Math 141 Exam #2 Key (Summer III 2018)

1a Let $u = \cos x$, so that

$$\int \sin^3 x \cos^{-2} x\, dx = \int \cos^{-2} x \sin x\, dx - \int \sin x\, dx = -\int \frac{1}{u^2}\, du + \cos x + c = \sec x + \cos x + c.$$

1b Let $u = \tan x$, so that

$$\int \sec^2 x \tan^4 x\, dx = \int u^4\, du = \frac{1}{5}u^5 + c = \frac{1}{5}\tan^5 x + c.$$

2a Let $x = \frac{3}{2}\tan\theta$, so integral becomes

$$\int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta \cdot \frac{3}{2}\sec^2\theta}{(9\tan^2\theta + 9)^{3/2}}\, d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\, d\theta = \frac{3}{16}\left[\int_0^{\pi/3} \sec\theta\tan\theta\, d\theta - \int_0^{\pi/3} \sin\theta\, d\theta\right] = \frac{3}{32}.$$

2b Let $r = \frac{5}{6}\cos\theta$, so

$$\begin{aligned}\int \sqrt{25 - 36r^2}\, dr &= -\int \sqrt{25\sin^2\theta} \cdot \frac{5}{6}\sin\theta\, d\theta = -\frac{25}{6}\int \sin^2\theta\, d\theta \\ &= -\frac{25}{6}\left(-\frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2}\right) + c = \frac{r\sqrt{25-36r^2}}{2} - \frac{25}{12}\cos^{-1}\frac{6r}{5} + c.\end{aligned}$$

3a From

$$\frac{4q^2 - 7q - 12}{q(q+2)(q-3)} = \frac{A}{q} + \frac{B}{q+2} + \frac{C}{q-3}$$

we have

$$4q^2 - 7q - 12 = (A + B + C)q^2 + (-A - 3B + 2C)q - 6A,$$

and hence $A = 2$, $B = \frac{9}{5}$, and $C = \frac{1}{5}$. Now,

$$\begin{aligned}\int_1^2 \frac{4q^2 - 7q - 12}{q(q+2)(q-3)}\, dq &= \int_1^2 \frac{2}{q}\, dq + \int_1^2 \frac{9/5}{q+2}\, dq + \int_1^2 \frac{1/5}{q-3}\, dq \\ &= 2\big[\ln|q|\big]_1^2 + \frac{9}{5}\big[\ln|q+2|\big]_1^2 + \frac{1}{5}\big[\ln|q-3|\big]_1^2 \\ &= 2\ln 2 + \frac{9}{5}\ln\frac{4}{3} - \frac{1}{5}\ln 2 = \frac{9}{5}\ln\left(\frac{8}{3}\right).\end{aligned}$$

3b From

$$\frac{1}{(t^2-1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

we obtain

$$\frac{1}{(t^2-1)^2} = (A+C)t^3 + (A+B-C+D)t^2 + (-A+2B-C-2D)t + (-A+B+C+D),$$

and hence the system

$$\begin{cases} A \qquad\quad + C \qquad\quad = 0 \\ A + \quad B - C + \quad D = 0 \\ -A + 2B - C - 2D = 0 \\ -A + \quad B + C + \quad D = 1 \end{cases}$$

The solution is $(A,B,C,D) = \left(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Integral becomes

$$\begin{aligned} \int \frac{1}{(t^2-1)^2}\,dt &= \frac{1}{4}\int\left(-\frac{1}{t-1} + \frac{1}{(t-1)^2} + \frac{1}{t+1} + \frac{1}{(t+1)^2}\right)dt \\ &= \frac{1}{4}\left(\ln\left|\frac{t+1}{t-1}\right| - \frac{1}{t-1} - \frac{1}{t+1}\right) + c. \end{aligned}$$

4 We have

$$\lim_{t\to-\infty}\int_t^0 e^{bx}dx = \lim_{t\to-\infty}\left[\frac{1}{b}e^{bx}\right]_t^0 = \lim_{t\to-\infty}\left(\frac{1-e^{bt}}{b}\right) = \frac{1}{b},$$

and so the integral converges.

