Math 141 Exam #1 Key (Summer 2019)

1 Let $y = f(x)$, so that

$$y = \frac{1-\sqrt{x}}{1+\sqrt{x}}.$$

Since $f^{-1}(y) = x$, solve for x to get

$$x = \left(\frac{1-y}{1+y}\right)^2 \quad \Rightarrow \quad f^{-1}(y) = \left(\frac{1-y}{1+y}\right)^2.$$

2 Since $f(0) = 2$ and $f'(x) = 3x^2 + 3\cos x - 2\sin x$, the Inverse Function Theorem gives

$$(f^{-1})'(2) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{3(0)^2 + 3\cos 0 - 2\sin 0} = \frac{1}{3}.$$

3a $f'(x) = \dfrac{1}{\sin^2 x} \cdot 2\sin x \cos x = 2\cot x.$

3b $g'(x) = e^{x^2-x}(x^2-x)' = (2x-1)e^{x^2-x}.$

3c $h'(t) = \dfrac{d}{dt} e^{\ln(3^{\cos 5t})} = \dfrac{d}{dt} e^{\ln(3)\cos 5t} = e^{\ln(3)\cos 5t} \dfrac{d}{dt}(\ln(3)\cos 5t) = -3^{\cos 5t}[5\ln(3)\sin 5t].$

3d We have

$$\begin{aligned} y = \log_2(4-6x) \quad &\Rightarrow \quad 2^y = 4 - 6x \quad \Rightarrow \quad y\ln 2 = \ln(4-6x) \\ &\Rightarrow \quad y = \frac{\ln(4-6x)}{\ln 2} \quad \Rightarrow \quad y' = -\frac{3}{(2-3x)\ln 2}. \end{aligned}$$

3e $F'(x) = \dfrac{d}{dx} e^{\ln(x^{x/9})} = \dfrac{d}{dx} e^{(x/9)\ln x} = e^{(x/9)\ln x} \dfrac{d}{dx}\left(\dfrac{x}{9}\ln x\right) = \dfrac{1}{9}x^{x/9}(1 + \ln x).$

3f We have

$$y = \frac{d}{dx} e^{\ln(\ln x)^{\cos x}} = \frac{d}{dx} e^{\cos x \cdot \ln(\ln x)} = (\ln x)^{\cos x}\left[\frac{\cos x}{x \ln x} - \sin x \cdot \ln(\ln x)\right].$$

3g $y' = \dfrac{1}{\sqrt{1-(2/x+1)^2}}\left(\dfrac{2}{x}+1\right)' = -\dfrac{2}{x^2\sqrt{1-(2/x+1)^2}} = -\dfrac{1}{|x|\sqrt{-(x+1)}}.$

3h $f'(x) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}.$

4 Here

$$y' = \frac{3x^2}{x^3 - 7},$$

so when $x = 2$ we have $y' = 12$. This is the slope of the tangent line through the point $(2, 0)$, so the equation is $y = 12x - 24$.

5a Letting $u = 8 - 3t$, we have

$$\int_1^2 \frac{1}{8-3t} dt = -\int_5^2 \frac{1/3}{u} du = -\frac{1}{3}[\ln u]_5^2 = \tfrac{1}{3} \ln \tfrac{5}{2}.$$

5b $\displaystyle\int \frac{(1+e^x)^2}{e^x} dx = \int (e^{-x} + 2 + e^x) dx = -e^{-x} + 2x + e^x + c.$

5c Let $u = \log_{10} x$, so $10^u = x$, which implies $u = \ln x / \ln 10$, and then $(1/x)dx = \ln 10\, du$. Now,

$$\int \frac{\log_{10} x}{x} dx = \ln 10 \int u\, du = \frac{\ln 10}{2} u^2 + c = \frac{\ln 10}{2} (\log_{10} x)^2 + c.$$

5d Let $u = \sin\theta$, so $du = \cos\theta\, d\theta$, and then

$$\int 3^{\sin\theta} \cos\theta\, d\theta = \int 3^u\, du = \int e^{u \ln 3} du.$$

Now let $v = u \ln 3$, so $du = dv / \ln 3$ and we finally obtain

$$\int e^{u\ln 3} du = \int \frac{e^v}{\ln 3} dv = \frac{e^v}{\ln 3} + c = \frac{e^{\ln(3)\sin\theta}}{\ln 3} + c = \frac{3^{\sin\theta}}{\ln 3} + c.$$

5e We have

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{16} \int_0^{\sqrt{3}/4} \frac{1}{(1/4)^2 + x^2} dx = \frac{1}{16} \left[\frac{1}{1/4} \tan^{-1} \frac{x}{1/4} \right]_0^{\sqrt{3}/4} = \frac{\pi}{12}.$$

5f Since $\cosh^2 x - \sinh^2 x = 1$, we can let $u = \sinh x$ to obtain

$$\int \frac{\cosh x}{\cosh^2 x - 1} dx = \int \frac{\cosh x}{\sinh^2 x} dx = \int \frac{1}{u^2} du = -\frac{1}{u} + c = -\frac{1}{\sinh x} + c.$$

6 Using implicit differentiation,

$$\begin{aligned} x^y = y^x \;\Rightarrow\; y \ln x = x \ln y \;&\Rightarrow\; \frac{d}{dx}(y \ln x) = \frac{d}{dx}(x \ln y) \\ &\Rightarrow\; \frac{y}{x} + y' \ln x = \ln y + \frac{xy'}{y} \;\Rightarrow\; y' = \frac{\ln y - y/x}{\ln x - x/y}. \end{aligned}$$

7a $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = 1.$

7b $\lim_{x\to 0} \sin 3x \csc 8x = \lim_{x\to 0} \frac{\sin 3x}{\sin 8x} = \lim_{x\to 0} \frac{3\cos 3x}{8\cos 8x} = \frac{3}{8}.$

7c $\lim_{x\to 0^+} (9x+1)^{\cot x} = \lim_{x\to 0^+} e^{\cot x \cdot \ln(9x+1)} = e^{\lim_{x\to 0^+} \frac{\ln(9x+1)}{\tan x}} = \exp\left(\lim_{x\to 0^+} \frac{\frac{9}{9x+1}}{\sec^2 x}\right) = \exp(9) = e^9.$