_{R} (\mathbf{F} \circ \mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \qquad (16.41)
\end{aligned}$$

A surface Σ is **closed** if it encloses a bounded region $D \subseteq \mathbb{R}^3$, which is to say that for any $\mathbf{x} \in \mathrm{Int}(D)$ and $\mathbf{y} \notin D$ there exists no continuous path from $\mathbf{x}$ to $\mathbf{y}$ that does not pass through Σ; that is, if C is a continuous curve having $\mathbf{x}$ and $\mathbf{y}$ as endpoints, then $C \cap \Sigma \neq \varnothing$. Examples

[15]This definition is admittedly intuitive, not rigorous. But it will suffice for our purposes.

of closed surfaces are ellipsoids (Figure 99) and tori (Figure 100). Planes, cones, cylinders, hemispheres and paraboloids are examples of surfaces that are not closed. Care should be taken not to conflate the concept of a closed surface with that of a closed set. A plane, as a subset of $\mathbb{R}^3$, is a closed set even though it is not a closed surface.

A closed surface Σ is said to have **positive orientation** if it is orientated such that all unit normal vectors point away from the region D bounded by Σ. If all unit normal vectors point toward D, then Σ has a **negative orientation**.

Example 16.48. Let $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$, and let Σ be a tetrahedron with center at the origin, face Σ_1 in the first octant given by $z = 10 - 2x - 5y$, and positive orientation. Find the flux of $\mathbf{F}$ across the face Σ_1.

Solution. To find a parametrization for Σ_1 we first find its xy-trace. This is a line segment in the first quadrant of the xy-plane given by $0 = 10 - 2x - 5y$, or equivalently $y = -\frac{2}{5}x + 2$, which forms part of the boundary of a region R' as shown in at left in Figure 99. This corresponds most naturally to a region R in the uv-plane given by

$$R = \{(u, v) : 0 \leq v \leq -2u/5 + 2 \text{ and } 0 \leq u \leq 5\}$$

and shown at right in Figure 99. We parameterize Σ_1 by

$$\mathbf{r}(u, v) = \langle u, v, 10 - 2u - 5v\rangle, \quad (u, v) \in R.$$

Now, $\mathbf{r}_u(u, v) = \langle 1, 0, -2\rangle$ and $\mathbf{r}_v(u, v) = \langle 0, 1, -5\rangle$, so that

$$(\mathbf{r}_u \times \mathbf{r}_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -5 \end{vmatrix} = \langle 2, 5, 1\rangle.$$

for all $(u, v) \in R$. It can be seen that if the initial point of the vector $\langle 2, 5, 1\rangle$ is any point on Σ_1, then it must be directed away from the region enclosed by the tetrahedron Σ, and so we choose $\mathbf{n} = \hat{\mathbf{n}}$ as given in (16.40). Thus, by (16.41), we obtain

$$\begin{aligned} \iint_{\Sigma_1} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)(u, v)\, dA \\ &= \iint_R \mathbf{F}(u, v, 10 - 2u - 5v) \cdot \langle 2, 5, 1\rangle\, dA \\ &= \int_0^5 \int_0^{-2u/5+2} \langle u, v, 10 - 2u - 5v\rangle \cdot \langle 2, 5, 1\rangle\, dvdu \end{aligned}$$

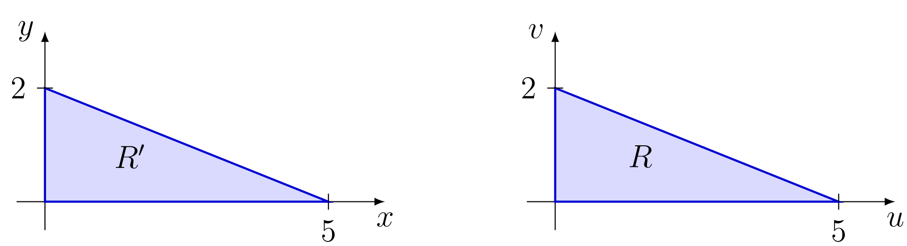


Figure 99.

$$= \int_0^5 \int_0^{-2u/5+2} 10\, dv du = \int_0^5 (20 - 4u)\, du = 50$$

as the flux of $\mathbf{F}$ across Σ_1. ■

In the next example we encounter a surface that is not closed, so that orientation must be specified in some way other than with the "positive" and "negative" designations.

Example 16.49. Let $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$, and let Σ be the cone $z^2 = x^2 + y^2$, $0 \le z \le 1$. Give Σ the orientation for which $\mathbf{n}$ has a negative z-component at each point of Σ where the cone is orientable. Find the flux of $\mathbf{F}$ across Σ.

