Math 141 Exam #1 Key (Summer III 2017)

1 By the Inverse Function Theorem we have $(f^{-1})'(f(x)) = 1/f'(x)$ for all x in an interval I on which f is one-to-one. Since $f(2) = 6$ and $f'(x) = 6x^2 + 1$, we have

$$(f^{-1})'(6) = (f^{-1})'(f(x) = \frac{1}{f'(2)} = \frac{1}{25}.$$

(Note that $f'(2) = 25 > 0$, so f is strictly increasing on some open interval I containing 2, and thus f is one-to-one on I.)

2a Applying the Product Rule,

$$y' = 2xe^{x^3} + x^2e^{x^3}(x^3)' = 2xe^{x^3} + 3x^4e^{x^3} = (2x + 3x^4)e^{x^3}.$$

2b Note that $\ln\sqrt{x} = \frac{1}{2}\ln x$, so by the Chain Rule

$$f'(x) = 2\ln\sqrt{x}\cdot\left(\tfrac{1}{2}\ln x\right)' = \frac{\ln\sqrt{x}}{x}.$$

2c Note that $g(x) = e^{\ln 3\cdot\sin x}$, so

$$g'(x) = e^{\ln 3\cdot\sin x}\frac{d}{dx}(\ln 3\cdot\sin x) = 3^{\sin x}\ln(3)\cos(x).$$

2d Since $\ln x^5 = 5\ln x$,

$$r'(x) = \frac{1}{5\ln x}(5\ln x)' = \frac{1}{x\ln x}.$$

2e By the Chain Rule:

$$\varphi'(z) = -\frac{1}{1 + (1/z)^2}\cdot(1/z)' = \frac{1}{1 + z^2}.$$

2f We have

$$y' = \operatorname{sech}^2(\ln x)\cdot(\ln x)' = \frac{\operatorname{sech}^2(\ln x)}{x}$$

3 Here $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x\ln x\right)$. At $x = 1$ we thus have $y' = \sin 1$. So the tangent line has slope $\sin 1$ and contains the point $(1, 1^{\sin 1}) = (1, 1)$. The equation of the line is then

$$y = (\sin 1)(x - 1) + 1.$$

4a Let $u = 4e^x + 6$, so $\frac{1}{4}du = e^x\,dx$, and we get

$$\int\frac{e^x}{4e^x + 6}dx = \int\frac{1/4}{u}du = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln(4e^x + 6) + c.$$

4b Let $u = \ln(\ln x)$, so $du = \frac{1}{x \ln x} dx$, and the integral becomes

$$\int \frac{1}{u} du = \ln|u| + c = \ln\big|\ln(\ln x)\big| + c.$$

4c Let $u = \sin x$, so $du = \cos x\, dx = \frac{1}{\sec x} dx$. Integral becomes

$$\int e^u du = e^u + c = e^{\sin x} + c.$$

4d Letting $u = \cosh t$, and noting that $\cosh t > 0$ for all $t \in \mathbb{R}$, we have

$$\int \frac{\sinh t}{1 + \cosh t} dt = \int \frac{1}{1+u} du = \ln|u+1| + c = \ln|\cosh t + 1| + c = \ln(\cosh t + 1) + c.$$

5a Let $u = \ln x$, so $du = \frac{1}{x} dx$. Integral becomes

$$\int_0^{\ln 2e} 3^u du = \left[\frac{3^u}{\ln 3}\right]_0^{\ln 2e} = \frac{3^{\ln 2e} - 1}{\ln 3}.$$

5b Let $u = e^x$, so $du = e^x dx$. Integral becomes

$$\int_{1/\sqrt{3}}^{1} \frac{1}{1+u^2} du = \left[\tan^{-1}(u)\right]_{1/\sqrt{3}}^{1} = \tan^{-1}(1) - \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

6 Use of L'Hôpital's Rule is not absolutely essential in this case, but it doesn't hurt either.

$$\begin{aligned}\lim_{x\to 0^+} x^{1/\ln x} &= \lim_{x\to 0^+} e^{\ln x^{1/\ln x}} = \lim_{x\to 0^+} e^{\ln x/\ln x} = \exp\left(\lim_{x\to 0^+} \frac{\ln x}{\ln x}\right) \\ &\overset{\text{LR}}{=} \exp\left(\lim_{x\to 0^+} \frac{1/x}{1/x}\right) = \exp(1) = e.\end{aligned}$$

7 Get a sum of squares:

$$\int_0^3 \frac{2}{(x+1)^2 + 1^2} dx = 2\left[\tan^{-1}(x+1)\right]_0^3 = 2\big[\tan^{-1}(4) - \tan^{-1}(1)\big] = 2\tan^{-1} 4 - \frac{\pi}{2}.$$

8 Using integration by parts,

$$\begin{aligned}\int t^2 e^{2t} dt &= \frac{1}{2} t^2 e^{2t} - \int t e^{2t} dt = \frac{1}{2} t^2 e^{2t} - \left(\frac{1}{2} t e^{2t} - \frac{1}{2}\int e^{2t} dt\right) \\ &= \frac{1}{2} t^2 e^{2t} - \frac{1}{2} t e^{2t} + \frac{1}{4} e^{2t} + c.\end{aligned}$$

9 Using integration by parts,

$$\int_0^{\pi/2} x\cos 2x\,dx = \left[\frac{x}{2}\sin 2x\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx = \frac{1}{4}(\cos\pi - \cos 0) = -\frac{1}{2}$$

10 The length of the curve given by a function $f : [a,b] \to \mathbb{R}$ is

$$\mathcal{L} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx,$$

and so, using the Fundamental Theorem of Calculus,

$$\begin{aligned}\mathcal{L} &= \int_e^{e^5} \sqrt{1 + \left(\frac{d}{dx}\int_e^x \sqrt{\ln^2 t - 1}\,dt\right)^2}\,dx = \int_e^{e^5}\sqrt{1 + \left(\sqrt{\ln^2 x - 1}\right)^2}\,dx \\ &= \int_e^{e^5}\sqrt{\ln^2 x}\,dx = \int_e^{e^5}\ln x\,dx = \big[x\ln x - x\big]_e^{e^5} = 4e^5.\end{aligned}$$