Math 141 Exam #2 Key (Summer 2018)

1 A long division along the way is needed:

$$\int \frac{8}{t^{-2}+1}dt = 8\int \frac{t^2}{t^2+1}dt = 8\int\left(1-\frac{1}{t^2+1}\right)dt = 8t - 8\tan^{-1}t + c.$$

2a Integration by parts twice, starting with $u = x^2$ and $v' = \sin 2x$, gives

$$\begin{aligned}\int_0^{\pi/4} x^2 \sin 2x\,dx &= \frac{\pi^2}{8} + \int_0^{\pi/4} x\cos 2x\,dx \\ &= \frac{\pi^2}{8} + \left[\frac{x}{2}\sin 2x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\sin 2x\,dx \\ &= \frac{\pi^2}{8} + \frac{1}{4}\big[\cos 2x\big]_0^{\pi/4} = \frac{\pi^2}{8} - \frac{1}{2}.\end{aligned}$$

2b Using integration by parts with $u = (\ln x)^2$, $v' = 1$ gives

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2x\ln x + 2x + c.$$

Note that $\int \ln x\,dx$ also requires integration by parts if you don't remember the formula.

3 That f'' is continuous on $[a,b]$ ensures that $xf''(x)$ is continuous (and hence integrable) on $[a,b]$, and so integration by parts and the assumption that $f'(a) = f'(b) = 0$ gives

$$\int_a^b xf''(x)\,dx = xf'(x)\big|_a^b - \int_a^b f'(x)\,dx = af'(a) - bf'(b) - \int_a^b f'(x)\,dx = -\int_a^b f'(x)\,dx.$$

Next, the continuity of f'' on $[a,b]$ implies that f' is differentiable on $[a,b]$, and so

$$\int_a^b f'(x)\,dx = f(x)\big|_a^b = f(b) - f(a)$$

by the Fundamental Theorem of Calculus. Putting our two findings together, we have

$$\int_a^b xf''(x)\,dx = -[f(b)-f(a)] = f(a) - f(b).$$

4a Let $u = \cos x$, so that

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^2 x}dx &= \int \frac{(1-\cos^2 x)^2 \sin x}{\cos^2 x}dx = -\int \frac{(1-u^2)^2}{u^2}du = \int (2-u^{-2}-u^2)\,du \\ &= 2u + u^{-1} - \frac{1}{3}u^3 + c = 2\cos x + \sec x - \frac{\cos^3 x}{3} + c.\end{aligned}$$

4b Since $\cot^2 = \csc^2 - 1$, we have

$$\int \cot^4 x\,dx = \int (\csc^2 x - 1)\cot^2 x\,dx = \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx.$$

Let $u = \cot x$ in the first integral, so that $du = -\csc^2 x\,dx$, and we get

$$\int \cot^2 x \csc^2 x\,dx = -\int u^2\,du = -\frac{1}{3}u^3 = -\frac{1}{3}\cot^3 x.$$

Thus, since $(\cot x)' = -\csc^2 x$,

$$\begin{aligned}\int \cot^4 x\,dx &= -\frac{1}{3}\cot^3 x - \int \cot^2 x\,dx = -\frac{1}{3}\cot^3 x - \int (\csc^2 x - 1)\,dx \\ &= -\frac{1}{3}\cot^3 x + \cot x + x + c.\end{aligned}$$

4c Let $u = \tan x$, so $du = \sec^2 x\,dx$, and with the identity $\sec^2 = \tan^2 + 1$ we obtain

$$\int \tan^9 x \sec^4 x\,dx = \int (u^{11} + u^9)\,du = \frac{1}{12}u^{12} + \frac{1}{10}u^{10} + c = \frac{1}{12}\tan^{12} x + \frac{1}{10}\tan^{10} x + c.$$

5a Let $y = \sin\theta$, so $dy = \cos\theta\,d\theta$. Now, $y \in \left[\frac{1}{2}, 1\right]$ implies $\frac{1}{2} \le \sin\theta \le 1$, and thus $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$. We have

$$\begin{aligned}\int_{1/2}^{1} \frac{\sqrt{1-y^2}}{y^2}dy &= \int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin^2\theta}\cos\theta\,d\theta = \int_{\pi/6}^{\pi/2} (\csc^2\theta - 1)\,d\theta = -\big[\cot\theta + \theta\big]_{\pi/6}^{\pi/2} \\ &= \left(\cot\frac{\pi}{6} + \frac{\pi}{6}\right) - \left(\cot\frac{\pi}{2} + \frac{\pi}{2}\right) = \sqrt{3} + \frac{\pi}{6} - 0 - \frac{\pi}{2} = \sqrt{3} - \frac{\pi}{3}.\end{aligned}$$

