Math 141 Exam #1 Key (Summer 2018)

1a We have $f'(x) = 3 + \cos x$, so $f'(x) > 0$ for all $x \in \mathbb{R}$, which implies f is strictly increasing everywhere and is therefore one-to-one.

1b Since $f(0) = 0$, by the Inverse Function Theorem we have

$$(f^{-1})'(0) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{3 + \cos(0)} = \frac{1}{4}.$$

2a Quotient rule:

$$y' = \frac{(\ln x + 1)(1/x) - (\ln x)(1/x)}{(\ln x + 1)^2} = \frac{1}{x(\ln x + 1)^2}.$$

2b Chain rule: $f'(x) = \cos(\cos e^x) \cdot (-\sin e^x) \cdot e^x$.

2c We have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(e^{(x-1)\ln(\cot x)}\right) = e^{(x-1)\ln(\cot x)} \frac{d}{dx}\left[(x-1)\ln(\cot x)\right] \\ &= (\cot x)^{x-1}\left[-\frac{(x-1)\csc^2 x}{\cot x} + \ln(\cot x)\right]. \end{aligned}$$

2d We have

$$h'(t) = \frac{d}{dt}\left(e^{t^{1/2}\ln t}\right) = e^{t^{1/2}\ln t}\frac{d}{dt}(t^{1/2}\ln t) = t^{(t^{1/2})}\left(t^{-1/2} + \frac{1}{2}t^{-1/2}\ln t\right).$$

2e Use algebra to find that

$$\ell(x) = \frac{4\ln(1 - x^3)}{\ln 3}.$$

Now,

$$\ell'(x) = \frac{12x^2}{(x^3 - 1)\ln 3}.$$

2f By the Chain Rule:

$$\varphi'(u) = \frac{1}{1 + (4u-4)^2} \cdot (4u - 4)' = \frac{4}{1 + (4u-4)^2}.$$

2g $y' = 2x\cosh^3(5x) + 15x^2\cosh^2(5x)\sinh(5x)$.

3 Here $y'(x) = 2e^x$, so slope of the tangent line at $x = \ln 6$ is $y'(\ln 6) = 2e^{\ln 6} = 12$. Also the tangent line contains the point $(\ln 6, 20)$ since $y(\ln 6) = 2e^{\ln 6} + 8 = 20$. Equation of the line:

$$y = 12x + (20 - 12\ln 6).$$

4a Let $u = x^6$, so

$$\int x^5 e^{x^6}\,dx = \int e^u du = \frac{1}{6}e^u + c = \frac{1}{6}e^{x^6} + c.$$

4b Let $u = \ln(\ln x)$, so $du = \frac{1}{x\ln x}dx$, and the integral becomes

$$\int \frac{1}{u}du = \ln|u| + c = \ln\big|\ln(\ln x)\big| + c.$$

4c Let $u = \cos x$, so $du = -\sin x\,dx = -\frac{1}{\csc x}dx$. Integral becomes

$$-\int e^u du = -e^u + c = -e^{\cos x} + c.$$

5a Letting $u = e^{t/2} + 1$, so integral becomes

$$\int_{1/e+1}^{e+1} \frac{2}{u}\,du = 2\ln|u|\Big]_{1/e+1}^{e+1} = 2\ln\left(\frac{e+1}{e^{-1}+1}\right) = 2.$$

5b Use a given formula:

$$\frac{1}{2}\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}\,dx = \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{3/2} = \frac{\pi}{12}.$$

5c Let $u = \cosh 4x$, so integral becomes

$$\frac{1}{4}\int_1^{\cosh 4} u^3\,du = \frac{1}{4}\left[\frac{1}{4}u^4\right]_1^{\cosh 4} = \frac{\cosh^4 4 - 1}{16}.$$

6 We have

$$\lim_{x\to\infty} e^{\frac{\ln(2x^3+1)}{\ln x}} = \exp\left(\lim_{x\to\infty}\frac{\ln(2x^3+1)}{\ln x}\right) \overset{\text{LR}}{=} \exp\left(\lim_{x\to\infty}\frac{6x^2/(2x^3+1)}{1/x}\right) = \exp\left(\lim_{x\to\infty}\frac{6x^3}{2x^3+1}\right) = e^3.$$