MATH 141 EXAM #1 KEY (SUMMER 2020)

1 Since $f(-3) = 12$ and $f'(x) = 2x - 2$, the Inverse Function Theorem gives

$$(f^{-1})'(12) = (f^{-1})'(f(-3)) = \frac{1}{f'(-3)} = \frac{1}{2(-3)-2} = -\frac{1}{8}.$$

2a $\cot x$

2b $-e^x \csc^2(e^x)$

2c $\dfrac{d}{dx}\left[e^{(x^2-x)\ln 3}\right] = 3^{x^2-x}(2x-1)\ln 3$

2d $\dfrac{d}{dx}\left[e^{\sin x \cdot \ln(1+x^2)}\right] = (1+x^2)^{\sin x}\left[\ln(1+x^2)\cos x + \dfrac{2x\sin x}{1+x^2}\right]$

2e Let $y = \log_8 |\tan x|$, so we must find y'. We have $8^y = |\tan x|$, and hence $y = \dfrac{\ln|\tan x|}{\ln 8}$. Now,

$$y' = \frac{\sec^2 x}{\ln 8 \tan x}.$$

2f Using the given formula and the Chain Rule: $\dfrac{1}{x\sqrt{1-\ln^2 x}}$.

2g $-\dfrac{e^t \sec^2 e^t}{|\tan e^t|\sqrt{(\tan e^t)^2 - 1}}$.

2h $\frac{1}{2}(\operatorname{sech} 8z)^{-1/2}(-\tanh 8z \operatorname{sech} 8z) \cdot 8 = -4\tanh 8z\sqrt{\operatorname{sech} 8z}$.

3 Here

$$y' = 2^{\sin x} \cdot \cos x \cdot \ln 2,$$

so when $x = \pi$ we have $y' = -\ln 2$. This is the slope of the tangent line through the point $(\pi, 1)$, so the equation is $y = (\pi - x)\ln 2 + 1$.

4a $\displaystyle\int_{-2}^{3} \frac{12}{18-5t}\,dt = -\frac{12}{5}\Big[\ln|18-5t|\Big]_{-2}^{3} = \frac{12}{5}\ln\left(\frac{28}{3}\right).$

4b Let $u = 1 + \cos x$ to get

$$-\int_2^1 \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^2 = \ln 2.$$

4c Let $u = \ln\ln y$, so $du = \frac{1}{y\ln y}dy$ and the integral becomes

$$\int u^4\,du = \frac{1}{5}u^5 + C = \frac{(\ln\ln y)^5}{5} + C.$$

4d We have

$$\int_{-1}^{1} 10^x\,dx = \int_{-1}^{1} e^{x\ln 10}\,dx = \left[\frac{1}{\ln 10}e^{x\ln 10}\right]_{-1}^{1} = \frac{1}{\ln 10}\left[10^x\right]_{-1}^{1} = \frac{99}{10\ln 10}.$$

4e Use one of the formulas on the back side of the exam:

$$\frac{3}{64}\int \frac{1}{(5/4)^2 + x^2}\,dx = \frac{3}{64}\left(\frac{4}{5}\tan^{-1}\frac{4x}{5}\right) + C = \frac{3}{80}\tan^{-1}\frac{4x}{5} + C.$$

4f Let $u = \operatorname{sech} w$, so $du = -\operatorname{sech} w \tanh w\,dw$ and we get

$$-\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\operatorname{sech}^2 w + C.$$

5 For $f(x) = \ln(x - \sqrt{x^2-1})$ we have

$$f'(x) = \frac{1}{x - \sqrt{x^2-1}}\left[1 - \frac{1}{2}(x^2-1)^{-1/2}(2x)\right] = -\frac{1}{\sqrt{x^2-1}},$$

and so, making the substitution $u = x^2 - 1$, the arc length is

$$\int_1^{\sqrt{2}} \sqrt{1 + [f'(x)]^2}\,dx = \int_1^{\sqrt{2}} \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{u}}\,du = 1.$$

One note: though this problem came from the review exercises in Chapter 7 of the text, the rightmost integral above (and hence all the integrals in this problem) is in fact improper. Improper integrals are not treated until the end of Chapter 8, so this is a mistake on the book's part. Fortunately, handling the integral in the usual way still gives the right answer here.

6a The limit has the form 3^∞, which is not indeterminate at all but rather equals $+\infty$.

6b With L'Hôpital's Rule,

$$\lim_{x\to 1^-} \frac{\ln x}{\cot(\pi x/2)} \overset{\text{LR}}{=} \lim_{x\to 1^-} \frac{1/x}{-\csc^2(\pi x/2)\cdot(\pi/2)} = -\frac{2}{\pi}\lim_{x\to 1^-}\frac{\sin^2(\pi x/2)}{x} = -\frac{2}{\pi}.$$

6c Limit equals

$$\lim_{x\to 0} e^{(b/x)\ln(1+a^x)} = \exp\left(\lim_{x\to 0}\frac{b\ln(1+a^x)}{x}\right) \overset{\text{LR}}{=} \exp\left(\lim_{x\to 0}\frac{b\cdot a^x\cdot\ln a}{1+a^x}\right) = e^{(b/2)\ln a} = a^{b/2}.$$