Math 141 Exam #2 Key (Summer 2019)

1a Let $u = \ln x$, $v' = 1$, so that $u' = 1/x$, $v = x$. Then:

$$\int \ln \sqrt{x}\, dx = \frac{1}{2} \int \ln x\, dx = \frac{1}{2}\left(x \ln x - \int dx\right) = \frac{x \ln x - x}{2} + C.$$

1b Let $u = \ln(\cos x)$, $v' = \sin x$, so $u' = -\tan x$, $v = -\cos x$. Then:

$$\Big[-\cos x \ln(\cos x)\Big]_0^{\pi/3} - \int_0^{\pi/3} \sin x\, dx = \frac{\ln 2 - 1}{2}.$$

2a Let $u = \cos 6t$, so

$$\int (\sin 6t)(1 - \cos^2 6t)^2(\cos^2 6t)\, dt = -\frac{1}{6} \int (1 - u^2)^2 u^2\, du = -\frac{1}{6}\left(\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right) + C,$$

and finally

$$-\frac{1}{42}\cos^7 6t + \frac{1}{15}\cos^5 6t - \frac{1}{18}\cos^3 6t + C.$$

2b Let $u = \tan\theta$, so

$$\int \tan^2\theta(\tan^2\theta + 1)\sec^2\theta\, d\theta = \int u^2(u^2+1)\, du = \frac{u^5}{5} + \frac{u^3}{3} + C = \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + C.$$

2c Since $1 + \cos 2x = 2\cos^2 x$, we get

$$\int_0^{\pi/6} \sqrt{2\cos^2 x}\, dx = \sqrt{2} \int_0^{\pi/6} \cos x\, dx = \frac{1}{\sqrt{2}}.$$

3a Let $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, and with a reduction of order formula we have

$$\int_0^{\pi/4} \cos^2\theta\, d\theta = \left[\frac{\cos\theta \sin\theta}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} d\theta = \frac{1}{4} + \frac{\pi}{8}.$$

3b Let $t = 4\sec\theta$, so $dt = 4\sec\theta\tan\theta\, d\theta$. Then:

$$\int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta - 16}} d\theta = \frac{1}{16}\int \cos\theta\, d\theta = \frac{1}{16}\sin\theta + C = \frac{\sqrt{t^2 - 16}}{16t} + C.$$

4a With partial fraction decomposition:

$$\int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right) dy = \left[2\ln|y| + \frac{9}{5}\ln|y+2| + \frac{1}{5}\ln|y-3|\right]_1^2 = \frac{9}{5}\ln\left(\frac{8}{3}\right).$$

4b We have

$$\int \left(\frac{2}{x} - \frac{x+1}{x^2+3}\right) dx = 2\int \frac{1}{x} dx - \int \frac{x}{x^2+3} dx - \int \frac{1}{x^2+3} dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C.$$

4c Let $u = \sqrt[3]{x}$, so $du = \frac{1}{3}x^{-2/3}dx$, and hence $dx = 3u^2\,du$. Integral becomes

$$\int_0^1 \frac{3u^2}{u+1}\,du = \int_0^1 \left(3u - 3 + \frac{3}{u+1}\right) du = \ln 8 - \frac{3}{2}.$$

5a Let $u = e^x$:

$$\int e^x \cdot e^{e^x}\,dx = \int e^u\,du = e^u + C = e^{e^x} + C.$$

5b Integration by parts: set $u = \arctan\sqrt{x}$ and $v' = 1$, so $u' = \dfrac{1}{2(x+1)\sqrt{x}}$ and $v = x$. Integral becomes:

$$x\arctan\sqrt{x} - \int \frac{1}{x+1}\cdot\frac{1}{2\sqrt{x}}\,dx.$$

Now let $w = \sqrt{x}$ in the new integral to get

$$x\arctan\sqrt{x} - \int \frac{w^2}{w^2+1}\,dw = x\arctan\sqrt{x} - \int \left(1 - \frac{1}{w^2+1}\right) dw,$$

where long division shows that

$$\frac{w^2}{w^2+1} = 1 - \frac{1}{w^2+1}.$$

Finally we obtain

$$(x+1)\arctan\sqrt{x} - \sqrt{x} + C.$$

5c Rewrite the radicand as a difference of squares, then make the substitution $u = x+1$ followed by the substitution $u = 2\sin\theta$:

$$\int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4-u^2}\,du = \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$$
$$= 4\int \cos^2\theta\,d\theta = 4\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int d\theta\right)$$
$$= 2\cos\theta\sin\theta + 2\theta + C = 2\cdot\frac{\sqrt{4-u^2}}{2}\cdot\frac{u}{2} + 2\sin^{-1}\frac{u}{2} + C$$
$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C.$$

6a Let $u = 1 + x^3$, so $du = 3x^2\,dx$, and integral becomes

$$\int_1^\infty \frac{1/3}{\sqrt{u}}\,du = \lim_{t\to\infty}\left[\frac{2}{3}\sqrt{u}\right]_1^t = \lim_{t\to\infty}\frac{2}{3}\left(\sqrt{t}-1\right) = \infty.$$

Integral diverges.

6b Integral converges:

$$\lim_{t\to -2^+}\left[\frac{4}{3}(x+2)^{3/4}\right]_t^{14} = \frac{4}{3}\lim_{t\to -2^+}\left[8-(t+2)^{3/4}\right] = \frac{32}{3}.$$

6c Integration by parts yields

$$\left[\frac{z}{2}e^{2z}\right]_{-\infty}^0 - \int_{-\infty}^0 \frac{1}{2}e^{2z}\,dz = \lim_{t\to -\infty}\left[\frac{z}{2}e^{2z}\right]_t^0 - \frac{1}{4}\lim_{t\to -\infty}\left[e^{2z}\right]_t^0 = -\frac{1}{4}.$$

Integral converges.

7 For all $x \in (0,1]$ we have

$$\frac{\sec^2 x}{x\sqrt{x}} \geq \frac{1}{x\sqrt{x}},$$

and since

$$\int_0^1 \frac{1}{x\sqrt{x}}\,dx = -\lim_{t\to 0^+}\left[\frac{2}{\sqrt{x}}\right]_t^1 = -2\lim_{t\to 0^+}\left(1-\frac{1}{\sqrt{t}}\right) = \infty,$$

the integral

$$\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$$

diverges by the Comparison Theorem.