Math 141 Quiz #4 (Fall 2020)

1 Find the Taylor polynomial p_2 with center $a = 8$ for $f(x) = \sqrt[3]{x}$.

From $f(x) = \sqrt[3]{x}$, $f'(x) = \frac{1}{3}x^{-2/3}$, and $f''(x) = -\frac{2}{9}x^{-5/3}$ we obtain $f(8) = 2$, $f'(8) = \frac{1}{12}$, and $f''(8) = -\frac{1}{144}$. Then

$$p_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(8)}{n!}(x-8)^n = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}.$$

2 Approximate $\sqrt[4]{79}$ using a Taylor polynomial p_n with $n = 3$.

The best center would be $a = 81$. Letting $f(x) = \sqrt[4]{x}$, we obtain $f(81) = 3$, $f'(81) = \frac{1}{108}$, $f''(81) = -\frac{1}{11{,}664}$, and $f'''(81) = \frac{7}{3{,}779{,}136}$. Now,

$$p_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(81)}{n!}(x-81)^n = 3 + \frac{x-81}{108} - \frac{(x-81)^2}{23,328} + \frac{7(x-81)^3}{22,674,816},$$

and so $\sqrt[4]{79} \approx p_3(79) \approx 2.981307544$.

3 Find the radius and interval of convergence for

$$\sum_{k=2}^{\infty} \frac{(x+3)^k}{k \ln k}.$$

Using the Ratio Test, with

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(x+3)^{k+1}}{(k+1)\ln(k+1)} \cdot \frac{k \ln k}{(x+3)^k} \right| = |x+3| \lim_{k\to\infty} \frac{k \ln k}{(k+1)\ln(k+1)}$$
$$= |x+3| \left(\lim_{k\to\infty} \frac{k}{k+1} \right) \left(\lim_{k\to\infty} \frac{\ln k}{\ln(k+1)} \right) = |x+3|(1)(1) = |x+3|,$$

we conclude that the series converges for $|x+3| < 1$, or $x \in (-4, -2)$. Radius of convergence is $R = 1$. When $x = -4$ the series becomes

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k},$$

which converges by the Alternating Series Test. When $x = -2$ the series becomes

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k},$$

but since

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \big[\ln(\ln x)\big]_2^{\infty} = \lim_{t\to\infty} \big[\ln(\ln t) - \ln(\ln 2)\big] = \infty,$$

we conclude by the Integral Test that the series converges. Therefore the interval of convergence is $[-4, -2)$.