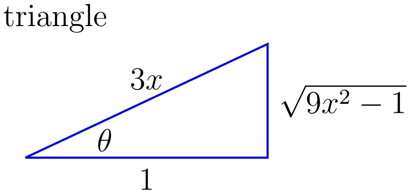
5b Let $x = \frac{1}{3}\sec\theta$ for $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$, so that $dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$. Since $x > \frac{1}{3}$ implies $\sec\theta > 1$ implies $\theta \in [0, \pi/2)$, we have $\tan\theta \ge 0$ and hence

$$\sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta.$$

Now,

$$\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx = 3\int \frac{\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\,d\theta = 3\int \cos\theta\,d\theta = 3\sin\theta + c.$$

From $\sec\theta = 3x$ we obtain the triangle



which makes clear that

$$\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx = 3 \cdot \frac{\sqrt{9x^2-1}}{3x} + c = \frac{\sqrt{9x^2-1}}{x} + c.$$

6a We have

$$\frac{8}{(y-4)^2(y+3)} = \frac{A}{y-4} + \frac{B}{(y-4)^2} + \frac{C}{y+3},$$

so

$$8 = (A+C)y^2 + (-A+B-8C)y + (-12A+3B+16C),$$

which yields the system of equations

$$\begin{cases} A \qquad\quad + \quad C = 0 \\ -A + \ B - \ 8C = 0 \\ -12A + 3B + 16C = 8 \end{cases}$$

The solution to the system is $(A, B, C) = \left(-\frac{8}{49}, \frac{8}{7}, \frac{8}{49}\right)$, so

$$\begin{aligned} \int \frac{8}{(y-4)^2(y+3)} &= -\frac{8}{49}\int \frac{1}{y-4}dy + \frac{8}{7}\int \frac{1}{(y-4)^2}dy + \frac{8}{49}\int \frac{1}{y+3}dy \\ &= -\frac{8}{49}\ln|y-4| - \frac{8}{7(y-4)} + \frac{8}{49}\ln|y+3| + c \\ &= \frac{8}{49}\ln\left|\frac{y+3}{y-4}\right| - \frac{8}{7(y-4)} + c. \end{aligned}$$

6b Again start with a decomposition, noting that x^2+2x+6 is an irreducible quadratic:

$$\begin{aligned} \int \frac{2}{(x-4)(x^2+2x+6)}\,dx &= \int \left(\frac{1/15}{x-4} + \frac{-x/15 - 2/5}{x^2+2x+6}\right) dx \\ &= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+6}{(x+1)^2+5}\,dx. \end{aligned} \tag{1}$$

For the remaining integral, let $u = x+1$ to obtain

$$\int \frac{x+6}{(x+1)^2+5}\,dx = \int \frac{u+5}{u^2+5}\,du = \int \frac{u}{u^2+5}\,du + 5\int \frac{1}{u^2+(\sqrt{5}\,)^2}\,du \tag{2}$$

Letting $w = u^2+5$ in the first integral at right in (2), and using a formula for the second, we next get

$$\begin{aligned} \int \frac{x+6}{(x+1)^2+5}\,dx &= \int \frac{1/2}{w}\,dw + 5\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln|w| + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c = \frac{1}{2}\ln(u^2+5) + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln[(x+1)^2+5] + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c \end{aligned}$$

Returning to (1),

$$\begin{aligned} \int \frac{2}{(x-4)(x^2+2x+6)}\,dx &= \frac{\ln|x-4|}{15} - \frac{1}{15}\left[\frac{\ln[(x+1)^2+5]}{2} + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\right] \\ &= \frac{\ln|x-4|}{15} - \frac{\ln(x^2+2x+6)}{30} - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c. \end{aligned}$$

7 We have

$$\lim_{t\to\infty}\int_0^t e^{-ax}dx = \lim_{t\to\infty}\left[-\frac{1}{a}e^{-ax}\right]_0^t = \lim_{t\to\infty}\left(\frac{1-e^{-at}}{a}\right) = \frac{1}{a},$$

and so the integral converges.

8 We have, with the substitution $u = t^4 - 1$,

$$\begin{aligned}\int_0^1 \frac{t^3}{t^4-1}\,dt &= \lim_{x\to 1^-}\int_{-1}^{x^4-1}\frac{1/4}{u}\,du = \frac{1}{4}\lim_{x\to 1^-}\big[\ln|u|\big]_{-1}^{x^4-1} \\ &= \frac{1}{4}\lim_{x\to 1^-}\ln|x^4-1| = \frac{1}{4}\lim_{x\to 1^-}\ln(1-x^4) = -\infty.\end{aligned}$$

The integral diverges.

9 The volume of the solid is, using the substitution $u = \ln x$,

$$\int_2^\infty \pi[f(x)]^2dx = \pi\lim_{t\to\infty}\int_2^t\frac{1}{x\ln^2 x}dx = \pi\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u^2}\,du = \pi\lim_{t\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln t}\right) = \frac{\pi}{\ln 2}.$$