Math 141 Exam #4 Key (Summer 2018)

1 The 4th-order Taylor polynomial consists of the first four terms of the Maclaurin series for $\ln(1+x)$ with $x = -0.1$:

$$\ln(0.9) = \ln(1-0.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-0.1)^n}{n} = -\sum_{n=1}^{\infty} \frac{0.1^n}{n}$$
$$\approx -\sum_{n=1}^{4} \frac{0.1^n}{n} = -\frac{1}{10} - \frac{1}{200} - \frac{1}{3000} - \frac{1}{40{,}000} = -0.105358\bar{3}.$$

2a Apply Ratio Test:

$$\lim_{n\to\infty} \left| \frac{3^{2(n+1)}x^{n+1}}{(n+1)^4} \cdot \frac{n^4}{3^{2n}x^n} \right| = |x| \lim_{n\to\infty} \frac{9n^4}{(n+1)^4} = 9|x|.$$

Series converges if $|x| < 1/9$, so interval of convergence contains $\left(-\frac{1}{9}, \frac{1}{9}\right)$. Check endpoints.

At $x = 1/9$: series becomes $\sum 1/n^4$, a convergent p-series. At $x = -1/9$: series becomes $\sum (-1)^n/n^4$, which converges by the Alternating Series Test.

Interval of convergence is $\left[-\frac{1}{9}, \frac{1}{9}\right]$.

2b Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(2x+1)^{n+1}}{(n+1)\cdot 8^{n+1}} \cdot \frac{n \cdot 8^n}{(2x+1)^n} \right| = |2x+1| \lim_{n\to\infty} \frac{n}{8n+8} = \frac{|2x+1|}{8}.$$

Series converges if $|2x+1| < 8$, so interval of convergence contains $\left(-\frac{9}{2}, \frac{7}{2}\right)$. Check endpoints.

At $x = \frac{7}{2}$ series becomes $\sum 1/n$, which diverges. At $x = -\frac{9}{2}$ series becomes $\sum (-1)^n/n$, which converges by the Alternating Series Test.

Interval of convergence is $\left[-\frac{9}{2}, \frac{7}{2}\right)$.

2c Ratio Test:

$$\lim \left| \frac{(n+1)!(x-10)^{n+1}}{n!(x-10)^n} \right| = \lim_{n\to\infty} (n+1)|x-10| = \begin{cases} \infty, & x \neq 10 \\ 0, & x = 10. \end{cases}$$

The series only converges at $\{10\}$.

3 Use the geometric series:

$$\frac{5x^2}{5+x^3} = x^2 \cdot \frac{1}{1-(-x^3/5)} = x^2 \sum_{n=0}^{\infty} \left(-\frac{x^3}{5}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{5^n}.$$

Interval of convergence is $|-x^3/5| < 1$, or $|x|^3 < 5$, and hence $\left(-\sqrt[3]{5}, \sqrt[3]{5}\right)$.

4 First we write

$$(x-3)^{1/3} = \left[-3\left(-\frac{x}{3}+1\right)\right]^{1/3} = \sqrt[3]{-3}\left(1-\frac{x}{3}\right)^{1/3}.$$

Now, using the binomial series,

$$f(x) = -3^{1/3}\sum_{n=0}^{\infty}\binom{1/3}{n}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\binom{1/3}{n}\frac{(-1)^{n+1}x^n}{3^{n-1/3}}.$$

This series converges for all x such that $|-x/3| < 1$, and so the interval of convergence is $(-3,3)$.

5 From the table provided we have $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all $x \in (-\infty,\infty)$, and so

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}$$

for all x. Now,

$$\int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}\right)dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}+c$$

for all x and arbitrary constant c. Thus, by the Fundamental Theorem of Calculus,

$$\int_0^{1/3} e^{-x^2}dx = \int_0^{1/3}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}\right)dx = \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}\right]_0^{1/3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\frac{1}{3}\right)^{2n+1}.$$

We have arrived at an alternating series $\sum(-1)^n b_n$ with

$$b_n = \frac{1}{n!(2n+1)}\left(\frac{1}{3}\right)^{2n+1}$$

for $n \geq 0$. The first few b_n values are

$$b_0 = \tfrac{1}{3},\quad b_1 = \tfrac{1}{81},\quad b_2 = \tfrac{1}{2430},\quad b_3 = \tfrac{1}{91{,}854},\quad b_4 = \tfrac{1}{4{,}251{,}528},$$

so by the Alternating Series Estimation Theorem the approximation

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\frac{1}{3}\right)^{2n+1} \approx b_0 - b_1 + b_2 - b_3 \approx 0.3213882901$$

will have an absolute error that is less than $b_4 \approx 2.35 \times 10^{-7} < 10^{-6}$. Hence the approximation

$$\int_0^{1/3} e^{-x^2}dx \approx \frac{1}{3} - \frac{1}{81} + \frac{1}{2430} - \frac{1}{91,854}$$

has an absolute error less than 10^{-6}.

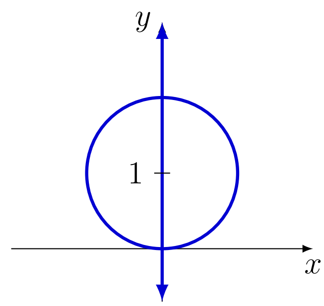
6 We have $x^2 = t+1$, so that $t-1 = x^2-2$, and then $y^2 = t-1 = x^2-2$. Noting that $y \geq 0$, it follows that

$$y = \sqrt{x^2-2}.$$

The domain of this function is $(\sqrt{2},\infty)$.

7 The set-up is thus:
$$(x,y) = \left(1 - \tfrac{1}{30}t\right)(4,-40) + \tfrac{1}{30}t\,(2,10)$$
for $0 \le t \le 30$. Equivalently we may write
$$(x,y) = \left(-\tfrac{1}{15}t + 4, \tfrac{5}{3}t - 40\right), \quad t \in [0,30].$$

8 Using a given trigonometric identity gives
$$r\cos\theta = 2\cos\theta\sin\theta \;\Rightarrow\; r^2(r\cos\theta) = 2(r\cos\theta)(r\sin\theta) \;\Rightarrow\; (x^2+y^2)x = 2xy,$$
or equivalently
$$x(x^2+y^2-2y) = 0.$$
The graph of this equation will include the vertical line $x = 0$, and also the curve
$$x^2 + y^2 - 2y = 0 \;\Rightarrow\; x^2 + (y-1)^2 = 1,$$
which is a circle centered at $(0,1)$ with radius 1.



9 With $f(\theta) = 8\sin\theta$ and $\theta_0 = 5\pi/6$, the slope is
$$\frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0} = \frac{8\sqrt{3}(1/2) + 4\sqrt{3}}{8\sqrt{3}(\sqrt{3}) - 4(1/2)} = \frac{4\sqrt{3}}{11}.$$

10 When $\theta = 0$ we have $r = 0$ (the "stem" of the leaf), and when $\theta = \pi/8$ we have $r = 2$ (the "tip" of the leaf). This covers half of a single leaf, and to cover the whole leaf we increase θ further to $\pi/4$ to again obtain $r = 0$. Using the identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, the area $\mathcal{A}$ of the leaf is
$$\mathcal{A} = \frac{1}{2}\int_0^{\pi/4} (2\sin 4\theta)^2\,d\theta = \int_0^{\pi/4} (1 - \cos 8\theta)\,d\theta = \left[\theta - \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/4} = \frac{\pi}{4}.$$