MATH 141 EXAM #3 KEY (SUMMER III 2018)

1a Since

$$\lim_{n\to\infty} \frac{4^n}{n^2} = \infty$$

(details omitted here), the series diverges by the Divergence Test.

1b Since

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!}\right| = 2\lim_{n\to\infty} \frac{n^n}{(n+1)^n} = 2\lim_{n\to\infty} \exp\left(n \cdot \ln\frac{n}{n+1}\right)\\ &= 2\exp\left(\lim_{n\to\infty} \frac{\ln n - \ln(n+1)}{1/n}\right) \overset{\text{LR}}{=} 2\exp\left(\frac{1/n - 1/(n+1)}{-1/n^2}\right)\\ &= 2\exp\left(-\lim_{n\to\infty}\frac{n}{n+1}\right) = 2\exp(-1) = \frac{2}{e} < 1,\end{aligned}$$

the series converges by the Ratio Test.

1c Since

$$\lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} \exp\left(-\frac{\ln n}{n}\right) = \exp(0) = 1 \neq 0,$$

the series diverges by the Divergence Test.

1d The series may be written as

$$\sum_{n=1}^{\infty} \frac{2\cdot 5\cdot 8 \cdots (3n-1)}{3\cdot 5\cdot 7\cdots (2n+1)}.$$

Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 5\cdot 8\cdots [3(n+1)-1]}{3\cdot 5\cdot 7\cdots[2(n+1)+1]} \cdot \frac{3\cdot 5\cdot 7\cdots(2n+1)}{2\cdot 5\cdot 8\cdots(3n-1)}\right| = \lim_{n\to\infty}\frac{3n+2}{2n+3} = \frac{3}{2} > 1,$$

the series diverges by the Ratio Test.

2a Let $b_n = n^2/(n^3+1)$. Clearly $b_n > 0$ for all $n \geq 1$, with $b_n \to 0$ as $n \to \infty$. It remains to show that (b_n) is a nonincreasing (i.e. monotone decreasing) sequence. Let $f(x) = x^2/(x^3+1)$, so $b_n = f(n)$ for each integer $n \geq 1$. Since

$$f'(x) = -\frac{x(x^3-2)}{(x^3+1)^2} < 0$$

for all $x \geq \frac{3}{2}$, it follows that f is decreasing on $[\frac{3}{2}, \infty)$, and hence (b_n) is decreasing for $n \geq 2$. Indeed, since $b_1 = \frac{1}{2} > \frac{4}{9} = b_2$, we see that (b_n) is decreasing for all $n \geq 1$. Therefore the series converges by the Alternating Series Test.

2b Since $\tan^{-1} n \to \pi/2$ as $n \to \infty$, the series diverges by the Divergence Test.

3 The 4th-order Taylor polynomial consists of the first four terms of the Maclaurin series for $\ln(1+x)$ with $x = -0.1$:

$$\ln(0.9) = \ln(1-0.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-0.1)^n}{n} = -\sum_{n=1}^{\infty} \frac{0.1^n}{n}$$

$$\approx -\sum_{n=1}^{4} \frac{0.1^n}{n} = -\frac{1}{10} - \frac{1}{200} - \frac{1}{3000} - \frac{1}{40{,}000} = 0.105358\bar{3}.$$

4a Apply Ratio Test:

$$\lim_{n\to\infty} \left| \frac{3^{2(n+1)} x^{n+1}}{(n+1)^4} \cdot \frac{n^4}{3^{2n} x^n} \right| = |x| \lim_{n\to\infty} \frac{9n^4}{(n+1)^4} = 9|x|.$$

Series converges if $|x| < 1/9$, so interval of convergence contains $\left(-\frac{1}{9}, \frac{1}{9}\right)$. Check endpoints.

At $x = 1/9$: series becomes $\sum 1/n^4$, a convergent p-series. At $x = -1/9$: series becomes $\sum (-1)^n/n^4$, which converges by the Alternating Series Test.

Interval of convergence is $\left[-\frac{1}{9}, \frac{1}{9}\right]$.

4b Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(2x+1)^{n+1}}{(n+1)\cdot 8^{n+1}} \cdot \frac{n \cdot 8^n}{(2x+1)^n} \right| = |2x+1| \lim_{n\to\infty} \frac{n}{8n+8} = \frac{|2x+1|}{8}.$$

Series converges if $|2x+1| < 8$, so interval of convergence contains $\left(-\frac{9}{2}, \frac{7}{2}\right)$. Check endpoints.

At $x = \frac{7}{2}$ series becomes $\sum 1/n$, which diverges. At $x = -\frac{9}{2}$ series becomes $\sum (-1)^n/n$, which converges by the Alternating Series Test.

Interval of convergence is $\left[-\frac{9}{2}, \frac{7}{2}\right)$.