Solution. We start by finding a suitable parametrization for Σ. Since $z^2 = x^2 + y^2$ for $z \ge 0$, we have

$$z = \sqrt{x^2 + y^2}.$$

Thus if we let $x = u$ and $y = v$, we arrive at the parametrization

$$\mathbf{r}(u, v) = \langle u, v, \sqrt{u^2 + v^2}\rangle, \quad (u, v) \in R,$$

where

$$R = \{(u, v) : 0 \le u^2 + v^2 \le 1\}$$

since $0 \le z \le 1$ implies that $0 \le \sqrt{u^2 + v^2} \le 1$.

Next, we need to find a suitable orientation for Σ, which entails determining some function $\mathbf{n} : R \to \mathbb{R}^3$ such that, for each $(u, v) \in R$, the unit vector $\mathbf{n}(u, v)$ has a negative z-component. (Recall that we consider $\mathbf{n}(u, v)$ to be located at the point $\mathbf{r}(u, v)$ on Σ.) From

$$\mathbf{r}_u(u, v) = \left\langle 1, 0, \frac{u}{\sqrt{u^2 + v^2}}\right\rangle \quad \text{and} \quad \mathbf{r}_v(u, v) = \left\langle 0, 1, \frac{v}{\sqrt{u^2 + v^2}}\right\rangle$$

we have

$$(\mathbf{r}_u \times \mathbf{r}_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{u}{\sqrt{u^2 + v^2}} \\ 0 & 1 & \dfrac{v}{\sqrt{u^2 + v^2}} \end{vmatrix} = \left\langle -\frac{u}{\sqrt{u^2 + v^2}}, -\frac{v}{\sqrt{u^2 + v^2}}, 1\right\rangle$$

Our two choices of orientation for Σ are $\hat{\mathbf{n}}$, as given by (16.40), and $-\hat{\mathbf{n}}$. In order to have negative z-components we choose $\mathbf{n} = -\hat{\mathbf{n}}$; that is,

$$\mathbf{n}(u, v) = -\hat{\mathbf{n}}(u, v) = -\frac{(\mathbf{r}_u \times \mathbf{r}_v)(u, v)}{\|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|} = \left\langle \frac{u}{\sqrt{2(u^2 + v^2)}}, \frac{v}{\sqrt{2(u^2 + v^2)}}, -\frac{1}{\sqrt{2}}\right\rangle$$

Finally we evaluate the appropriate flux integral, substituting $-\hat{\mathbf{n}}$ for $\mathbf{n}$:

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS = -\iint_\Sigma \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -\iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \hat{\mathbf{n}}(u, v)\|(\mathbf{r}_u \times \mathbf{r}_v)(u, v)\|\, dA$$

$$= -\iint_R \mathbf{F}\big(u, v, \sqrt{u^2 + v^2}\,\big) \cdot (\mathbf{r}_u \times \mathbf{r}_v)(u, v)\, dA$$

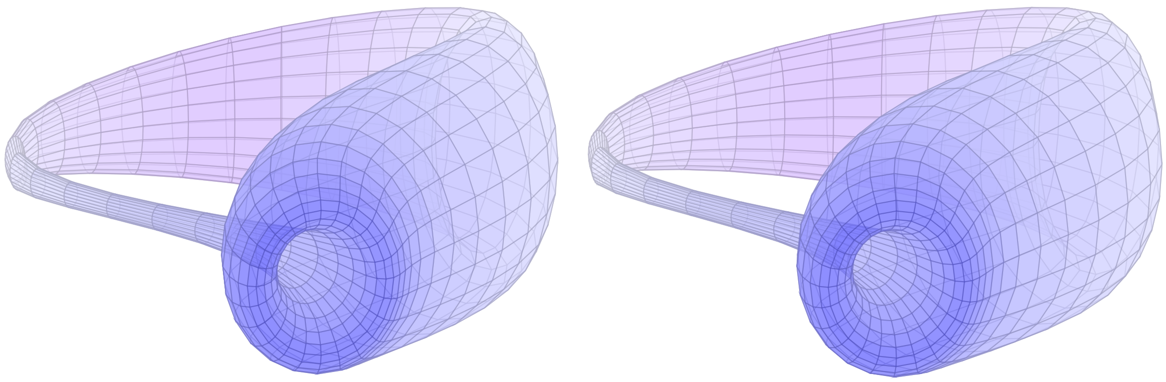


FIGURE 100. Stereoscopic image of the Klein bottle immersed in $\mathbb{R}^3$.

