Math 141 Exam #3 Key (Summer 2019)

1a $\lim_{n\to\infty} a_n = 2$, since $\sum(0.98)^n$ is a convergent geometric series and so $(0.98)^n \to 0$ as $n \to \infty$ by the Divergence Test. (There are many other ways to argue this, the simplest argument being that $\lim_{n\to\infty} x^n = 0$ whenever $|x| < 1$ is an established fact.)

1b $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \frac{3n^2+4}{n^2+4} = \ln 3.$

2 In the following we multiply by e^{n+1} to simplify things:

$$a_{n+1} > a_n \quad \Leftrightarrow \quad 3 - 2(n+1)e^{-(n+1)} > 3 - 2ne^{-n} \quad \Leftrightarrow \quad n+1 < ne \quad \Leftrightarrow \quad e > 1 + \frac{1}{n}.$$

The last equality is obviously true for any $n \geq 1$, and therefore $a_{n+1} > a_n$ is true for all $n \geq 1$; that is, the sequence (a_n) is increasing. This means (a_n) is bounded below, and since $a_n \to 3$ as $n \to \infty$, we find that 3 is an upper bounded on the sequence, and hence (a_n) is a bounded sequence.

3 We have

$$10 - 2 + 0.4 - 0.08 + \cdots = \frac{1}{10^{-1}} - \frac{2}{10^0} + \frac{2^2}{10^1} - \frac{2^3}{10^2} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{10^{n-1}} = 10 \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n,$$

a convergent geometric series equalling $\frac{25}{3}$.

4 The kth partial sum is

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \left(e^{1/n} - e^{1/(n+1)}\right) \\ &= (e - e^{1/2}) + (e^{1/2} - e^{1/3}) + (e^{1/3} - e^{1/4}) + \cdots + (e^{1/k} - e^{1/(k+1)}) \\ &= e - e^{1/(k+1)}, \end{aligned}$$

and so the series converges:

$$\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right) = \lim_{k\to\infty} \sum_{n=1}^{k} \left(e^{1/n} - e^{1/(n+1)}\right) = \lim_{k\to\infty} \left(e - e^{1/(k+1)}\right) = e - 1.$$

5 The series is geometric:

$$\sum_{n=1}^{\infty} \left(\frac{1}{c+3}\right)^n = 4 \quad \Leftrightarrow \quad \frac{1}{c+3}\left(\frac{1}{1 - \frac{1}{c+3}}\right) = 4 \quad \Leftrightarrow \quad c = -\frac{7}{4}.$$

6 The series is $\sum_{n=1}^{\infty} \frac{1}{4n-1}$, and since

$$\int_1^{\infty} \frac{1}{4x-1} dx = \frac{1}{4} \lim_{t\to\infty} \left[\ln(4x-1)\right]_1^t = \frac{1}{4} \lim_{t\to\infty} \left[\ln(4t-1) - \ln 3\right] = \infty$$

the series diverges by the Integral Test.

7a Since $\dfrac{n\sin^2 n}{n^3+1} \le \dfrac{n}{n^3+1} \le \dfrac{n}{n^3} = \dfrac{1}{n^2}$ and $\sum 1/n^2$ converges, the Direct Comparison Test implies the given series converges also.

7b Since $\sum(4/3)^n$ is a divergent geometric series and

$$\lim_{n\to\infty} \frac{\dfrac{4^n+1}{3^n-2}}{(4/3)^n} = \lim_{n\to\infty}\left(\frac{4^n+1}{3^n-2}\cdot\frac{3^n}{4^n}\right) = \lim_{n\to\infty}\frac{12^n+3^n}{12^n-2\cdot 4^n} = \lim_{n\to\infty}\frac{1+(1/4)^n}{1-2\cdot(1/3)^n} = 1 \in (0,\infty),$$

the Limit Comparison Test implies the given series diverges also.

8 For all n we have $d_n/10^n \le 9/10^n$, and since $\sum 9/10^n$ is a convergent geometric series, the Direct Comparison Test implies that the given series $\sum_{n=1}^{\infty} d_n/10^n$ also converges.

9a Since $\displaystyle\lim_{n\to\infty}\frac{n^2}{n^2+n+1} = 1 \ne 0$, the series diverges by the Divergence Test.

9b For $b_n = \sqrt{n}/(2n+3)$ we find that

$$\begin{aligned} b_{n+1} < b_n \quad &\Leftrightarrow \quad \frac{\sqrt{n+1}}{2(n+1)+3} < \frac{\sqrt{n}}{2n+3} \quad \Leftrightarrow \quad (2n+3)^2(n+1) < (2n+5)^2 n \\ &\Leftrightarrow \quad 9 < 4n^2 + 4n, \end{aligned}$$

and since $9 < 4n^2 + 4n$ is true for all $n \ge 2$, so too is $b_{n+1} < b_n$, and thus (b_n) is an (eventually) decreasing sequence. Moreover

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\sqrt{n}}{2n+3} = \lim_{n\to\infty}\frac{1/\sqrt{n}}{2+3/n} = 0,$$

and since $b_n > 0$ for all n, the Alternating Series Test implies that the given series converges.

10a Ratio Test: since

$$\lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{[2(n+1)+1]!}\cdot\frac{(2n+1)!}{(-3)^n}\right| = \lim_{n\to\infty}\frac{3}{(2n+2)(2n+3)} = 0 < 1,$$

the series converges.

10b Root test: since

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty}\frac{1}{\ln n} = 0 < 1,$$

the series converges.

11a $\sum(-1)^n/\ln n$ converges by the Alternating Series Test; however, $\sum|(-1)^n/\ln n| = \sum 1/\ln n$ diverges by the Direct Comparison Test since $1/(\ln n) > n$ for all sufficiently large n, and $\sum 1/n$ is known to diverge. Therefore $\sum(-1)^n/\ln n$ converges conditionally.

11b The series converges by the Alternating Series Test, and since

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n \arctan n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{\arctan n}{n^2}$$

with $(\arctan n)/n^2 \leq (\pi/2)/n^2$ and $\sum 1/n^2$ convergent, we use the Direct Comparison Test to conclude that the given series is absolutely convergent.