Math 141 Exam #3 Key (Summer III 2017)

1 For all $n \geq 1$ we have $\sin(1/n)/n^2 \leq 1/n^2$, and since $\sum 1/n^2$ is a convergent p-series, the Direct comparison Test implies that the given series also converges.

2 We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{[(n+1)!]^2}{[2(n+1)]!}\cdot\frac{(2n)!}{(n!)^2}\right) = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} \in [0,1),$$

and so the series converges by the Ratio Test.

3 Use the Ratio Test:

$$\rho = \lim_{n\to\infty}\left|\frac{[-(n+1)]^n}{(n+1)!}\cdot\frac{n!}{(-n)^{n-1}}\right| = \lim_{n\to\infty}\frac{(n+1)^{n-1}}{n^{n-1}} = \lim_{n\to\infty} e^{(n-1)\ln\frac{n+1}{n}} = e > 1,$$

and so the series diverges.

4 For integers $n > e^2$ we have $\ln n > 2$, so that $n^{\ln n} > n^2$ and hence $1/n^{\ln n} < 1/n^2$. Since $\sum 1/n^2$ is a convergent p-series, it follows by the Direct Comparison Test that $\sum 1/n^{\ln n}$ also converges.

5 For $n \geq 1$ we have

$$b_n = \frac{1}{\sqrt{n}} > 0,$$

with $b_n \to 0$ as $n \to \infty$. Also,

$$b_{n+1} \leq b_n \quad\Leftrightarrow\quad \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \quad\Leftrightarrow\quad n+1 \geq n \quad\Leftrightarrow\quad 1 \geq 0,$$

and since the last inequality is clearly true, the sequence (b_n) is nonincreasing. Therefore the series converges by the Alternating Series Test. It is conditionally convergent since the series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

is a divergent p-series.

6 By the Alternating Series Estimation Theorem, the smallest integer k for which the partial sum

$$\sum_{n=1}^{k-1}\frac{(-1)^n n}{n^5+2}$$

approximates the value of the series with an error of less than 10^{-3} must be the first integer k for which $k/(k^5+2) < 10^{-3}$. Since $5/(5^5+2) \approx 0.0016 > 10^{-3}$ and $6/(6^5+2) \approx 0.00077 < 10^{-3}$, we find that $k = 6$ fits the bill. Thus we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^5+2} \approx \sum_{n=1}^{5}\frac{(-1)^n n}{n^5+2}$$

with absolute error less than 10^{-3}.

7 The 3rd-order Taylor polynomial for e^x is $1+x+x^2/2+x^3/6$, and so

$$e^{0.11} \approx 1+0.11+\frac{1}{2}(0.11)^2+\frac{1}{6}(0.11)^3 \approx 1.11627807.$$

8a Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}}\cdot\frac{\sqrt{n^2+3}}{x^n}\right| = |x|\lim_{n\to\infty}\left|\sqrt{\frac{n^2+3}{n^2+2n+4}}\right| = |x|.$$

Series converges if $|x|<1$, so interval of convergence contains $(-1,1)$. Check endpoints.

At $x=1$: series becomes $\sum 1/\sqrt{n^2+3}$, and since

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and the series $\sum 1/n$ is known to diverge, the series $\sum 1/\sqrt{n^2+3}$ diverges by the Direct Comparison Test.

At $x=-1$: series becomes $\sum(-1)^n/\sqrt{n^2+3}$, which can be shown to converge by the Alternating Series Test.

Therefore the original series has interval of convergence $[-1,1)$.

8b Apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|2+\frac{2}{n}\right||x-1| = |x-1|\lim_{n\to\infty}\left(2+\frac{2}{n}\right) = 2|x-1|.$$

Series converges if $2|x-1|<1$, so interval of convergence contains $\left(\frac{1}{2},\frac{3}{2}\right)$. Check endpoints.

At $x=\frac{3}{2}$: Series becomes

$$\sum\left(2+\frac{2}{n}\right)^n\left(\frac{1}{2}\right)^n = \sum\left(1+\frac{1}{n}\right)^n,$$

and since $\left(1+\frac{1}{n}\right)^n \to e$ as $n\to\infty$, the series diverges by the Divergence Test.

At $x=\frac{1}{2}$: Series becomes

$$\sum\left(2+\frac{2}{n}\right)^n\left(-\frac{1}{2}\right)^n = \sum(-1)^n\left(1+\frac{1}{n}\right)^n,$$

which also diverges by the Divergence Test.

Therefore the original series has interval of convergence $\left(\frac{1}{2},\frac{3}{2}\right)$.

8c Apply Ratio Test, using L'Hôpital's Rule to find the limit:

$$\lim\left|\frac{x^{n+1}\ln(n+1)}{x^n\ln n}\right| = |x|\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \overset{\text{LR}}{=} |x|\lim_{n\to\infty}\frac{1/(n+1)}{1/n} = |x|.$$

Thus the series converges if $|x|<1$, so interval of convergence contains $(-1,1)$. At the endpoints we obtain either the series $\sum \ln n$ or $\sum(-1)^n\ln n$, both of which diverge by the Divergence Test. Therefore the original series has interval of convergence $(-1,1)$.

9 Use the geometric series:

$$\frac{2x^2}{1+x^3} = 2x^2 \cdot \frac{1}{1-(-x^3)} = 2x^2 \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n 2x^{3n+2}.$$

Interval of convergence is $|-x^3| < 1$, and hence $(-1, 1)$.

10 Using the binomial series,

$$f(x) = \sum_{n=0}^{\infty} \binom{1/4}{n} x^{3n} = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \cdots .$$

Interval of convergence is $(-1, 1)$.

11 We have

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\,dx = \int_0^{0.1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n-1}}{n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \int_0^{0.1} x^{n-1}dx$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n^2} = 0.1 - \frac{(0.1)^2}{2^2} + \frac{(0.1)^3}{3^2} - \cdots .$$

Since $(0.1)^3/3^2 < 10^{-5}$, the estimate

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\,dx \approx 0.1 - \frac{(0.1)^2}{2^2} = 0.0975$$

will have an absolute error less than 10^{-5}.

12 We have $(2x)^2 + (3y)^2 = 36\cos^2\theta + 36\sin^2\theta = 36$, and thus $x^2/9 + y^2/4 = 1$.

13 $(4\sin\theta - 2, 4\cos\theta + 9)$.

14 Recalling $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$, we use the identity $\sin 2\theta = 2\sin\theta\cos\theta$ to obtain

$$x = 2 \cdot \frac{y}{r} \cdot \frac{x}{r} \quad \Rightarrow \quad x = \frac{2xy}{x^2+y^2} \quad \Rightarrow \quad x(x^2+y^2) = 2xy.$$

15 The polar curves intersect for θ such that $\cos\theta = \frac{1}{2}$, giving $\theta = \pm\frac{\pi}{3}$. Thus the area is

$$\int_{-\pi/3}^{\pi/3} \frac{\cos^2\theta - (1/2)^2}{2}\,d\theta = \frac{3\sqrt{3} + 2\pi}{24}.$$