$$= -\iint_R \left\langle u, v, \sqrt{u^2+v^2} \right\rangle \cdot \left\langle -\frac{u}{\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1 \right\rangle \, dA$$
$$= -\iint_R (0)\, dA = 0.$$

The flux of $\mathbf{F}$ across Σ is therefore 0. ■

We end this section with Figure 100, which depicts what is perhaps the second most famous example of a nonorientable surface: the Klein bottle. In the figure it appears that the Klein bottle intersects itself, but in fact this does not happen. The problem is that the true Klein bottle inhabits $\mathbb{R}^4$, which we cannot visualize even with stereoscopy! What Figure 100 really shows is an "immersion" of the Klein bottle in three-dimensional space, which while not a wholly accurate representation is still much better than no representation at all. The parametrization for the surface as pictured is quite complicated:

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in [0,\pi] \times [0, 2\pi],$$

with

$$\begin{aligned}
x(u,v) &= -\tfrac{2}{15}(3\cos v - 30\sin u + 90\cos^4 u \sin u - 60\cos^6 u \sin u + 5\cos u \cos v \sin u)\cos u \\
y(u,v) &= -\tfrac{1}{15}(3\cos v - 3\cos^2 u \cos v - 48\cos^4 u \cos v + 48\cos^6 u \cos v - 60\sin u \\
&\quad + 5\cos u \cos v \sin u - 5\cos^3 u \cos v \sin u - 80\cos^5 u \cos v \sin u \\
&\quad + 80\cos^7 u \cos v \sin u)\sin u \\
z(u,v) &= \tfrac{2}{15}(3 + 5\cos u \sin u)\sin v
\end{aligned}$$

16.8 – Stokes' Theorem

As with our definition of a smooth surface, the following definition for the notion of a boundary of a surface will be sufficient to suit our immediate needs, but it is not the most general definition.

Definition 16.50. *Let $\Sigma \subseteq \mathbb{R}^n$ be a surface with parametrization $\mathbf{r}(u,v)$, $(u,v) \in R$. If $R \subseteq \mathbb{R}^2$ is closed and $\mathbf{r}$ is one-to-one and continuous on R, then the **boundary** of Σ is the set $\partial\Sigma = \mathbf{r}(\partial R)$. That is,*

$$\partial\Sigma = \{\mathbf{r}(u,v) : (u,v) \in \partial R\}.$$

Thus we see that $\partial\Sigma \subseteq \Sigma$, which is to say the boundary of a surface Σ is taken to be part of the surface.

The boundary of a surface should not be confused with the similar-sounding notion of a bounded surface. A surface $\Sigma \subseteq \mathbb{R}^n$ is **bounded** if there exists some $\mathbf{x} \in \mathbb{R}^n$ and real number $r > 0$ such that $\Sigma \subseteq B_r(\mathbf{x})$. That is, a bounded surface is just a surface that is a bounded set. A closed surface, defined in the previous section, can now be said to be a bounded surface without boundary!

Consider a simple closed curve C in the xy-plane. It forms the boundary of a connected and simply-connected region $R \subseteq \mathbb{R}^2$ that is a closed set. If we define $\mathbf{r} : R \to \mathbb{R}^3$ by $\mathbf{r}(u,v) = \langle u,v,0\rangle$ for all $(u,v) \in R$, which is a continuous one-to-one function, then R is seen to be a bounded smooth parameterized surface $\Sigma \subseteq \mathbb{R}^3$ with boundary

$$\partial\Sigma = \{(u,v,0) : (u,v) \in \partial R\} = \{(u,v,0) : (u,v) \in C\}.$$

From

$$\mathbf{r}_u(u,v) = \langle 1,0,0\rangle \quad \text{and} \quad \mathbf{r}_v(u,v) = \langle 0,1,0\rangle$$

we obtain an orientation for Σ:

$$\hat{\mathbf{n}}(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}(u,v) = \frac{\mathbf{k}}{\|\mathbf{k}\|} = \mathbf{k}. \tag{16.42}$$

Suppose C admits a smooth parametrization $\boldsymbol{\rho}(t)$, $t \in [a,b]$, and let $\mathbf{T}$ and $\mathbf{N}$ denote the unit tangent vector and principal unit normal vector, respectively. The orientation for C that is **consistent** with the orientation (16.42) for Σ is the one for which $(\mathbf{T} \times \mathbf{N})(t) = \mathbf{k}$ for all $t \in [a,b]$. This has the effect of keeping Σ to one's left were one to walk "upright" (i.e. with the top of one's head pointed in the $\mathbf{k}$ direction) along the curve C in the direction in which t is increasing.

