Math 141 Exam #3 Key (Summer 2018)

1a Use L'Hôpital's rule:

$$\lim_{n\to\infty} n\tan\frac{\pi}{n} = \lim_{n\to\infty}\frac{\tan(\pi/n)}{1/n} \overset{\text{LR}}{=} \lim_{n\to\infty}\frac{(-\pi/n^2)\sec^2(\pi/n)}{-1/n^2} = \lim_{n\to\infty}\pi\sec^2\frac{\pi}{n} = \pi\sec^2 0 = \pi.$$

1b We have

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n^4-2n}-n^2\right) &= \lim_{n\to\infty}\frac{(\sqrt{n^4-2n}-n^2)(\sqrt{n^4-2n}+n^2)}{\sqrt{n^4-2n}+n^2} = \lim_{n\to\infty}\frac{-2n}{\sqrt{n^4-2n}+n^2}\\ &= \lim_{n\to\infty}\frac{-2/n}{\sqrt{1-2/n^3}+1} = \frac{0}{\sqrt{1-0}+1} = 0.\end{aligned}$$

2 Since $-\pi/2 < \tan^{-1} n < \pi/2$ for any integer n, we have

$$-\frac{10\pi}{n^2+8} < \frac{20\tan^{-1} n}{n^2+8} < \frac{10\pi}{n^2+8}$$

for all n, and since

$$\lim_{n\to\infty}\left(\pm\frac{10\pi}{n^2+8}\right) = 0,$$

the Squeeze Theorem implies that

$$\lim_{n\to\infty}\frac{20\tan^{-1} n}{n^2+8} = 0.$$

3 Reindex to obtain

$$\sum_{n=2}^{\infty}\frac{3}{7^n} = \sum_{n=0}^{\infty}\frac{3}{7^{n+2}} = \frac{3}{7^2}\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n = \frac{3}{49}\cdot\frac{1}{1-1/7} = \frac{1}{14}.$$

4 The nth partial sum is

$$\begin{aligned}s_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + [\ln n - \ln(n-1)] + [\ln(n+1) - \ln n]\\ &= -\ln 1 + \ln(n+1) = \ln(n+1),\end{aligned}$$

and so

$$\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\ln(n+1) = \infty.$$

That is, the series diverges.

5 Find the smallest integer value of n for which $\frac{1}{2n^4} < \frac{1}{10{,}000}$. Since

$$\frac{1}{2n^4} < \frac{1}{10,000} \quad \Rightarrow \quad n^4 > 5000,$$

and 9 is the first integer for which $9^4 > 5000$, estimation with the first eight terms will suffice:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^4} \approx \sum_{n=1}^{8} \frac{(-1)^n}{2n^4}$$

has absolute error less than 10^{-4}.

6a For all $n \geq 1$ we have

$$0 < \frac{4}{2+3^n n} \leq \frac{4}{3^n n} \leq \frac{4}{3^n},$$

and since $\sum 4/3^n$ is a convergent geometric series, we conclude by the Direct Comparison Test that the given series converges.

6b Since

$$\lim_{n\to\infty} \frac{4^n}{n^2} = +\infty,$$

the series diverges by the Divergence Test.

6c For all $n \geq 1$ we have

$$0 \leq \frac{\tan^{-1} n}{n^2} \leq \frac{\pi}{2n^2},$$

and since $\sum 1/n^2$ is a convergent p-series, it follows that $\sum \pi/2n^2$ is likewise convergent, and therefore the given series converges by the Direct Comparison Test.

6d Since

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} \right| = 2 \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = 2 \lim_{n\to\infty} \exp\left(n \cdot \ln \frac{n}{n+1} \right) \\ &= 2 \exp\left(\lim_{n\to\infty} \frac{\ln n - \ln(n+1)}{1/n} \right) \overset{\text{LR}}{=} 2 \exp\left(\frac{1/n - 1/(n+1)}{-1/n^2} \right) \\ &= 2 \exp\left(-\lim_{n\to\infty} \frac{n}{n+1} \right) = 2\exp(-1) = \frac{2}{e} < 1,\end{aligned}$$

the series converges by the Ratio Test.

6e Since

$$\lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} \exp\left(-\frac{\ln n}{n} \right) = \exp(0) = 1 \neq 0,$$

the series diverges by the Divergence Test.

6f Since

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{1\cdot 3\cdot 5\cdots[2(n+1)-1]}{[2(n+1)-1]!}\cdot\frac{(2n-1)!}{1\cdot 3\cdot 5\cdots(2n-1)}\right)$$
$$= \lim_{n\to\infty}\frac{2(n+1)-1}{2n(2n+1)} = \lim_{n\to\infty}\frac{2n+1}{4n^2+2n} = 0,$$

the series converges by the Ratio Test.

7a Since $(1/n^{5/4})$ is a decreasing sequence of nonnegative values such that $1/n^{5/4} \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test. Since $\sum 1/n^{5/4}$ is a convergent p-series, the given series is also absolutely convergent.

7b Since

$$\lim_{n\to\infty}\frac{n}{\ln n} = +\infty$$

the series diverges by the Divergence Test.