4c Ratio Test:

$$\lim \left| \frac{(n+1)!(x-10)^{n+1}}{n!(x-10)^n} \right| = \lim_{n\to\infty} (n+1)|x-10| = \begin{cases} \infty, & x \neq 10 \\ 0, & x = 10. \end{cases}$$

The series only converges at $\{10\}$.

5 Use the geometric series:

$$\frac{5x^2}{5+x^3} = x^2 \cdot \frac{1}{1-(-x^3/5)} = x^2 \sum_{n=0}^{\infty} \left(-\frac{x^3}{5}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{5^n}.$$

Interval of convergence is $|-x^3/5| < 1$, or $|x|^3 < 5$, and hence $\left(-\sqrt[3]{5}, \sqrt[3]{5}\right)$.

6 Use the geometric series:

$$\frac{x}{1+x^3} = x \cdot \frac{1}{1-(-x)^3} = x \cdot \sum_{n=0}^{\infty}[(-x)^3]^n = \sum_{n=0}^{\infty}(-1)^n x^{3n+1}$$

for all $|x| < 1$. We have

$$\begin{aligned}\int_0^{0.3} \frac{x}{1+x^3}\,dx &= \int_0^{0.3}\left(\sum_{n=0}^{\infty}(-1)^n x^{3n+1}\right)dx = \sum_{n=0}^{\infty}(-1)^n\left(\int_0^{0.3} x^{3n+1}\,dx\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 0.3^{3n+2}}{3n+2},\end{aligned}$$

which is an alternating series $\sum(-1)^n b_n$ with

$$b_n = \frac{0.3^{3n+2}}{3n+2}$$

for $n \geq 0$. The first few b_n values are

$$b_0 = 0.045, \quad b_1 = 4.86 \times 10^{-4}, \quad b_2 \approx 8.20 \times 10^{-6}, \quad b_3 \approx 1.61 \times 10^{-7} < 10^{-6},$$

so by the Alternating Series Estimation Theorem the approximation

$$\int_0^{0.3}\frac{x}{1+x^3}\,dx \approx \sum_{n=0}^{2}\frac{(-1)^n(0.3)^{3n+2}}{3n+2} = b_0 - b_1 + b_2$$

will have an absolute error that is less than 10^{-6}.

7 We have $x^2 = t+1$, so that $t-1 = x^2-2$, and then $y^2 = t-1 = x^2-2$. Noting that $y \geq 0$, it follows that

$$y = \sqrt{x^2-2}.$$

The domain of this function is $(\sqrt{2}, \infty)$.

8 The set-up is thus:

$$(x,y) = \left(1 - \tfrac{1}{30}t\right)(4,-40) + \tfrac{1}{30}t\,(2,10)$$

for $0 \leq t \leq 30$. Equivalently we may write

$$(x,y) = \left(-\tfrac{1}{15}t + 4, \tfrac{5}{3}t - 40\right), \quad t \in [0,30].$$

9 Using a given trigonometric identity gives

$$r\cos\theta = 2\cos\theta\sin\theta \;\Rightarrow\; r^2(r\cos\theta) = 2(r\cos\theta)(r\sin\theta) \;\Rightarrow\; (x^2+y^2)x = 2xy,$$

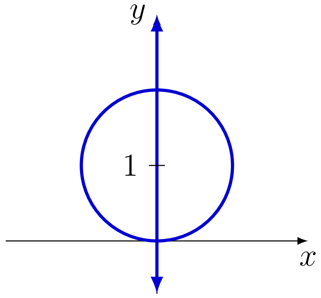
or equivalently

$$x(x^2+y^2-2y) = 0.$$

The graph of this equation will include the vertical line $x = 0$, and also the curve

$$x^2+y^2-2y = 0 \;\Rightarrow\; x^2+(y-1)^2 = 1,$$

which is a circle centered at $(0,1)$ with radius 1.



10 With $f(\theta) = 8\sin\theta$ and $\theta_0 = 5\pi/6$, the slope is

$$\frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0} = \frac{8\sqrt{3}(1/2) + 4\sqrt{3}}{8\sqrt{3}(\sqrt{3}) - 4(1/2)} = \frac{4\sqrt{3}}{11}.$$

11 When $\theta = 0$ we have $r = 0$ (the "stem" of the leaf), and when $\theta = \pi/8$ we have $r = 2$ (the "tip" of the leaf). This covers half of a single leaf, and to cover the whole leaf we increase θ further to $\pi/4$ to again obtain $r = 0$. Using the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, the area $\mathcal{A}$ of the leaf is

$$\mathcal{A} = \frac{1}{2}\int_0^{\pi/4} (2\sin 4\theta)^2\,d\theta = \int_0^{\pi/4} (1 - \cos 8\theta)\,d\theta = \left[\theta - \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/4} = \frac{\pi}{4}.$$