Math 141 Exam #2 Key (Summer III 2017)

1 Let $u = \cos x$, so that

$$\int \sin^3 x\sqrt{\cos x}\,dx = -\int (1-u^2)\sqrt{u}\,du = \int (u^{5/2} - u^{1/2})\,du$$
$$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + c = \frac{2}{7}\cos^{7/2} x - \frac{2}{3}\cos^{3/2} x + c.$$

2 Write

$$\int \frac{\sqrt{9p^2-4}}{p}\,dp = 2\int \frac{\sqrt{(3p/2)^2-1}}{p}\,dp,$$

and let $\sec\theta = 3p/2$. Note: $\theta \in [0, \frac{\pi}{2})$ if $p \in (-\infty, -\frac{2}{3}]$, in which case $\tan\theta \le 0$, and $\theta \in (\frac{\pi}{2}, \pi]$ if $p \in [\frac{2}{3}, \infty)$, in which case $\tan\theta \ge 0$. Now the integral becomes

$$2\int \frac{\sqrt{\sec^2\theta - 1}}{\frac{2}{3}\sec\theta} \cdot \frac{2}{3}\sec\theta\tan\theta\,d\theta = 2\int \tan\theta\sqrt{\tan^2\theta}\,d\theta = 2\int \tan\theta|\tan\theta|\,d\theta.$$

The integral is indefinite, so $|\tan\theta| = \tan\theta$ or $|\tan\theta| = -\tan\theta$ depending on whether $p \ge \frac{2}{3}$ or $p \le -\frac{2}{3}$, respectively. If $p \ge \frac{2}{3}$, then the integral becomes

$$2\int \tan^2\theta\,d\theta = 2\left(\tan\theta - \int d\theta\right) = 2\tan\theta - 2\theta + c$$
$$= 2\tan\big(\sec^{-1}(3p/2)\big) - 2\sec^{-1}(3p/2) + c$$
$$= \sqrt{9p^2-4} - 2\sec^{-1}\left(\frac{3p}{2}\right) + c.$$

The opposite answer obtains if $p \le -\frac{2}{3}$, and so

$$\int \frac{\sqrt{9p^2-4}}{p}\,dp = \begin{cases} \sqrt{9p^2-4} - 2\sec^{-1}(3p/2) + c, & \text{if } p \ge \frac{2}{3} \\ -\sqrt{9p^2-4} + 2\sec^{-1}(3p/2) + c, & \text{if } p \le -\frac{2}{3}. \end{cases}$$

3a By partial fractions technique:

$$\frac{12r}{(r-4)^2} = \frac{A}{r-4} + \frac{B}{(r-4)^2} \quad\Rightarrow\quad 12r = A(r-4) + B \quad\Rightarrow\quad A = 12,\ B = 48.$$

So,

$$\int \frac{12r}{(r-4)^2}\,dr = \int \left(\frac{12}{r-4} + \frac{48}{(r-4)^2}\right)dr = 12\ln|r-4| - \frac{48}{r-4} + c.$$

3b We have

$$\frac{x+1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2},$$

which yields $A = -3/4$, $B = -1/2$, $C = 3/4$. Then

$$\int \frac{x+1}{x^2(x-2)}\,dx = \int \left(\frac{-3/4}{x} - \frac{1/2}{x^2} + \frac{3/4}{x-2} \right) dx = \frac{3}{4} \ln \left| \frac{x-2}{x} \right| + \frac{1}{2x} + c.$$

4a Making the substitution $u = \ln y$,

$$\int_2^\infty \frac{dy}{y \ln y} = \lim_{t\to\infty} \int_2^t \frac{dy}{y \ln y} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du = \lim_{t\to\infty} \big[\ln|\ln t| - \ln|\ln 2| \big] = \infty.$$

The integral diverges.