More generally, if C is a simple closed *planar* curve in $\mathbb{R}^3$ with smooth parametrization $\boldsymbol{\rho}(t)$, $t \in [a,b]$, and the surface Σ is the planar region enclosed by C, then clearly Σ is orientable with orientation $\mathbf{n}$ that is a constant function. That is, $\mathbf{n}$ assigns the same unit vector at every point $\mathbf{x} \in \Sigma$, which we may as well denote by $\mathbf{n}$. The orientation for C that is consistent with the orientation $\mathbf{n}$ for Σ is the one for which $(\mathbf{T} \times \mathbf{N})(t) = \mathbf{n}$ for all $t \in [a,b]$.

To further generalize, imagine that the oriented surface $(\Sigma, \mathbf{n})$ is a rubber membrane, and it is stretched to form a new surface Σ' that has boundary C (which remains unchanged). If, during the deformation, the vectors $\mathbf{n}$ at each point on Σ change location and direction in *continuous* fashion so as to form an orientation $\mathbf{n}'$ for Σ', then the orientation for C (which remains unchanged) will be consistent with $\mathbf{n}'$ just as it was consistent with $\mathbf{n}$. From here we could generalize one more time by giving up the condition that C be a planar curve.

It is a fact that if C is a piecewise-smooth simple closed oriented curve that forms the boundary of an orientable surface Σ, then the orientation of C is consistent with an orientation $\mathbf{n}$ of Σ if and only if $-C$ has orientation consistent with the orientation $-\mathbf{n}$ of Σ.

In what follows we take $\mathbf{n}$ to be either the orientation $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$ for any orientable surface Σ. In the case when Σ is piecewise-smooth $\mathbf{n}$ will necessarily be a piecewise-defined function as discussed in the previous section.

Theorem 16.51 (**Stokes' Theorem**)**.** *Let $\Sigma \subseteq \mathbb{R}^3$ be a piecewise-smooth orientable surface with piecewise-smooth simple closed boundary $\partial\Sigma$ parameterized by $\mathbf{r}$. If the orientation for Σ is consistent with the orientation for $\partial\Sigma$, and if $\mathbf{F}$ is a vector field that is continuously differentiable on an open set containing Σ, then*

$$\oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Green's Theorem is a special case of Stoke's Theorem in which Σ is a connected and simply connected region R on the xy-plane such that ∂R is a piecewise-smooth closed simple planar curve. A general proof of Stokes' Theorem is beyond our current capabilities, but a proof for the case when Σ is parameterized by a function $\mathbf{r}(u, v)$ defined on a rectangle in the uv-plane is forthcoming.

Example 16.52. Let $\mathbf{F}(x, y, z) = \langle 2y, -z, x\rangle$, and let C be the circle $x^2 + y^2 = 9$ in the plane $z = 0$ with positive orientation. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by evaluating the surface integral in Stokes' Theorem using an appropriate choice of surface Σ.

Solution. Here C is the circle in the xy-plane centered at the origin with radius 3. Since C must have positive (i.e. counterclockwise) orientation, a parametrization for C is

$$\mathbf{r}(t) = 3\langle \cos t, \sin t, 0\rangle, \quad t \in [0, 2\pi].$$

A convenient choice for a surface Σ that has C as its boundary would be the planar region enclosed by C, which is the circular disk in the xy-plane with radius 3. A parametrization for Σ is

$$\boldsymbol{\rho}(u, v) = \langle v \cos u, v \sin u, 0\rangle, \quad (u, v) \in R = [0, 2\pi] \times [0, 3].$$

Clearly Σ is orientable (i.e. it has two identifiable "sides"). We have

$$(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & 0 \end{vmatrix} = \langle 0, 0, -v\rangle$$

for any $(u, v) \in \mathrm{Int}(R) = (0, 2\pi) \times (0, 3)$, and so

$$\hat{\mathbf{n}}(u, v) = \frac{\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v}{\|\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v\|}(u, v) = \frac{\langle 0, 0, -v\rangle}{v} = \langle 0, 0, -1\rangle = -\mathbf{k}.$$

Thus $\hat{\mathbf{n}} : \mathrm{Int}(R) \to \mathbb{R}^3$ is continuous on $\mathrm{Int}(R)$, and it has continuous extension to R by setting $\hat{\mathbf{n}}(u, v) = -\mathbf{k}$ for all $(u, v) \in \partial R = C$. Indeed, since $\hat{\mathbf{n}}$ is a constant function equal to $-\mathbf{k}$ on all R, we simply define $\hat{\mathbf{n}} = -\mathbf{k}$.