5 We have, with the substitution $u = t^4 - 1$,

$$\begin{aligned} \int_0^1 \frac{t^3}{t^4-1}\,dt &= \lim_{x\to1^-}\int_{-1}^{x^4-1}\frac{1/4}{u}\,du = \frac{1}{4}\lim_{x\to1^-}\big[\ln|u|\big]_{-1}^{x^4-1} \\ &= \frac{1}{4}\lim_{x\to1^-}\ln|x^4-1| = \frac{1}{4}\lim_{x\to1^-}\ln(1-x^4) = -\infty. \end{aligned}$$

The integral diverges.

6 The volume of the solid is, using the substitution $u = \ln x$,

$$\int_2^\infty \pi[f(x)]^2dx = \pi\lim_{t\to\infty}\int_2^t \frac{1}{x\ln^2 x}dx = \pi\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u^2}\,du = \pi\lim_{t\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln t}\right) = \frac{\pi}{\ln 2}.$$

7a Use L'Hôpital's rule:

$$\lim_{n\to\infty} n\tan\frac{\pi}{n} = \lim_{n\to\infty}\frac{\tan(\pi/n)}{1/n} \overset{\text{LR}}{=} \lim_{n\to\infty}\frac{(-\pi/n^2)\sec^2(\pi/n)}{-1/n^2} = \lim_{n\to\infty}\pi\sec^2\frac{\pi}{n} = \pi\sec^2 0 = \pi.$$

7b We have

$$\lim_{n\to\infty}\left(\sqrt{n^4-2n}-n^2\right)=\lim_{n\to\infty}\frac{(\sqrt{n^4-2n}-n^2)(\sqrt{n^4-2n}+n^2)}{\sqrt{n^4-2n}+n^2}=\lim_{n\to\infty}\frac{-2n}{\sqrt{n^4-2n}+n^2}$$
$$=\lim_{n\to\infty}\frac{-2/n}{\sqrt{1-2/n^3}+1}=\frac{0}{\sqrt{1-0}+1}=0.$$

8 Reindex to obtain

$$\sum_{n=2}^{\infty}\frac{3}{7^n}=\sum_{n=0}^{\infty}\frac{3}{7^{n+2}}=\frac{3}{7^2}\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n=\frac{3}{49}\cdot\frac{1}{1-1/7}=\frac{1}{14}.$$

9 The nth partial sum is

$$s_n=(\ln 2-\ln 1)+(\ln 3-\ln 2)+\cdots+[\ln n-\ln(n-1)]+[\ln(n+1)-\ln n]$$
$$=-\ln 1+\ln(n+1)=\ln(n+1),$$

and so

$$\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\ln(n+1)=\infty.$$

That is, the series diverges.

10a With the substitution $u=1+x^{3/2}$ we have

$$\int_1^{\infty}\frac{\sqrt{x}}{1+x^{3/2}}\,dx=\lim_{t\to\infty}\int_1^{1+t^{3/2}}\frac{2/3}{u}\,du=\lim_{t\to\infty}\frac{2}{3}\big[\ln|u|\big]_1^{1+t^{3/2}}=\frac{2}{3}\lim_{t\to\infty}\ln(1+t^{3/2})=\infty.$$

Since the integral diverges, the series diverges by the Integral Test.

10b Making the substitution $u=\ln x$, so $x=e^u$ and we have

$$\int_2^{\infty}\frac{\sqrt{x}}{1+x^{3/2}}\,dx=\lim_{t\to\infty}\int_2^{\ln t}\frac{u}{e^u}\,du=-\lim_{t\to\infty}\big[(1+u)e^{-u}\big]_1^{1+t^{3/2}}=\frac{1+\ln 2}{2}.$$

Since the integral converges, the series converges by the Integral Test.