Math 141 Exam #1 Key (Summer III 2018)

1a We have $f'(x) = 3 + \cos x$, so $f'(x) > 0$ for all $x \in \mathbb{R}$, which implies f is strictly increasing everywhere and is therefore one-to-one.

1b Since $f(0) = 0$, by the Inverse Function Theorem we have

$$(f^{-1})'(0) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{3+\cos(0)} = \frac{1}{4}.$$

2a Quotient rule:

$$y' = 2 \cdot \frac{(\ln x + 1)(1/x) - (\ln x)(1/x)}{(\ln x + 1)^2} = \frac{2}{x(\ln x + 1)^2}.$$

2b Chain rule: $f'(x) = \cos(\cos e^x) \cdot (-\sin e^x) \cdot e^x$.

2c We have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(e^{(x-1)\ln(\cot x)}\right) = e^{(x-1)\ln(\cot x)} \frac{d}{dx}\big[(x-1)\ln(\cot x)\big] \\ &= (\cot x)^{x-1}\left[-\frac{(x-1)\csc^2 x}{\cot x} + \ln(\cot x)\right]. \end{aligned}$$

2d We have

$$h'(t) = \frac{d}{dt}\left(e^{t^{1/2}\ln t}\right) = e^{t^{1/2}\ln t}\frac{d}{dt}(t^{1/2}\ln t) = t^{(t^{1/2})}\left(t^{-1/2} + \frac{1}{2}t^{-1/2}\ln t\right).$$

2e Use algebra to find that

$$\ell(x) = \frac{4\ln(1-x^3)}{\ln 3}.$$

Now,

$$\ell'(x) = \frac{12x^2}{(x^3-1)\ln 3}.$$

2f By the Chain Rule:

$$\varphi'(z) = \frac{1}{1+(1/z)^2} \cdot (1/z)' = \frac{-1/z^2}{1+(1/z)^2} = -\frac{1}{z^2+1}.$$

3a Use integration by parts with $u = \ln(3-2x)$ and $v' = 1$ to get

$$\begin{aligned} \int \ln(3-2x)\,dx &= x\ln(3-2x) - \int \frac{-2x}{3-2x}\,dx = x\ln(3-2x) - \int\left(1 + \frac{3}{2x-3}\right)dx \\ &= x\ln(3-2x) - x - \tfrac{3}{2}\ln|2x-3| + c \end{aligned}$$

$$= \left(x - \tfrac{3}{2}\right)\ln(3-2x) - x + c,$$

where for the last equality we note that $3 - 2x > 0$ must be the case, and so $|2x-3| = 3-2x$.

3b Let $u = e^x - e^{-x}$, so $du = (e^x + e^{-x})dx$, and we obtain

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int \frac{1}{u}du = \ln|u| + c = \ln\left|e^x - e^{-x}\right| + c.$$

3c Let $u = \cos x$, so $du = -\sin x\,dx = -\frac{1}{\csc x}dx$. Integral becomes

$$-\int e^u du = -e^u + c = -e^{\cos x} + c.$$

4a Letting $u = e^{t/2} + 1$, so integral becomes

$$\int_{1/e+1}^{e+1} \frac{2}{u}\,du = 2\ln|u|\Big]_{1/e+1}^{e+1} = 2\ln\left(\frac{e+1}{e^{-1}+1}\right) = 2.$$

4b Use a given formula:

$$\frac{1}{2}\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}\,dx = \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{3/2} = \frac{\pi}{12}.$$

4c Let $u = \sinh 2y$, so $du = 2\cosh 2y\,dy$ integral becomes

$$\frac{1}{2}\int_0^{\sinh 2} u^6\,du = \frac{1}{2}\left[\frac{1}{7}u^7\right]_0^{\sinh 2} = \frac{\sinh^7 2}{14}.$$

5 We have

$$\lim_{x\to\infty} e^{\frac{\ln(2x^8-3)}{\ln x}} = \exp\left(\lim_{x\to\infty}\frac{\ln(2x^8-3)}{\ln x}\right) \overset{\text{LR}}{=} \exp\left(\lim_{x\to\infty}\frac{16x^7/(2x^8-3)}{1/x}\right) = \exp\left(\lim_{x\to\infty}\frac{16x^8}{2x^8-3}\right) = e^8.$$

6 A long division along the way is needed:

$$\int \frac{8}{t^{-2}+1}dt = 8\int \frac{t^2}{t^2+1}dt = 8\int\left(1 - \frac{1}{t^2+1}\right)dt = 8t - 8\tan^{-1}t + c.$$

7a Use integration by parts with $u = x$ and $v' = 1/\sqrt{x+1}$, so that $u' = 1$ and $v = 2\sqrt{x+1}$:

$$\int \frac{x}{\sqrt{x+1}}\,dx = 2x\sqrt{x+1} - \tfrac{4}{3}(x+1)^{3/2} + c.$$

7b Integration by parts twice, starting with $u = x^2$ and $v' = \sin 2x$, gives

$$\begin{aligned}\int_0^{\pi/4} x^2 \sin 2x\,dx &= \frac{\pi^2}{8} + \int_0^{\pi/4} x\cos 2x\,dx \\ &= \frac{\pi^2}{8} + \left[\frac{x}{2}\sin 2x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \sin 2x\,dx \\ &= \frac{\pi^2}{8} + \frac{1}{4}\big[\cos 2x\big]_0^{\pi/4} = \frac{\pi^2}{8} - \frac{1}{2}.\end{aligned}$$

7c Using integration by parts with $u = (\ln x)^2$, $v' = 1$ gives

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2x\ln x + 2x + c.$$

Note that $\int \ln x\,dx$ also requires integration by parts if you don't remember the formula.

8 That f'' is continuous on $[a, b]$ ensures that $xf''(x)$ is continuous (and hence integrable) on $[a, b]$, and so integration by parts and the assumption that $f'(a) = f'(b) = 0$ gives

$$\int_a^b xf''(x)\,dx = xf'(x)\Big|_a^b - \int_a^b f'(x)\,dx = af'(a) - bf'(b) - \int_a^b f'(x)\,dx = -\int_a^b f'(x)\,dx.$$

Next, the continuity of f'' on $[a, b]$ implies that f' is differentiable on $[a, b]$, and so

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$$

by the Fundamental Theorem of Calculus. Putting our two findings together, we have

$$\int_a^b xf''(x)\,dx = -[f(b) - f(a)] = f(a) - f(b).$$