It must be determined which orientation $\mathbf{n}$ of Σ, $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$, is consistent with the orientation of C. That is, which orientation $\mathbf{n}$ of Σ is such that $(\mathbf{T}\times\mathbf{N})(t) = \mathbf{n}$ for all $t \in [0, 2\pi]$, where $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and principal unit normal vectors for C, respectively. We calculate

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \langle -\sin t, \cos t, 0\rangle$$

and

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle -\cos t, -\sin t, 0\rangle,$$

and so

$$(\mathbf{T}\times\mathbf{N})(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle 0, 0, 1\rangle = \mathbf{k}.$$

Since $(\mathbf{T}\times\mathbf{N})(t) = \mathbf{k} = -\hat{\mathbf{n}}$, we give Σ the orientation $-\hat{\mathbf{n}}$.

Next, we have

$$(\nabla\times\mathbf{F})(x, y, z) = \langle 1, -1, -2\rangle$$

Finally, substituting $-\hat{\mathbf{n}}$ for $\mathbf{n}$, we obtain

$$\begin{aligned} \oint_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{r} &= -\iint_{\Sigma} (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\, dS \\ &= -\iint_{R} (\nabla\times\mathbf{F})(\boldsymbol{\rho}(u,v))\cdot\hat{\mathbf{n}}(u,v)\|(\boldsymbol{\rho}_u\times\boldsymbol{\rho}_v)(u,v)\|\, dA \\ &= -\iint_{R} (\nabla\times\mathbf{F})(\boldsymbol{\rho}(u,v))\cdot(\boldsymbol{\rho}_u\times\boldsymbol{\rho}_v)(u,v)\, dA \\ &= -\iint_{R} \langle 1, -1, -2\rangle\cdot\langle 0, 0, -v\rangle\, dA \\ &= -\int_0^3\int_0^{2\pi} 2v\, du dv = -18\pi \end{aligned}$$

by Stokes' Theorem and Definition 16.47. ■

Example 16.53. Let $\mathbf{F}(x, y, z) = \langle y^2, -z^2, x\rangle$, and let C be the circle

$$\mathbf{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t\rangle, \quad t \in [0, 2\pi].$$

Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ by evaluating the surface integral in Stokes' Theorem using an appropriate choice of surface Σ.

Solution. Opposite points on the circle are $\mathbf{x} = \mathbf{r}(t)$ and $\mathbf{y} = \mathbf{r}(t+\pi)$ for any $0 \le t \le \pi$, and the midpoint of the segment $[\mathbf{x}, \mathbf{y}]$ is $\frac{1}{2}(\mathbf{x}+\mathbf{y}) = \mathbf{0}$ while the length is $|\mathbf{x}-\mathbf{y}| = 10$. Thus C is a circle centered at the origin with radius 5, and a convenient choice for Σ would be simply the planar region enclosed by C, as shown in Figure 101. One parametrization for Σ is

$$\boldsymbol{\rho}(u, v) = \langle 3v\cos u, 4v\cos u, 5v\sin u\rangle, \quad (u, v) \in [0, 2\pi]\times[0, 1].$$

Note the origin is obtained when $v = 0$, C is obtained when $v = 1$, and concentric circles inside C are obtained for $0 < v < 1$. The parameter u simply stands for t.

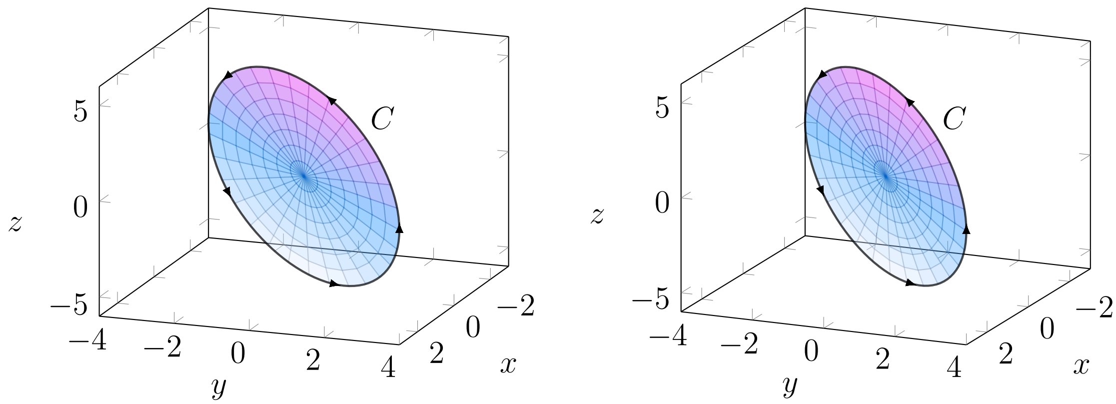


FIGURE 101. The planar region Σ with boundary the circle C.