4b With the substitution $u = 10 - x$,

$$\int_0^{10} \frac{1}{\sqrt[4]{10-x}}\,dx = \lim_{t\to 10^-} \int_0^t \frac{1}{\sqrt[4]{10-x}}\,dx = \lim_{t\to 10^-} \int_{10}^{10-t} \frac{-1}{\sqrt[4]{u}}\,du$$

$$= \lim_{t\to 10^-} \left[-\frac{4}{3} u^{3/4} \right]_{10}^{10-t} = -\frac{4}{3} \lim_{t\to 10^-} \left[(10-t)^{3/4} - 10^{3/4} \right]$$

$$= \frac{4(10^{3/4})}{3}.$$

The integral converges.

5 Area is

$$\mathcal{A} = \int_0^\infty (e^{-bx} - e^{-ax})\,dx = \lim_{t\to\infty} \int_0^t (e^{-bx} - e^{-ax})\,dx = \lim_{t\to\infty} \left[-\frac{1}{b} e^{-bx} + \frac{1}{a} e^{-ax} \right]_0^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{b} e^{-bt} + \frac{1}{a} e^{-at} + \frac{1}{b} - \frac{1}{a} \right) = \frac{1}{b} - \frac{1}{a}.$$

6a Recurrence relation: $a_{n+1} = -a_n$, $a_1 = 4$.

6b Explicit formula: $a_n = (-1)^{n+1} 4$, $n \geq 1$.

7 We have

$$\lim_{n\to\infty} \left(\frac{1}{n} \right)^{1/\ln n} = \lim_{n\to\infty} e^{\ln(1/n)^{1/\ln n}} = \lim_{n\to\infty} e^{\ln(1/n)/\ln n} = \lim_{n\to\infty} e^{-\ln n/\ln n} = e^{-1}.$$

8 In the limit process wherein $n \to \infty$ only the expression for $n > 5000$ is relevant. Thus:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n e^{-n} = \lim_{n\to\infty} \frac{n}{e^n} \overset{\text{LR}}{=\!=} \lim_{n\to\infty} \frac{1}{e^n} = 0.$$

9 Make sure to reindex to use the usual formula:

$$\sum_{n=1}^{\infty} 3^{-2n} = \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n = \sum_{n=0}^{\infty} \frac{1}{9}\left(\frac{1}{9}\right)^n = \frac{1/9}{1-1/9} = \frac{1}{8}.$$

10 Partial fraction decomposition yields

$$\frac{2}{(n+6)(n+7)} = \frac{2}{n+6} - \frac{2}{n+7},$$

and so

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \left(\frac{2}{n+6} - \frac{2}{n+7}\right) \\ &= \left(\tfrac{2}{7} - \tfrac{2}{8}\right) + \left(\tfrac{2}{8} - \tfrac{2}{9}\right) + \left(\tfrac{2}{9} - \tfrac{2}{10}\right) + \cdots + \left(\tfrac{2}{k+5} - \tfrac{2}{k+6}\right) + \left(\tfrac{2}{k+6} - \tfrac{2}{k+7}\right) \\ &= \tfrac{2}{7} - \tfrac{2}{k+7}. \end{aligned}$$

From this we see that

$$\sum_{n=1}^{\infty} \frac{2}{(n+6)(n+7)} = \lim_{n\to\infty} s_k = \lim_{k\to\infty} \left(\frac{2}{7} - \frac{2}{k+7}\right) = \frac{2}{7}.$$

11 We have

$$\int_1^{\infty} \frac{1}{e^x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{e^x} dx = \lim_{t\to\infty} \left[-e^{-x}\right]_1^t = -\lim_{t\to\infty} (e^{-t} - e^{-1}) = e^{-1}.$$

Since the integral converges, the series also converges by the Integral Test.

12 Let $u = \tan^{-1} x$, so

$$\begin{aligned} \int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\, dx &= \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1} t} u\, du = \lim_{t\to\infty} \left[\frac{u^2}{2}\right]_{\pi/4}^{\tan^{-1} t} \\ &= \lim_{t\to\infty} \left[\frac{(\tan^{-1} t)^2}{2} - \frac{\pi^2}{32}\right] = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}. \end{aligned}$$

The integral converges, and therefore so too does the series.