Math 141 Exam #4 Key (Summer 2017)

1 4th-order Taylor polynomial is $p_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$. Now, converting to radians,

$$\cos(3^\circ) = \cos\left(\tfrac{\pi}{60}\right) \approx p_4\left(\tfrac{\pi}{60}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{60}\right)^2 + \frac{1}{24}\left(\frac{\pi}{60}\right)^4 \approx 0.9986295348.$$

2a Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right| = |x| \lim_{n\to\infty}\left|\sqrt{\frac{n^2+3}{n^2+2n+4}}\right| = |x|.$$

Series converges if $|x| < 1$, so interval of convergence contains $(-1, 1)$. Check endpoints.

At $x = 1$: series becomes $\sum 1/\sqrt{n^2+3}$, and since

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and the series $\sum 1/n$ is known to diverge, the series $\sum 1/\sqrt{n^2+3}$ diverges by the Direct Comparison Test.

At $x = -1$: series becomes $\sum(-1)^n/\sqrt{n^2+3}$, which can be shown to converge by the Alternating Series Test.

Therefore the original series has interval of convergence $[-1, 1)$.

2b Apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|2 + \frac{2}{n}\right| |x-1| = |x-1| \lim_{n\to\infty}\left(2 + \frac{2}{n}\right) = 2|x-1|.$$

Series converges if $2|x-1| < 1$, so interval of convergence contains $\left(\frac{1}{2}, \frac{3}{2}\right)$. Check endpoints.

At $x = \frac{3}{2}$: Series becomes

$$\sum\left(2 + \frac{2}{n}\right)^n \left(\frac{1}{2}\right)^n = \sum\left(1 + \frac{1}{n}\right)^n,$$

and since $\left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$, the series diverges by the Divergence Test.

At $x = \frac{1}{2}$: Series becomes

$$\sum\left(2 + \frac{2}{n}\right)^n \left(-\frac{1}{2}\right)^n = \sum(-1)^n\left(1 + \frac{1}{n}\right)^n,$$

which also diverges by the Divergence Test.

Therefore the original series has interval of convergence $\left(\frac{1}{2}, \frac{3}{2}\right)$.

2c Apply Ratio Test, using L'Hôpital's Rule to find the limit:

$$\lim\left|\frac{x^{n+1}\ln(n+1)}{x^n \ln n}\right| = |x| \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \stackrel{\text{LR}}{=} |x| \lim_{n\to\infty}\frac{1/(n+1)}{1/n} = |x|.$$

Thus the series converges if $|x| < 1$, so interval of convergence contains $(-1, 1)$. At the endpoints we obtain either the series $\sum \ln n$ or $\sum(-1)^n \ln n$, both of which diverge by the Divergence Test. Therefore the original series has interval of convergence $(-1, 1)$.

3 Use the geometric series:

$$\frac{2x^2}{1+x^3} = 2x^2 \cdot \frac{1}{1-(-x^3)} = 2x^2 \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n 2x^{3n+2}.$$

Interval of convergence is $|-x^3| < 1$, and hence $(-1, 1)$.

4 Using the binomial series,

$$f(x) = \sum_{n=0}^{\infty} \binom{1/4}{n} x^n = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \cdots.$$

Interval of convergence is $(-1, 1)$.

5 We have

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\, dx = \int_0^{0.1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n-1}}{n}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \int_0^{0.1} x^{n-1} dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n^2} = 0.1 - \frac{(0.1)^2}{2^2} + \frac{(0.1)^3}{3^2} - \cdots.$$

Since $(0.1)^3/3^2 < 10^{-5}$, the estimate

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\, dx \approx 0.1 - \frac{(0.1)^2}{2^2} = 0.0975$$

will have an absolute error less than 10^{-5}.

6 Use the identity $\tan^2 + 1 = \sec^2$ to find that $y^2 + 1 = \sec^2 t$, so that

$$x = \sec^2 t - 1 = (y^2 + 1) - 1 = y^2.$$

Thus we have $x = y^2$ with domain $y \in (-\infty, \infty)$, recalling that $y = \tan t$ for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

7 The set-up is thus:

$$(x, y) = \left(1 - \tfrac{1}{3}t\right)(3, -4) + \tfrac{1}{3}t\,(2, 0)$$

for $0 \le t \le 3$. Equivalently we may write

$$(x, y) = \left(-\tfrac{1}{3}t + 3, \tfrac{4}{3}t - 4\right), \quad t \in [0, 3]$$

8 Change to $r \sin\theta = e^{r\cos\theta}$, and hence $y = e^x$.

9 Using the formula $\int_\alpha^\beta \frac{1}{2}\left[f^2(\theta) - g^2(\theta)\right] d\theta$ with $f(\theta) = 2\sqrt{\sin 2\theta}$ and $g(\theta) = 0$, we find the area to be

$$\int_0^{\pi/2} 2\sin 2\theta\, d\theta = 2.$$