It is clear that Σ is orientable. Let $R = [0, 2\pi] \times [0, 1]$. Then $\text{Int}(R) = (0, 2\pi) \times (0, 1)$, and for any $(u, v) \in \text{Int}(R)$ we have

$$(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3v \sin u & -4v \sin u & 5v \cos u \\ 3 \cos u & 4 \cos u & 5 \sin u \end{vmatrix} = \langle -20v, 15v, 0 \rangle,$$

and so

$$\hat{\mathbf{n}}(u, v) = \frac{\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v}{\|\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v\|}(u, v) = \frac{\langle -20v, 15v, 0 \rangle}{25v} = \left\langle -\frac{4}{5}, \frac{3}{5}, 0 \right\rangle.$$

Thus $\hat{\mathbf{n}} : \text{Int}(R) \to \mathbb{R}^3$ is continuous on $\text{Int}(R)$, and it has continuous extension to R simply by setting $\hat{\mathbf{n}}(u, v) = \langle -4/5, 3/5, 0 \rangle$ for all $(u, v) \in \partial R$.

It must be determined which orientation, $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$, is consistent with the orientation of C. The unit tangent and principal unit normal vectors for C are

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{5}\langle -3 \sin t, -4 \sin t, 5 \cos t \rangle$$

and

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{5}\langle -3 \cos t, -4 \cos t, -5 \sin t \rangle.$$

Now,

$$(\mathbf{T} \times \mathbf{N})(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{5} \sin t & -\frac{4}{5} \sin t & \cos t \\ -\frac{3}{5} \cos t & -\frac{4}{5} \cos t & -\sin t \end{vmatrix} = \left\langle \frac{4}{5}, -\frac{3}{5}, 0 \right\rangle,$$

which we see agrees with $-\hat{\mathbf{n}}(u, v)$ for all $(u, v) \in R$, and so we give Σ the orientation $-\hat{\mathbf{n}}$.

Next, $\mathbf{F} = \langle f, g, h \rangle$ with

$$f(x, y, z) = y^2, \quad g(x, y, z) = -z^2, \quad h(x, y, z) = x,$$

and so from (16.33) we have

$$(\nabla \times \mathbf{F})(x, y, z) = \langle 2z, -1, -2y \rangle$$

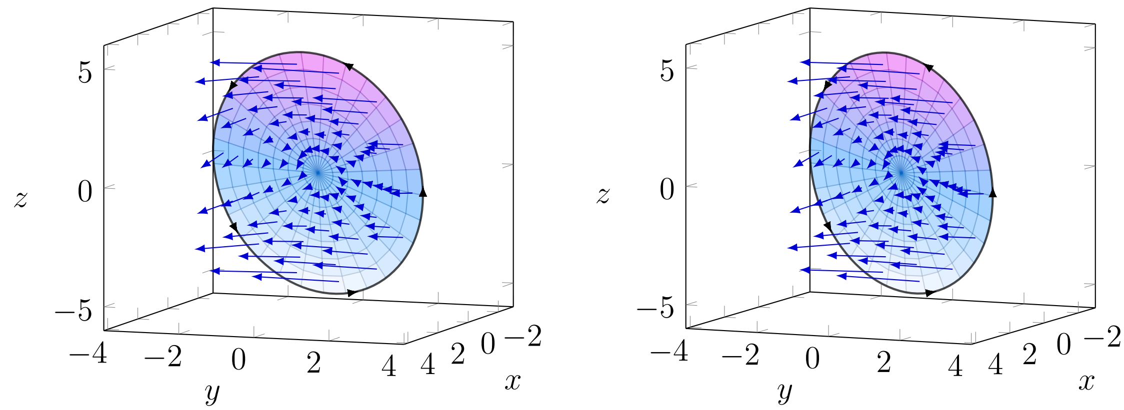


Figure 102. The vector field $\mathbf{F}$ restricted to Σ.

