MATH 141 EXAM #2 KEY (SUMMER 2020)

1a Let $u = x + 6$, so $du = dx$ and then the integral becomes

$$3\int \frac{u-6}{\sqrt{u}}du = 3\int (u^{1/2} - 6u^{-1/2})\,du = 2u^{3/2} - 36u^{1/2} + c = 2(x+6)^{3/2} - 36(x+6)^{1/2} + c.$$

1b Let $u = \sec\theta$, so $dx = \sec\theta\tan\theta\,d\theta$ and the integral becomes

$$\int_{\pi/4}^{\pi/3} \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \cdot \sec\theta\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta$$

$$= \big[\tan\theta\big]_{\pi/4}^{\pi/3} - \int_{\pi/4}^{\pi/3} d\theta = \sqrt{3} - 1 - \frac{\pi}{12},$$

using the given reduction formula for $\int \tan^n\theta\,d\theta$.

1c Partial fraction decomposition and some algebra gives

$$\frac{2x^2+7x+4}{x(x^2+2x+2)} = \frac{2}{x} + \frac{3}{x^2+2x+2} = \frac{2}{x} + \frac{3}{1^2+(x+1)^2},$$

so integral becomes

$$\int \left[\frac{2}{x} + \frac{3}{1^2+(x+1)^2}\right] dx = 2\ln|x| + 3\tan^{-1}(x+1) + c.$$

1d Let $u = \cos t$, so $du = -\sin t\,dt$ and integral becomes

$$\int \frac{(1-\cos^2 t)\sin t}{\cos^5 t}dt = \int \frac{u^2-1}{u^5}du = -\frac{1}{2}u^{-2} + \frac{1}{4}u^{-4} + c = -\frac{\sec^2 t}{2} + \frac{\sec^4 t}{4} + c.$$

1e Integration by parts: Let $u = \tan^{-1}7x$, $v' = x$, so $u' = \frac{7}{49x^2+1}$, $v = \frac{1}{2}x^2$. Then integral becomes

$$\frac{1}{2}x^2\tan^{-1}7x - \frac{7}{2}\int \frac{x^2}{49x^2+1}dx.$$

Do long division on the new integrand to get

$$\frac{x^2}{2}\tan^{-1}7x - \frac{1}{14}\int \left[1 - \frac{7}{(7x)^2+1}\right] dx = \frac{x^2}{2}\tan^{-1}7x - \frac{x}{14} + \frac{1}{98}\tan^{-1}7x + c.$$

1f Integration by parts: Let $u = \ln x$, $v' = x^2$, so $u' = 1/x$, $v = x^3/3$. Integral becomes

$$\left[\frac{1}{3}x^3\ln x\right]_1^e - \frac{1}{3}\int_1^e x^2\,dx = \frac{e^3}{3} - \frac{e^3-1}{9} = \frac{2e^3+1}{9}.$$

1g With the substitution $x = 2\tan\theta$ we get

$$\frac{1}{2}\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + c$$
$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + c = \frac{1}{2}\ln\left(\frac{\sqrt{x^2+4}+x}{2}\right) + c.$$

Note the absolute value is evaluated in the end, since in general

$$\sqrt{x^2+4} > \sqrt{x^2} = |x| = -x,$$

and so $\sqrt{x^2+4}+x > 0$ holds.

1h With long division:

$$\frac{x^4+1}{x^3+9x} = x + \frac{1-9x^2}{x^3+9x}.$$

Partial fraction decomposition:

$$x + \frac{1-9x^2}{x^3+9x} = x + \frac{\frac{1}{9}}{x} - \frac{\frac{82}{9}x}{x^2+9}.$$

Now integral becomes:

$$\int \left(x + \frac{1}{9x} - \frac{82x}{9(x^2+9)}\right)dx = \frac{1}{2}x^2 + \frac{1}{9}\ln|x| - \frac{41}{9}\ln(x^2+9) + c.$$

2 Volume is

$$\int_1^2 \pi\left(\frac{1}{\sqrt{x(3-x)}}\right)^2 dx = \pi\int_1^2 \frac{1}{x(3-x)}\,dx = \pi\int_1^2\left(\frac{1}{3x}+\frac{1}{3(3-x)}\right)dx$$
$$= \frac{\pi}{3}\int_1^2\left(\frac{1}{x}-\frac{1}{x-3}\right)dx = \frac{\pi}{3}\big[\ln|x| - \ln|x-3|\big]_1^2 = \frac{\pi}{3}\ln 4.$$

3a Applying partial fraction decomposition, we find the integral converges:

$$\lim_{t\to\infty}\int_1^t\left(\frac{1}{v}-\frac{1}{v+1}\right)dv = \lim_{t\to\infty}\big[\ln|v| - \ln|v+1|\big]_1^t = \lim_{t\to\infty}\left[\ln\left|\frac{t}{t+1}\right| - \ln\frac{1}{2}\right] = \ln 2.$$

3b The integral converges:

$$\lim_{t\to-3^+}\int_t^1 (2y+6)^{-2/3}dy = \lim_{t\to-3^+}\left[\frac{3}{2}(2y+6)^{1/3}\right]_t^1 = \frac{3}{2}\lim_{t\to-3^+}\left[2-(2t+6)^{1/3}\right] = 3.$$

3c First, note that

$$\int_0^\infty \frac{e^x}{\sqrt{e^{2x}-1}}\,dx = \int_0^1 \frac{e^x}{\sqrt{e^{2x}-1}}\,dx + \int_1^\infty \frac{e^x}{\sqrt{e^{2x}-1}}\,dx,$$

with both integrals being improper. Working with the second integral, let $u = e^x$, so $du = e^x\,dx$ and the integral becomes

$$\lim_{t\to\infty} \int_1^t \frac{e^x}{\sqrt{e^{2x}-1}}\,dx = \lim_{t\to\infty} \int_e^{e^t} \frac{1}{\sqrt{u^2-1}}\,du = \int_e^{\infty} \frac{1}{\sqrt{u^2-1}}\,du.$$

Next let $u = \sec\theta$ to get

$$\begin{aligned}\lim_{t\to\infty} \int_e^t \frac{1}{\sqrt{u^2-1}}\,du &= \lim_{t\to\infty} \int_{\sec^{-1} e}^{\sec^{-1} t} \sec\theta\,d\theta = \lim_{t\to\infty} \big[\ln|\sec\theta + \tan\theta|\big]_{\sec^{-1} e}^{\sec^{-1} t} \\ &= \lim_{t\to\infty} \Big[\ln\big(t + \sqrt{t^2-1}\,\big) - \ln\big(e + \sqrt{e^2-1}\,\big)\Big] = \infty.\end{aligned}$$

Since this last integral diverges, so does the original integral $\int_{-\infty}^{\infty}$.

4 $e^{-ax} = e^{-bx}$ only holds when $x = 0$, so the area is

$$\int_0^{\infty} (e^{-bx} - e^{-ax})\,dx = \lim_{t\to\infty} \left[-\frac{1}{b}e^{-bx} + \frac{1}{a}e^{-ax}\right]_0^t = \frac{1}{b} - \frac{1}{a}.$$

5 For all $x \geq 1$ we have

$$\frac{2+\cos x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}},$$

and since

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \big[2\sqrt{x}\big]_1^t = \lim_{t\to\infty} \big(2\sqrt{t} - 2\big) = \infty,$$

the Comparison Theorem implies that the given integral diverges.