Finally by Stokes' Theorem, substituting $-\hat{\mathbf{n}}$ for $\mathbf{n}$, we obtain

$$\begin{aligned}
\oint_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{r} &= -\iint_{\Sigma} (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\, dS \\
&= -\iint_{R} (\nabla\times\mathbf{F})(\boldsymbol{\rho}(u,v))\cdot\hat{\mathbf{n}}(u,v)\|(\boldsymbol{\rho}_u\times\boldsymbol{\rho}_v)(u,v)\|\, dA \\
&= -\iint_{R} \left[(\nabla\times\mathbf{F})(3v\cos u, 4v\cos u, 5v\sin u)\cdot\left\langle \tfrac{4}{5}, -\tfrac{3}{5}, 0\right\rangle\right](25v)\, dA \\
&= -\iint_{R} 25v\langle 10v\sin u, -1, -8v\cos u\rangle\cdot\left\langle \tfrac{4}{5}, -\tfrac{3}{5}, 0\right\rangle\, dA \\
&= \int_0^1\int_0^{2\pi} (15v + 200v^2\sin u)\, du dv = \int_0^1 30\pi v\, dv = 15\pi,
\end{aligned}$$

where Definition 16.47 is used for the second equality. In Figure 102 it can be seen how the positive answer 15π makes sense: the vectors generated by $\mathbf{F}$ are consistent with the orientation of C throughout the region Σ. ■

16.9 – Divergence Theorem

The final theorem that we will work with is the Divergence Theorem, which is also sometimes called Gauss' Theorem. In the last couple sections we have come up against the limitations of our analytical machinery as never before, especially in relation to concepts such as smooth surfaces and surface orientation. As a result we have had to rely increasingly on intuition and less on precision and rigor, which is a sure indicator that an entirely new and more sophisticated approach much be taken to soar to greater heights in the calculus-based branches of mathematics. Such an approach will be embarked upon in the sequel to this text on the subject of mathematical analysis.

Theorem 16.54 (**Divergence Theorem**). *Let $D \subseteq \mathbb{R}^3$ be a connected and simply connected region such that ∂D is a piecewise-smooth closed orientable surface. If D has positive orientation and $\mathbf{F}$ is a vector field that is continuously differentiable on an open set containing D, then*

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_{D} \nabla \cdot \mathbf{F}\, dV. \tag{16.43}$$

The integral at left in (16.43) is just another symbol for the flux of the vector field $\mathbf{F}$ across the surface ∂D as given by Definition 16.47; that is,

$$\text{Flux of } \mathbf{F} \text{ across } \partial D = \iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS = \oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S}.$$

The symbol $\oiint$ is used to represent a surface integral over a surface Σ that is *closed*, which again is any surface that fully encloses a region in the space $\mathbb{R}^3$. It is never wrong to replace $\oiint$ with $\iint$, and many texts never use the symbol.

Since the closed surface ∂D in the Divergence Theorem has the *positive* orientation, the integral at left in (16.43) gives specifically the net *outward* flux of $\mathbf{F}$ across ∂D; that is,

$$\text{Net outward flux of } \mathbf{F} \text{ across } \partial D = \iiint_{D} \nabla \cdot \mathbf{F}\, dV,$$

and so

$$\text{Net inward flux of } \mathbf{F} \text{ across } \partial D = -\iiint_{D} \nabla \cdot \mathbf{F}\, dV.$$

The following example shows some of the utility of the Divergence Theorem.

Example 16.55. Use the Divergence Theorem to find the net outward flux of the field

$$\mathbf{F}(x, y, z) = \langle 2z - y, x, -2x \rangle$$

across the sphere of radius 1 centered at the origin.

Solution. Let $D = \overline{B}_1(\mathbf{0})$, which is the solid ball with radius 1 and center at the origin. Clearly D is connected and simply connected. The boundary of D is

$$\partial D = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\},$$

which is the sphere with radius 1 and center at the origin. Thus ∂D is a smooth, closed, and orientable surface. The field $\mathbf{F}$ has scalar components

$$f(x,y,z) = 2z - y, \quad g(x,y,z) = x, \quad h(x,y,z) = -2x,$$

which have continuous first partials on $\mathbb{R}^3$. Finally, to find the "outward" flux of $\mathbf{F}$ across the sphere means to give the sphere the positive orientation $\mathbf{n} = \hat{\mathbf{n}}$. With all of the hypotheses of the Divergence Theorem being satisfied, we calculate

$$(\nabla \cdot \mathbf{F})(x,y,z) = \partial_x(2z - y) + \partial_y(x) + \partial_z(-2x) = 0,$$

and finally

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D (0) dV = 0.$$

That is, the net outward flux is zero, a result far more easily obtained using the Divergence Theorem than it would have been using Definition 16.47! ■

Example 16.56. Use the Divergence Theorem to find the net outward flux of the field

$$\mathbf{F}(x,y,z) = \langle 3y^2z^3, 9x^2yz^2, -4xy^2 \rangle$$

across the surface Σ that is the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

Solution. Let $D = [-1,1] \times [-1,1] \times [-1,1]$, which is a solid box with vertices at $(\pm 1, \pm 1, \pm 1)$; that is,

$$D = \{(x,y,z) : -1 \le x \le 1,\ -1 \le y \le 1,\ -1 \le z \le 1\}.$$

Clearly D is connected and simply connected, and $\partial D = \Sigma$ is a piecewise-smooth, closed, and orientable surface. The field $\mathbf{F}$ has scalar components

$$f(x,y,z) = 3y^2z^3, \quad g(x,y,z) = 9x^2yz^2, \quad h(x,y,z) = -4xy^2,$$

which have continuous first partials on $\mathbb{R}^3$. Finally, to find the "outward" flux of $\mathbf{F}$ across the cube means to give the cube the positive orientation $\mathbf{n} = \hat{\mathbf{n}}$. All of the hypotheses of the Divergence Theorem are satisfied, so we calculate

$$(\nabla \cdot \mathbf{F})(x,y,z) = \partial_x(3y^2z^3) + \partial_y(9x^2yz^2) + \partial_z(-4xy^2) = 9x^2z^2,$$

and then obtain

$$\oiint_{\Sigma} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 9x^2z^2 dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 9x^2z^2 dx\, dy\, dz = 8.$$

That is, the net outward flux is 8. To derive this result using Definition 16.47 would have required evaluating a separate surface integral for each of the six faces of the cube! ■

Suppose that R_1 and R_2 are connected and simply connected compact sets in the uv-plane such that $R_1 \cap R_2 = \varnothing$, and let $\mathbf{r} : R_1 \cup R_2 \to \mathbb{R}^3$ be a one-to-one function that is piecewise-smooth in the sense of Definition 16.40(2); that is, $\mathbf{r}$ is smooth on R_1 and also smooth on R_2. Thus, $\Sigma_1 = \mathbf{r}(R_1)$ and $\Sigma_2 = \mathbf{r}(R_2)$ are smooth surfaces in $\mathbb{R}^3$. Suppose that $\Sigma_1 \cap \Sigma_2 = \varnothing$. (In fact, this necessarily follows from the assumption that $\mathbf{r}$ is one-to-one and continuous on the disjoint closed sets R_1 and R_2.) Then $\Sigma_1 \cup \Sigma_2$ is a piecewise-smooth surface consisting of two

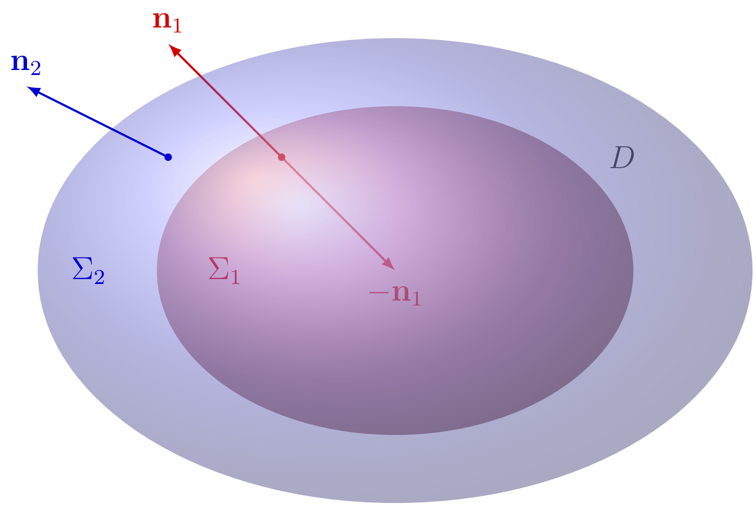


FIGURE 103. The surface $\Sigma = \Sigma_1 \cup \Sigma_2$, with the region D enclosed between Σ_1 and Σ_2.

components. Assuming $\mathbf{F}$ and $\mathbf{n}$ satisfy the conditions of Definition 16.47, we apply Proposition 15.8 to obtain

$$\begin{aligned}\iint_{\Sigma_1 \cup \Sigma_2} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_{R_1 \cup R_2} (\mathbf{F} \circ \mathbf{r}) \cdot \mathbf{n} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA \\ &= \iint_{R_1} (\mathbf{F} \circ \mathbf{r}) \cdot \mathbf{n} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA + \iint_{R_2} (\mathbf{F} \circ \mathbf{r}) \cdot \mathbf{n} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA \\ &= \iint_{\Sigma_1} \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{\Sigma_2} \mathbf{F} \cdot \mathbf{n}\, dS.\end{aligned}$$